Analysis of Switching Networks

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Using a simplified model, an analysis of switching networks is presented. Methods for finding characteristics of a network such as blocking probability, retrial and connection-time distributions are given. The problem of equivalent crosspoint minimization is also considered.

1. INTRODUCTION

The application of probability theory to telephone traffic problems owes its origin to the pioneering work of A. K. Erlang, T. C. Fry, E. C. Molina and others.[1, 2, 3] Since then much has been written on these and other related problems. On the other hand, except for several recent papers on the subject,[4, 5, 6, 7] the literature on the application of probability theory to large size switching systems has been comparatively meager, mainly because the sheer size and complexity of these systems tend to render exact analysis unmanageable.

To fix ideas, let us peer into a telephone central office and point our attention at a single link (or crosspoint) in the system. As time progresses the link becomes busy and idle in some fashion and gives rise to a sequence of pairs of observations:

$$(t_0 \text{, busy}), \qquad (t_1 \text{, change to idle}), \qquad (t_2 \text{, change to busy}), \cdots$$

Let x_t be a function such that x_t is 1 if the link is idle and is 0 if the link is busy, then the sequence of pairs of observations corresponds to the behavior of x_t as t changes. A plot of the values of x_t versus t would look perhaps as shown in Fig. 1.1. The function x_t (or the sequence of pairs of observations) is one of a large family of possible functions for the link. Since there are in general several thousand such links in a telephone central office, a complete description would involve several thousand families of functions x_t. We may add that the situation is made somewhat worse by the fact that these links are not independent of each other, for example, the establishment of a telephone conversation involves in general not one but several links in series.

In order to derive useful results, we shall consider a simplified model in which the links are assumed to be independent and we shall further restrict the families of functions x_t to only those which obey certain rules. The mathematical model used for a switching network is a probability linear-graph introduced in this paper. The basic steps we follow in the analysis of switching networks consist of (1) representing a switching network by a probability linear-graph and (2) using the graph as a probability model to calculate all characteristics of interest of the switching network. Once step (1) is effected, step (2) falls into the domain of probability theory in which standard techniques are available.

In Section 2 of this paper, the more basic aspects of switching network analysis are considered and properties of probability linear-graphs are discussed. Those who are mainly interested in numerical results may begin directly with Section 3 where representation of switching networks by graphs, probability of blocking of various networks and the problem of equivalent crosspoint minimization are presented. Using the graph model, methods for calculating average retrial time for blocked calls and network blocking due to excessive time required in the breakdown of crosspoints in a gas tube network are then given in Section 4.

2. PROBABILITY LINEAR-GRAPHS

It would be appropriate for us to associate with each link a class of binary-valued functions ω such that $\omega(t) = 0$ or 1 (i.e. the link is either busy or idle) at any given time t. It is more convenient, however, for us to associate with each link a random variable x_t (Reference 8, Chapter 17) such that at any given time t, $x_t = 0$ or 1. Furthermore, we assume that the random variable x_t has a stationary probability distribution.*

Definition 2.1. A (two-terminal) probability linear-graph G (or simply a graph if no confusion arises) is a finite, oriented, connected, cycle-free

* More strictly, we let Ω be the space of all binary-valued functions ω such that $\omega(t) = 0$ or 1, $-\infty < t < \infty$. Let

$$\{x_t, -\infty < t < \infty\}$$

be a stochastic process with discrete probability distributions

$$f_{t_1, t_2, \cdots t_n}(\delta_1, \delta_2, \cdots, \delta_n) = Pr[x_{t_i} = \delta_i, i = 1, 2, \cdots, n]$$

$$\delta_i = 0 \text{ or } 1, i = 1, 2, \cdots, n$$

where x_s is the s^{th} coordinate function of Ω; that is $x_s(\omega) = \omega(s)$. Furthermore, we assume that this process is strictly stationary; that is

$$f_{t_1, t_2, \cdots, t_n}(\delta_1, \delta_2, \cdots, \delta_n) = f_{t_1+h, t_2+h, \ldots, t_n+h}(\delta_1, \delta_2, \cdots, \delta_n)$$

