Growing Waves Due to Transverse Velocities

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This paper treats propagation of slow waves in two-dimensional neutralized electron flow in which all electrons have the same velocity in the direction of propagation but in which there are streams of two or more velocities normal to the direction of propagation. In a finite beam in which electrons are reflected elastically at the boundaries and in which equal dc currents are carried by electrons with transverse velocities $+u_1$ *and* $-u_1$*, there is an antisymmetrical growing wave if*

$$\omega_p^2 \sim (\pi u_1/W)^2$$

and a symmetrical growing wave if

$$\omega_p^2 \sim \tfrac{4}{3}(\pi u_1/W)^2$$

Here ω_p *is plasma frequency for the total charge density and* W *is beam width.*

INTRODUCTION

It is well-known that there can be growing waves in electron flow when the flow is composed of several streams of electrons having different velocities in the direction of propagation of the waves.[1-5] While Birdsall[6] considers the case of growing waves in electron flow consisting of streams which cross one another, the growing waves which he finds apparently occur when two streams have different components of velocity in the direction of propagation.

This paper shows that there can be growing waves in electron flow consisting of two or more streams with the same component of velocity in the direction of wave propagation but with different components of velocity transverse to the direction of propagation. Such growing waves can exist when the electric field varies in strength across the flow. Such waves could result in the amplification of noise fluctuations in electron flow. They could also be used to amplify signals.

Actual electron flow as it occurs in practical tubes can exhibit transverse velocities. For instance, in Brillouin flow,[7, 8] if we consider electron motion in a coordinate system rotating with the Larmor frequency we see that electrons with transverse velocities are free to cross the beam repeatedly, being reflected at the boundaries of the beam. The transverse velocities may be completely disorganized thermal velocities, or they may be larger and better-organized velocities due to aberrations at the edges of the cathode or at lenses or apertures. Two-dimensional Brillouin flow allows similar transverse motions.

It would be difficult to treat the case of Brillouin or Brillouin-like flow with transverse velocities. Here, simpler cases with transverse velocities will be considered. The first case treated is that of infinite ion-neutralized two-dimensional flow with transverse velocities. The second case treated is that of two-dimensional flow in a beam of finite width in which the electrons are elastically reflected at the boundaries of the beam. Growing waves are found in both cases, and the rate of growth may be large.

In the case of the finite beam both an antisymmetric mode and a symmetric mode are possible. Here, it appears, the current density required for a growing wave in the symmetric mode is about 4/3 times as great as the current density required for a growing wave in the antisymmetric mode. Hence, as the current is increased, the first growing waves to arise might be antisymmetric modes, which could couple to a symmetrical resonator or helix only through a lack of symmetry or through high-level effects.

1. *Infinite two-dimensional flow*

Consider a two-dimensional problem in which the potential varies sinusoidally in the y direction, as $\exp(-j\beta z)$ in the z direction and as exp $(j\omega t)$ with time. Let there be two electron streams, each of a negative charge ρ_0 and each moving with the velocity u_0 in the z direction, but with velocities u_1 and $-u_1$ respectively in the y direction. Let us denote ac quantities pertaining to the first stream by subscripts 1 and ac quantities pertaining to the second stream by subscripts 2. The ac charge density will be denoted by ρ, the ac velocity in the y direction by $\dot{y}$, and the ac velocity in the z direction by $\dot{z}$. We will use linearized or small-signal equations of motion.[9] We will denote differentiation with respect to y by the operator D.

The equation of continuity gives

$$j\omega\rho_1 = -D(\rho_1 u_1 + \rho_0 \dot{y}_1) + j\beta(\rho_1 u_0 + \rho_0 \dot{z}_1) \tag{1.1}$$

$$j\omega\rho_2 = -D(-\rho_2 u_1 + \rho_0 \dot{y}_2) + j\beta(\rho_2 u_0 + \rho_0 \dot{z}_2) \tag{1.2}$$

Let us define

$$d_1 = j(\omega - \beta u_0) + u_1 D \tag{1.3}$$

$$d_2 = j(\omega - \beta u_0) - u_1 D \tag{1.4}$$

We can then rewrite (1.1) and (1.2) as

$$d_1 \rho_1 = \rho_0(-D\dot{y}_1 + j\beta\dot{z}_1) \tag{1.5}$$

$$d_2 \rho_2 = \rho_0(-D\dot{y}_2 + j\beta\dot{z}_2) \tag{1.6}$$

We will assume that we are dealing with slow waves and can use a potential V to describe the field. We can thus write the linearized equations of motion in the form

$$d_1 \dot{z}_1 = -j\frac{e}{m}\beta V \tag{1.7}$$

$$d_2 \dot{z}_z = -j\frac{e}{m}\beta V \tag{1.8}$$

$$d_1 \dot{y}_1 = \frac{e}{m} DV \tag{1.9}$$

$$d_2 \dot{y}_2 = \frac{e}{m} DV \tag{1.10}$$

From (1.5) to (1.10) we obtain

$$d_1^2 \rho_1 = -\frac{e}{m}\rho_0(D^2 - \beta^2)V \tag{1.11}$$

$$d_2^2 \rho_2 = -\frac{e}{m}\rho_0(D^2 - \beta^2)V \tag{1.12}$$

Now, Poisson's equation is

$$(D^2 - \beta^2)V = -\frac{\rho_1 + \rho_2}{\epsilon} \tag{1.13}$$

From (1.11) to (1.13) we obtain

$$(D^2 - \beta^2)V = -\tfrac{1}{2}\omega_p^2\left(\frac{1}{d_1^2} + \frac{1}{d_2^2}\right)(D^2 - \beta^2)V \tag{1.14}$$

$$\omega_p^2 = \frac{-2\dfrac{e}{m}\rho_0}{\epsilon} \tag{1.15}$$

Here ω_p is the plasma frequency for the charge of both beams.

Either

$$(D^2 - \beta^2)V = 0 \tag{1.16}$$

or else

$$1 = \frac{-\omega_p^2}{2} \frac{(d_1^2 + d_2^2)}{d_1^2 d_2^2} \tag{1.17}$$

We will consider this second case.

We should note from (1.3) and (1.4) that

$$d_1^2 = u_1^2D^2 - (\omega - \beta u_0)^2 + 2jD(\omega - \beta u_0)u_1 \tag{1.18}$$

$$d_2^2 = u_1^2D^2 - (\omega - \beta u_0)^2 - 2jD(\omega - \beta u_0)u_1 \tag{1.19}$$

$$d_1^2 + d_2^2 = 2[u_1^2D^2 - (\omega - \beta u_0)^2] \tag{1.20}$$

$$d_1^2d_2^2 = [u_1^2D^2 + (\omega - \beta u_0)^2]^2 \tag{1.21}$$

Thus, (1.17) becomes

$$1 = \frac{-\omega_p^2[u_1^2D^2 - (\omega - \beta u_0)^2]}{[u_1^2D^2 + (\omega - \beta u_0)^2]^2} \tag{1.22}$$

If the quantities involved vary sinusoidally with y as $\cos \gamma y$ or $\sin \gamma y$, then

$$D^2 = -\gamma^2 \tag{1.23}$$

Our equation becomes

$$1 = \frac{\omega_p^2}{\gamma^2 u_1^2} \frac{\left[1 + \left(\dfrac{\omega - \beta u_0}{\gamma u_1}\right)^2\right]}{\left[1 - \left(\dfrac{\omega - \beta u_0}{\gamma u_1}\right)^2\right]^2} \tag{1.24}$$

What happens if we have many transverse velocities? If we refer back to (1.14) we see that we will have an equation of the form

$$1 = \sum - \tfrac{1}{2}\omega_{pn}^2 \left(\frac{d_{1n}^2 + d_{2n}^2}{d_{1n}^2\, d_{2n}^2}\right) \tag{1.25}$$

Here ω_{pn}^2 is a plasma frequency based on the density of electrons having transverse velocities $\pm u_n$. Equation (1.25) can be written

$$1 = \sum \frac{\omega_{pn}^2}{\gamma^2 u_n^2} \frac{\left[1 + \dfrac{(\omega - \beta u_0)^2}{\gamma^2 u_n^2}\right]}{\left[1 - \dfrac{(\omega - \beta u_0)^2}{\gamma^2 u_n^2}\right]^2} \tag{1.26}$$

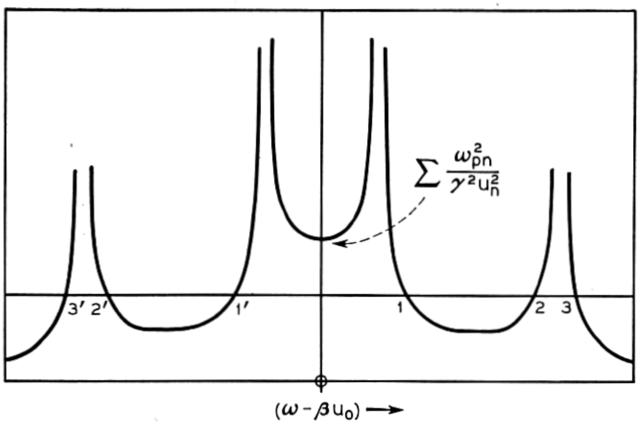

Fig. 1

Suppose we plot the left-hand and the right-hand sides of (1.26) versus $(\omega - \beta u_0)$. The general appearance of the left-hand and right-hand sides of (1.26) is indicated in Fig. 1 for the case of two velocities u_n. There will always be two unattenuated waves at values of $(\omega - \beta u_0)^2 > \gamma^2 u_e^2$ where u_e is the extreme value of u_n; these correspond to intersections 3 and 3′ in Fig. 2. The other waves, two per value of u_n, may be unattenuated or a pair of increasing and decreasing waves, depending on the values of the parameters. If

$$\sum \frac{\omega_{pn}^2}{\gamma^2 u_n^2} > 1$$

there will be at least one pair of increasing and decreasing waves.

It is not clear what will happen for a Maxwellian distribution of velocities. However, we must remember that various aberrations might give a very different, strongly peaked velocity distribution.