For each fixed $\omega \in \Omega$, x_t is a binary-valued function whose domain is $-\infty < t < \infty$. We then follow Reference 9 and call x_t a random variable indexed by t with a stationary probability distribution. For the existence and consistency of these stochastic processes, see Reference 9, Chapter I.

linear-graph* with at least two nodes such that (i) there exists a pair of nodes, called respectively an originating and terminating node, of G; this assignment being determined initially, (ii) with each link† of G is associated a random variable $x_t^{(i)}$, the index i running over all links of G such that $x_t^{(i)}$ are mutually independent and have stationary probability distributions

$$f_{t_1, t_2, \cdots, t_n}{}^{(i)}(\delta_1, \delta_2, \cdots, \delta_n) = Pr[x_{t_j}{}^{(i)} = \delta_j, j = 1, 2, \cdots, n]$$

$$\delta_j = 0 \text{ or } 1, \qquad j = 1, 2, \cdots, n$$

In what follows, by a directed path of G is meant a directed path between the originating and terminating nodes of G. Moreover, whenever there is no confusion, the letter x will be used to denote the vector

$$(x_t^{(1)}, x_t^{(2)}, \cdots)$$

Given a probability linear-graph, the usual problem is to find the

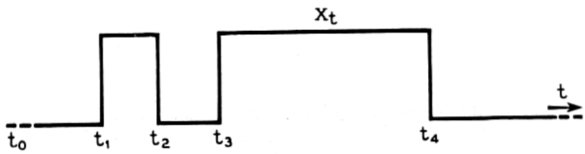

Fig. 1.1 — Observations of idle and busy conditions of a link.

probability distribution of some binary-valued function φ of independent random variables $x_t^{(i)}$. We begin this section by looking first at the simplest cases.

2.1 *Series-Parallel Graphs.*

Consider the series graph shown in Figure 2.1 with $N + 1$ nodes and N links. Let φ_t be the function given by

$$\varphi_t(x) = \prod_{i=1}^{N} x_t^{(i)}$$

Physically, if we interpret $x_t^{(i)} = 1$ as the event of link i being idle at time t, then $\varphi_t(x) = 1$ corresponds to the event that there is at least one directed path of G (in this case exactly one) idle at time t. In view of this interpretation, φ_t is called the *connection function* of G. The prob-

* A linear-graph is finite if both the sets of nodes and branches are non-empty and finite; it is connected if there exists at least one path (chain of branches) between each pair of nodes; it is oriented if the branches are all directed; it is cycle-free if there exists no *directed* path between any node and itself.

† Here, directed branches are called links.

abilities

$$P_\tau(1) = Pr[\varphi_\tau(x) = 1] \quad \text{and} \quad P_\tau(0) = Pr[\varphi_\tau(x) = 0] = 1 - P_\tau(1) \tag{2.1}$$

are called respectively the *linking* and *blocking* probabilities of G. Clearly,

$$\begin{aligned} P_\tau(1) &= Pr[\varphi_\tau(x) = 1] = Pr[x_\tau^{(i)} = 1, i = 1, 2, \cdots, N] \\ &= \prod_{i=1}^{N} Pr[x_\tau^{(i)} = 1] = \prod_{i=1}^{N} f_\tau^{(i)}(1) \end{aligned} \tag{2.2}$$

Next, let

$$P_{\tau,\tau+t}(\delta_1, \delta_2) = Pr[\varphi_\tau(x) = \delta_1, \varphi_{\tau+t}(x) = \delta_2] \tag{2.3}$$

then

$$\begin{aligned} P_{\tau,\tau+t}(1, 1) &= Pr[x_\tau^{(i)} = 1, x_{\tau+t}^{(i)} = 1; i = 1, 2, \cdots, N] \\ &= \prod_{i=1}^{N} f_{\tau,\tau+t}(1, 1) \end{aligned} \tag{2.4}$$

Since $x_t^{(i)}$ have stationary distributions, $P_\tau(\delta)$, $f_\tau^{(i)}(\delta)$ and $P_{\tau,\tau+t}(\delta_1, \delta_2)$