Let us consider the amount of gain in the case of one pair of transverse velocities, $\pm u_1$. The equation is now

$$\frac{\gamma^2 u_1^2}{\omega_p^2} = \frac{\left[1 + \left(\dfrac{\omega - \beta u_0}{\gamma u_1}\right)^2\right]}{\left[1 - \left(\dfrac{\omega - \beta u_0}{\gamma u_1}\right)^2\right]^2} \tag{1.27}$$

Let

$$\beta = \frac{\omega}{u_0} + j\,\frac{\gamma u_1 \epsilon}{u_0} \tag{1.28}$$

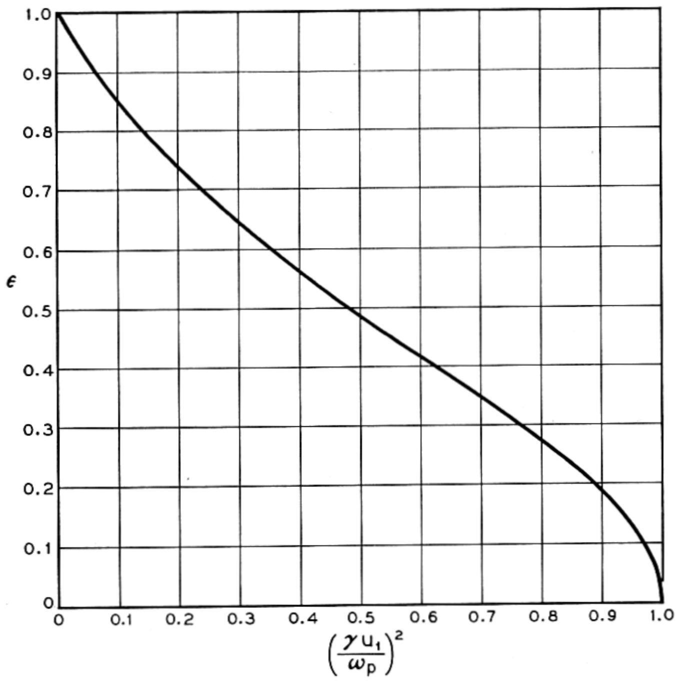

Fig. 2

This relation defines ϵ. Equation (1.27) becomes

$$\frac{\gamma^2 u_1^2}{\omega_p^2} = \frac{1 - \epsilon^2}{(1 + \epsilon^2)^2} \tag{1.29}$$

In Fig. 2, ϵ is plotted versus the parameter $\gamma^2 u_1^2/\omega_p^2$. We see that as the parameter falls below unity, ϵ increases, at first rapidly, and then more slowly, reaching a value of ± 1 as the parameter goes to zero (as ω_p^2 goes to infinity, for instance).

It will be shown in Section 2 of this paper that these results for infinite flow are in some degree an approximation to the results for flow in narrow beams. It is therefore of interest to see what results they yield if applied to a beam of finite width.

If the beam has a length L, the voltage gain is

$$e^{\epsilon\gamma(u_1/u_0)L} \tag{1.30}$$

The gain G in db is

$$G = 8.7 \frac{\gamma u_1 L}{u_0} \epsilon \text{ db} \tag{1.31}$$

Let the width of the beam be W. We let

$$\gamma = \frac{n\pi}{W} \tag{1.32}$$

Thus, for $n = 1$, there is a half-cycle variation across the beam. From (1.31) and (1.32)

$$G = 27.3 \left(\frac{u_1}{u_0}\frac{L}{d}\right) n\epsilon \text{ db} \tag{1.33}$$

Now L/u_0 is the time it takes the electrons to go from one end of the beam to the other, while W/u_1 is the time it takes the electrons to cross the beam. If the electrons cross the beam N times

$$N = \frac{u_1}{u_0}\frac{L}{W} \tag{1.34}$$

Thus,

$$G = 27.3\, Nn\epsilon \text{ db} \tag{1.35}$$

While for a given value of ϵ the gain is higher if we make the phase vary many times across the beam, i.e., if we make n large, we should note that to get any gain at all we must have

$$\begin{aligned} \omega_p^2 &> \gamma^2 u_1^2 \\ \omega_p^2 &> \left(\frac{n\pi u_1}{W}\right)^2 \end{aligned} \tag{1.36}$$

If we increase ω_p^2, which is proportional to current density, so that ω_p^2 passes through this value, the gain will rise sharply just after ω_p^2 passes through this value and will rise less rapidly thereafter.

2. A Two-Dimensional Beam of Finite Width.

Let us assume a beam of finite width in the y-direction; the boundaries lying at $y = \pm y_0$. It will be assumed also that electrons incident upon these boundaries are elastically reflected, so that electrons of the incident stream (1 or 2) are converted into those of the other stream (2 or 1). The condition of elastic reflection implies that

$$\dot{y}_1 = -\dot{y}_2 \tag{2.1}$$

$$\dot{z}_1 = \dot{z}_2 \quad \text{at } y = \pm y_0 \tag{2.2}$$

and, in addition, that

$$\rho_1 = \rho_2 \quad \text{at } y = \pm y_0 \tag{2.3}$$

since there is no change in the number of electrons at the boundary.

The equations of motion and of continuity (1.7–1.12) may be satisfied by introducing a single quantity, ψ, such that

$$V = d_1^2\, d_2^2 \psi \tag{2.4}$$

$$\dot{z}_1 = -j \frac{e}{m} \beta\, d_1\, d_2^2 \psi \tag{2.5}$$

$$\dot{z}_2 = -j \frac{e}{m} d_2^2\, d_1 \psi \tag{2.6}$$

$$\dot{y}_1 = \frac{e}{m} d_1\, d_2 D\psi \tag{2.7}$$

$$\dot{y}_2 = \frac{e}{m} d_1^2\, d_2 D\psi \tag{2.8}$$

$$\rho_1 = -\frac{e}{m} \rho_0 (D^2 - \beta^2)\, d_2^2 \psi \tag{2.9}$$

$$\rho_2 = -\frac{e}{m} \rho_0 (D^2 - \beta^2)\, d_1^2 \psi \tag{2.10}$$

Then, if we introduce the symbol, Ω, for $\omega - \beta u_0$

$$\dot{y}_1 + \dot{y}_2 = 2j \frac{e}{m} d_1\, d_2 D\Omega\psi \tag{2.11}$$

$$\dot{z}_1 - \dot{z}_2 = 2j \frac{e}{m} d_1\, d_2 u_1 D\psi \tag{2.12}$$

$$\rho_1 - \rho_2 = 2j \frac{e}{m} \rho_0 (D^2 - \beta^2) u_1 \Omega D\psi \tag{2.13}$$

It is clear that if

$$D\psi = D^3\psi = 0 \qquad y = \pm y_0 \tag{2.14}$$

the conditions for elastic reflection will be satisfied. The equation satisfied by ψ may now be found from Poisson's equation, (1–13), and is

$$(D^2 - \beta^2)\, d_1^2\, d_2^2 \psi = \frac{e\rho_0}{m\epsilon} (D^2 - \beta^2)(d_1^2 + d_2^2)\psi$$

or

$$(D^2 - \beta^2)[(u_1^2 D^2 + \Omega^2)^2 + \omega_p^2 (u_1^2 D^2 - \Omega^2)] = 0 \tag{2.15}$$

which is of the sixth degree in D. So far four boundary conditions have been imposed. The remaining necessary pair arise from matching the internal fields to the external ones. For $y > y_0$

$$V = V_0 e^{-j\beta z} \cdot e^{-\beta y} \tag{2.16}$$

and

$$\frac{\partial V}{\partial y} + \beta V = 0 \qquad \text{at } y = y_0$$

Similarly

$$\frac{\partial V}{\partial y} - \beta V = 0 \qquad \text{at } y = -y_0 \tag{2.17}$$

The most familiar procedure now would be to look for solutions of (2.15) of the form, e^{cy}. This would give the sextic for c

$$(c^2 - \beta^2)[(u_1^2 c^2 + \Omega^2)^2 + \omega_p^2(u_1^2 c^2 - \Omega^2)] = 0 \tag{2.18}$$

with the roots $c = \pm\beta, \pm c_1, \pm c_2$, let us say. We could then express ψ as a linear combination of these six solutions and adjust the coefficients to satisfy the six boundary equations. In this way a characteristic equation for β would be obtained. From the symmetry of the problem this has the general form $F(\beta, c_1) = F(\beta, c_2)$, where c_1 and c_2 are found from (2.18). The discussion of the problem in these terms is rather laborious and, if we are concerned mainly with examining qualitatively the onset of increasing waves, another approach serves better.

From the symmetry of the equations and of the boundary conditions we see that there are solutions for ψ (and consequently for V and ρ) which are even in y and again some which are odd in y. Consider first the even solutions. We will assume that there is an even function, $\psi_1(y)$, periodic in y with period $2y_0$, which coincides with $\psi(y)$ in the open interval, $-y_0 < y < y_0$ and that $\psi_1(y)$ has a Fourier cosine series representation:

$$\psi_1(y) = \sum_1^\infty c_n \cos \lambda_n y \qquad \lambda_n = \frac{n\pi}{y_0} \qquad n = 0, 1, 2, \cdots \tag{2.19}$$

ψ inside the interval satisfies (2.15), so we assume that $\psi_1(y)$ obeys

$$(D^2 - \beta^2)[(u_1^2 D^2 + \Omega^2)^2 + \omega_p^2(u_1^2 D^2 - \Omega^2)]\psi_1 = \sum_{m=-\infty}^{+\infty} \delta(y - \overline{2m+1}y_0) \tag{2.20}$$

where δ is the familiar δ-function. Since $D\psi$ and $D^3\psi$ are required to vanish at the ends of the interval and ψ, $D^2\psi$ and $D^4\psi$ are even it follows that all of these functions are continuous. We assume that $\psi_1 = \psi$, $D\psi_1 = D\psi$, $D^2\psi_1 = D^2\psi$, $D^3\psi_1 = D^3\psi$ and $D^4\psi_1 = D^4\psi$ at the ends of the intervals. From (2.20), $u_1^4 D^5 \psi_1 \to -\tfrac{1}{2}$ as $y \to y_0$.

Since

$$\sum_{-\infty}^{+\infty} \delta(y - \overline{2m + 1}y_0) = \frac{1}{2y_0} + \frac{1}{y_0} \sum_{1}^{\infty} (-1)^n \cos \lambda_n y \tag{2.21}$$

we obtain from (2.20)

$$\begin{aligned} 2y_0\psi_1 = -\Bigg(&\frac{1}{\beta^2\Omega^2(\Omega^2 - \omega_p^2)} \\ &+ 2\sum_{1}^{\infty} (-1)^n \frac{\cos \lambda_n y}{(\beta^2 + \lambda_n^2)[(\Omega^2 - u_1^2\lambda_n^2)^2 - \omega_p^2(\Omega^2 + u_1^2\lambda_n^2)]}\Bigg) \end{aligned} \tag{2.22}$$

Since

$$\frac{\partial V}{\partial y} + \beta V = (D + \beta)(u_1^2 D^2 + \Omega^2)^2 \psi,$$

using (2.4), the condition for matching to the external field,

$$\frac{\partial V}{\partial y} + \beta V = 0,$$

yields, using $D\psi = D^3\psi = 0$ and $u_1^4 D^5 \psi = -\tfrac{1}{2}$, the relation

$$(u_1^2 D^2 + \Omega^2)^2 \psi_1 = \tfrac{1}{2}\beta \quad \text{at } y = y_0 .$$

Applying this to (2.22), we then obtain, finally,

$$\begin{aligned} \frac{y_0}{\beta} = &\frac{1}{\beta^2[\Omega^2 - \omega_p^2]} \\ &+ 2\sum_{1}^{\infty} \frac{(\Omega^2 - u_1^2\lambda_n^2)^2}{(\beta^2 + \lambda_n^2)[(\Omega^2 - u_1^2\lambda_n^2)^2 - \omega_p^2(\Omega^2 + u_1^2\lambda_n^2)]} \end{aligned} \tag{2.23}$$

For the odd solution we use a function, $\psi_2(y)$, equal to $\psi(y)$ in $-y_0 < y < y_0$ and representable by a sine series. To ensure the vanishing of $D\psi$ and $D^3\psi$ at $y = \pm y_0$ it is appropriate to use the functions, $\sin \mu_n y$, where $\mu_n = (n + \tfrac{1}{2})\pi/y_0$. The period is now $4y_0$ and we define $\psi_2(y)$ in $y_0 < y < 3y_0$ by the relation $\psi_2(y) = \psi(2y_0 - y)$ and in $-3y_0 < y < -y_0$ by $\psi_2(y) = \psi(-2y_0 - y)$. Thus, we write

$$\psi_2(y) = \sum_{0}^{\infty} d_n \sin \mu_n y \qquad \mu_n = (n + \tfrac{1}{2})\pi/y_0$$

$\psi_2(y)$ will be supposed to satisfy

$$(D^2 - \beta^2)[(u_1^2D^2 + \Omega^2)^2 + \omega_p^2(u_1^2D^2 - \Omega^2)]\psi_2 = \sum_{m=-\infty}^{+\infty} [\delta(y - \overline{4m+1}y_0) - \delta(y - \overline{4m-1}y_0)] \tag{2.24}$$

The extended definition of ψ_2 (outside $-y_0 < y < y_0$) is such that we may again take $\psi_1 = \psi, \cdots, D^4\psi_1 = D^4\psi$ at the ends of the interval. $u_1^4D^5\psi_1$ is still equal to $-\frac{1}{2}$ at $y = y_0$. Now

$$\sum_{-\infty}^{+\infty} [\delta(y - \overline{4m+1}y_0) - \delta(y - \overline{4m-1}y_0)] = \frac{1}{y_0} \sum (-1)^n \sin \mu_n y \tag{2.25}$$

so from (2.24) we may find

$$y_0\psi_2 = -\sum_0^{\infty} \frac{(-1)^n \sin \mu_n y}{(\beta^2 + \mu_n^2)[(\Omega^2 - u_1^2\mu_n^2)^2 - \omega_p^2(\Omega^2 + u_1^2\mu_n^2)]} \tag{2.26}$$

Matching to the external field as before gives

$$(u_1^2D^2 + \Omega^2)^2\psi_2 = \frac{1}{2\beta} \qquad \text{at } y = y_0$$

and applied to (2.26) we have

$$-\frac{y_0}{2\beta} = \sum_0^{\infty} \frac{(\Omega^2 - u_1^2\mu_n^2)^2}{(\beta^2 + \mu_n^2)[(\Omega^2 - u_1^2\mu_n^2)^2 - \omega_p^2(\Omega^2 + u_1^2\mu_n^2)]} \tag{2.27}$$

The equations (2.23) and (2.27) for the even and odd modes may be rewritten using the following reduced variables.

$$z = \frac{\beta y_0}{\pi}$$

$$k = \frac{\omega y_0}{\pi u_1} - \frac{u_0}{u_1} z$$

$$\delta^2 = \frac{\omega_p^2 y_0^2}{\pi^2 u_1^2}$$

(2.23) becomes

$$\frac{k^2}{k^2 - \delta^2} + 2 \sum_{n=1}^{\infty} \frac{z^2}{z^2 + n^2} \cdot \frac{(n^2 - k^2)^2}{(n^2 - k^2)^2 - \delta^2(n^2 + k^2)} = -\pi z \tag{2.28}$$

and (2.27) transforms to

$$2 \sum_{n=0}^{\infty} \frac{z^2}{z^2 + (n + \frac{1}{2})^2} \cdot \frac{[(n + \frac{1}{2})^2 - k^2]^2}{[(n + \frac{1}{2})^2 - k^2]^2 - \delta^2[(n + \frac{1}{2})^2 + k^2]} = -\pi z \tag{2.29}$$

We shall assume in considering (2.28) and (2.29) that the beam is sufficiently wide for the transit of an electron from one side to the other to take a few RF cycles. The number of cycles is in fact, $\omega y_0/\pi u_1$, and, hence, from the definition of z, we see that for values of k less than 2, perhaps, z is certainly positive.

Let us consider (2.29) first since it proves to be the simpler case. If we transfer the term πz to the right hand side, it follows from the observation that z is positive (for modest values of k), that it is necessary to make the sum negative. The sum may be studied qualitatively by sketching in the $k^2 - \delta^2$ plane the lines on which the individual terms go to infinity, given by

$$\delta^2 = \frac{[(n + \frac{1}{2})^2 - k^2]^2}{(n + \frac{1}{2})^2 + k^2} \tag{2.30}$$

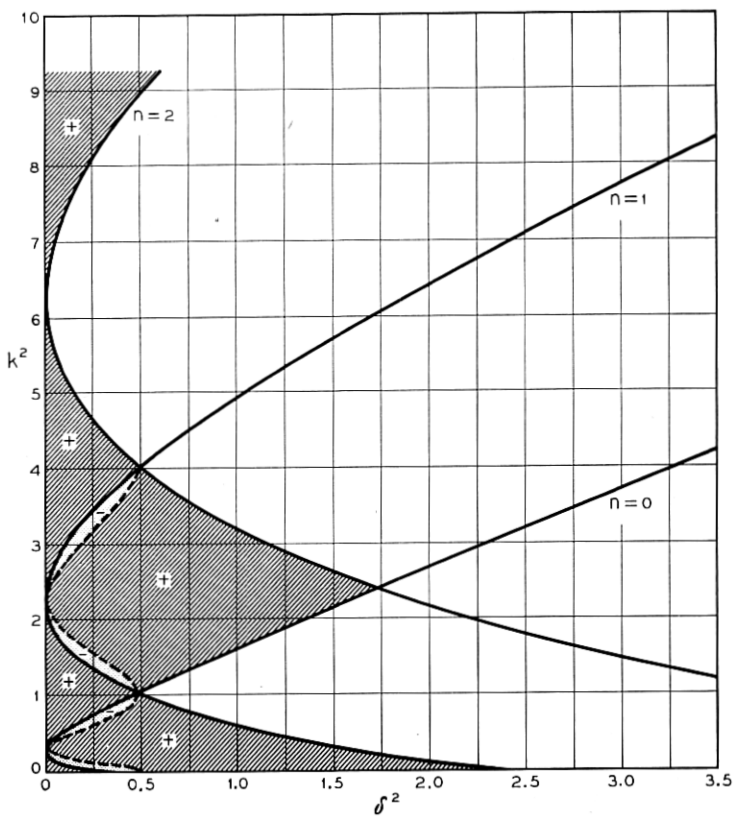

Fig. 3

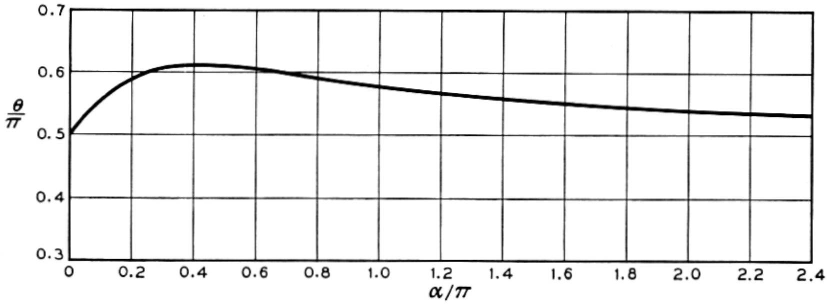

Fig. 4

Fig. 3 shows a few such curves ($n = 0, 1, 2$). To the right of such curves the individual term in question is negative, except on the line, $k^2 = (n + \frac{1}{2})^2$, where it attains the value of zero. Approaching the curves from the right the terms go to $-\infty$. On the left of the curves the function is positive and goes to $+\infty$ as the curve is approached from the

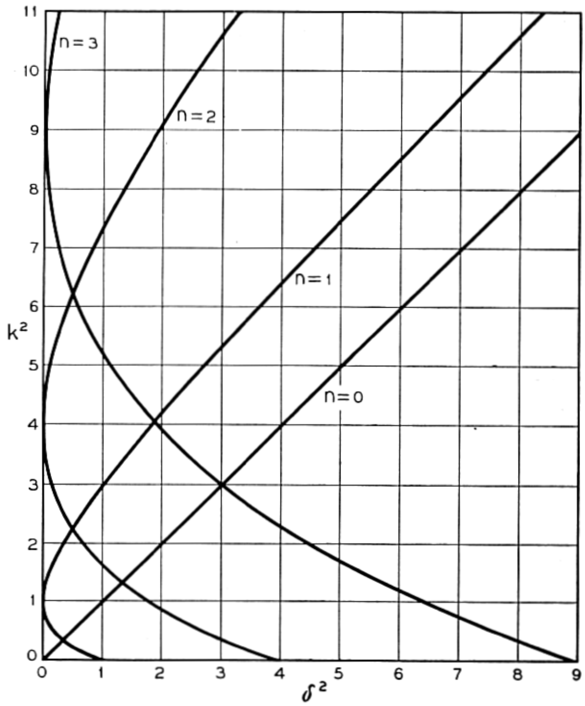

Fig. 5

left. Clearly in the regions marked + which lie to the left of every curve given by (2.30), the sum is positive and we cannot have roots. Let us examine the sum in the region to the right of the $n = 0$ curve and to the left of all others. On the line, $k^2 = \frac{1}{4}$, the sum is positive, since the first term is zero. On any other line, $k^2 =$ constant, the sum goes from $+\infty$ at the $n = 1$ curve monotonically to $-\infty$ at the $n = 0$ curve, so that somewhere it must pass through 0. This enables us to draw the zero-sum contours qualitatively in this region and they are indicated in Fig. 3. We are now in a position to follow the variation in the sum as k varies at fixed δ^2. It is readily seen that for $\delta^2 < 0.25$, because $-\pi z$ is negative in the region under consideration, there will be four real roots, two for positive, two for negative k. For δ^2 slightly greater than 0.25, the sum has

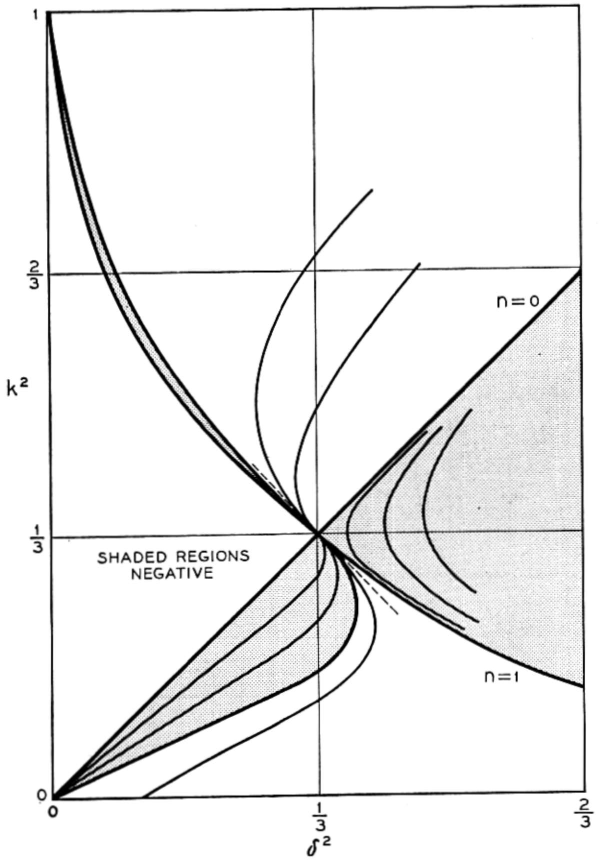

Fig. 6A

a deep minimum for $k = 0$, so that there are still four real roots unless z is very large. For z fixed, as δ^2 increases, the depth of the minimum decreases and there will finally occur a δ^2 for which the minimum is so shallow that two of the real roots disappear. Call $z(0)$ the value of z for $k = 0$, write the sum as $\Sigma(\delta^2, k^2)$ and suppose that $\Sigma(\delta_0^2, 0) = -\pi z(0)$, then for small k we have

$$\Sigma(\delta^2, k^2) = -\pi z(0) + (\delta^2 - \delta_0^2)\frac{\partial\Sigma}{\partial\delta^2} + k^2\frac{\partial\Sigma}{\partial k^2} = -\pi z(0) - \frac{u_1}{u_0}k$$

$$k^2 - \frac{u_1/u_0}{\dfrac{\partial\Sigma}{\partial k^2}}k = \frac{\dfrac{\partial\Sigma}{\partial\delta^2}}{\dfrac{\partial\Sigma}{\partial k^2}}(\delta^2 - \delta_0^2)$$

$$k = \frac{u_1/u_0}{2\dfrac{\partial\Sigma}{\partial k^2}} \pm \sqrt{\frac{\dfrac{\partial\Sigma}{\partial\delta^2}}{\dfrac{\partial\Sigma}{\partial k^2}}(\delta^2 - \delta_0^2) + \left(\frac{u_1/u_0}{2\dfrac{\partial\Sigma}{\partial k^2}}\right)^2}$$

The roots become complex when

$$\delta^2 = \delta_0^2 - \frac{(u_1/u_0)^2}{4\dfrac{\partial\Sigma}{\partial\delta^2}\dfrac{\partial\Sigma}{\partial k^2}}$$

Since u_1/u_0 may be considered small (say 10 per cent) it is sufficient to look for the values of δ_0^2.

When $k^2 = 0$ we have

$$\begin{aligned}-\pi z &= 2\sum\frac{z^2}{z^2 + (n + \tfrac{1}{2})^2}\cdot\frac{(n + \tfrac{1}{2})^2}{(n + \tfrac{1}{2})^2 - \delta^2}\\ &= \frac{2z^2}{z^2 + \delta^2}\sum_0^\infty\left(\frac{\delta^2}{(n + \tfrac{1}{2})^2 - \delta^2} + \frac{z^2}{(n + \tfrac{1}{2})^2 + z^2}\right)\\ &= \frac{\pi z^2}{z^2 + \delta^2}(\delta\tan\pi\delta + z\tanh\pi z)\end{aligned}$$

Fig. 4 shows the solution of this equation for various $z(0)$ or $\omega y_0/\pi u_0$. Clearly the threshold δ is rather insensitive to variations in $\omega y_0/\pi u_0$.

Equation (2.28) may be examined by a similar method, but here some complications arise. Fig. 5 shows the infinity curves for $n = 0, 1, 2, 3$; the $n = 0$ term being of the form $k^2/k^2 - \delta^2$. The lowest critical region in δ^2 is the neighborhood of the point $k^2 = \delta^2 = \tfrac{1}{3}$, which is the intersection of the $n = 0$ and $n = 1$ lines. To obtain an idea of the behavior of the left hand side (l.h.s.) of (2.28) in this area we first see how the point $k^2 = \delta^2 = \frac{1}{3}$ can be approached so that the l.h.s. remains finite. If we put $k^2 = \frac{1}{3} + \varepsilon$ and $\sigma^2 = \frac{1}{3} + c\varepsilon$ and expand the first two dominant terms of (2.28), then adjust c to keep the result finite as $\varepsilon \to 0$ we find

$$c = \frac{1}{4}\frac{3z^2 - 5}{3z^2 + 1}$$

c varies from $-\frac{5}{4}$ to $\frac{1}{4}$ as z goes from 0 to ∞, changing sign at $z^2 = \frac{5}{3}$. Every curve for which the l.h.s. is constant makes quadratic contact with the line $\delta^2 - \frac{1}{3} = c(k^2 - \frac{1}{3})$ at $k^2 = \delta^2 = \frac{1}{3}$. If we remember that the l.h.s. is positive for $k^2 = 0$, $0 < \delta^2 < 1$ and for $k^2 = 1$, $0 < \delta^2 < 1$,

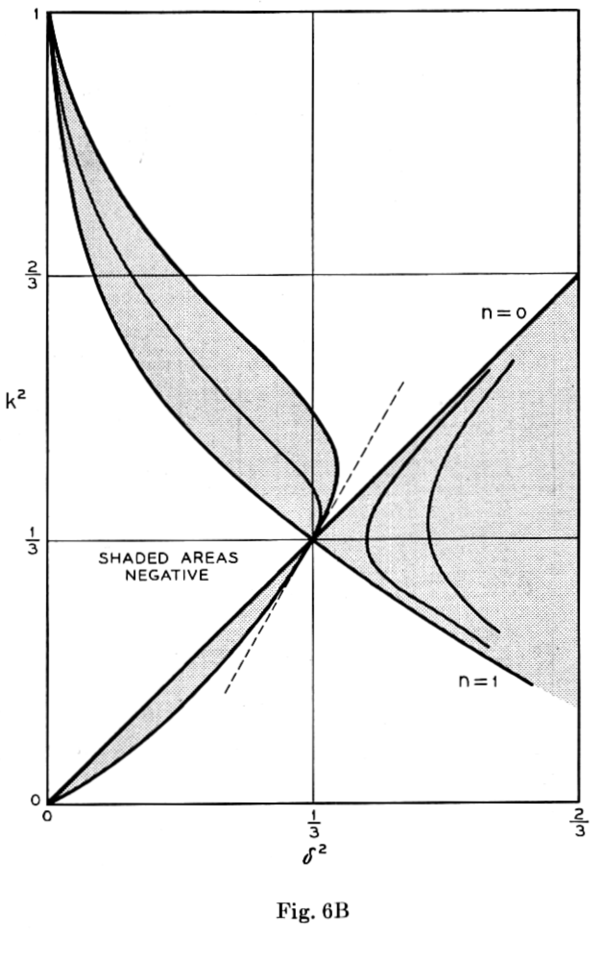

Fig. 6B

since there are no negative terms in the sum for these ranges and again that the l.h.s. must change sign between the $n = 0$ and $n = 1$ lines for any k^2 in the range $0 < k^2 < 1$ (since it varies from $\mp\infty$ to $\pm\infty$), this information may be combined with that about the immediate vicinity of $\delta^2 = k^2 = \frac{1}{3}$ to enable us to draw a line on which the l.h.s. is zero. This is indicated in Figs. 6A and 6B for small z and large z respectively. It will be seen that the zero curve and, in fact, all curves on which the l.h.s. is equal to a negative constant are required to have a vertical tangent at some point. This point may be above or below $k^2 = \frac{1}{3}$ (depending upon the sign of c or the size of z) but always at a $\delta^2 > \frac{1}{3}$. For $\delta^2 < \frac{1}{3}$ there are no regions where roots can arise as we can readily see by considering how the l.h.s. varies with k^2 at fixed δ^2. For a fixed $\delta^2 > \frac{1}{3}$ we have, then, either for $k^2 > \frac{1}{3}$ or $k^2 < \frac{1}{3}$, according to the size of z, a negative minimum which becomes indefinitely deep as $\delta^2 \rightarrow \frac{1}{3}$. Thus, since the negative terms on the right-hand side are not sensitive to small changes in δ^2, we must expect to find, for a fixed value of the l.h.s., two real solutions of (2.28) for some values of δ^2 and no real solutions for some larger value of δ^2, since the negative minimum of the l.h.s. may be made as shallow as we like by increasing δ^2. By continuity then we expect to find pairs of complex roots in this region. Rather oddly these roots, which will exist certainly for δ^2 sufficiently close to $\frac{1}{3} + 0$, will disappear if δ^2 is sufficiently increased.

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