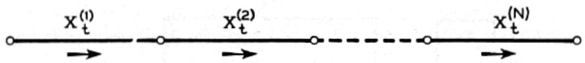

FIG. 2.1 — A series graph with N links.

are all independent of τ so that they may be written respectively $P(\delta)$, $f^{(i)}(\delta)$ and $P_t(\delta_1, \delta_2)$. Moreover,

$$P(1) = P_t(0, 1) + P_t(1, 1) = P_t(1, 0) + P_t(1, 1)$$

so that

$$\begin{aligned} P_t(0, 1) &= P_t(1, 0) = P(1) - P_t(1, 1) \\ &= \prod_{i=1}^{N} f^{(i)}(1) - \prod_{i=1}^{N} f_t^{(i)}(1, 1) \end{aligned} \tag{2.5}$$

Finally , let the means of $x_t^{(i)}$ and φ_t be denoted by

$$q^{(i)} = E(x_t^{(i)}), \qquad Q = E(\varphi_t) \tag{2.6}$$

Then, for a series graph, the following relations obtain:

$$Q = \prod_i q^{(i)} \tag{2.7}$$

$$P_t(1, 1) = \prod_i f_t^{(i)}(1, 1) \tag{2.8}$$

$$P_t(0, 1) = P_t(1, 0) = \prod_i q^{(i)} - \prod_i f_t^{(i)}(1, 1) \tag{2.9}$$

$$P_t(0, 0) = 1 - 2\prod_i q^{(i)} + \prod_i f_t^{(i)}(1, 1) \tag{2.10}$$

In the case of a parallel graph of N links (Figure 2.2), the connection function φ_t is given by*

$$\varphi_t(x) = 1 - \prod_i (1 - x_t^{(i)}).$$

Using the same notations as before, we obtain, for the parallel graph

$$Q = 1 - \prod_i (1 - q^{(i)}) \tag{2.11}$$

$$P_t(0, 0) = \prod_i f_t^{(i)}(0, 0) \tag{2.12}$$

$$P_t(0, 1) = P_t(1, 0) = \prod_i (1 - q^{(i)}) - \prod_i f_t^{(i)}(0, 0) \tag{2.13}$$

$$P_t(1, 1) = 1 - 2\prod_i (1 - q^{(i)}) + \prod_i f_t^{(i)}(0, 0) \tag{2.14}$$

2.2 *General Probability Linear-Graphs*

Let us first make the notion of a connection function of a graph more precise.

Let G be a probability linear-graph with N links. Let B be the set of all directed paths of G. Then each $\beta_i \in B$ is composed of series links of G with associated link random variables $x_t^{(i_1)}, x_t^{(i_2)}, \cdots$. To each $\beta_j \in B$, assign a new random variable $y_t^{(j)}$ as a function of $x_t^{(i_1)}, x_t^{(i_2)}, \cdots$ such that $y_t^{(j)} = 1$ if and only if $x_t^{(i_k)} = 1, k = 1, 2, \cdots$ and otherwise $y_t^{(j)} = 0$.

Definition 2.2. The graph G^* composed of all parallel links $\beta, \beta \in B$ with link random variables $y_t^{(1)}, y_t^{(2)}, \cdots$ is said to be the canonical form of G.

Definition 2.3. A binary-valued function φ_τ (itself a random variable) of the vector $y = (y_t^{(1)}, y_t^{(2)}, \cdots)$ such that $\varphi_\tau(y) = 0$ if and only if

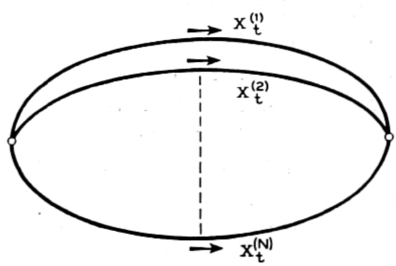

FIG. 2.2 — A parallel graph with N links.

* Note that by the change of variables $x_t'^{(i)} = 1 - x_t^{(i)}$ and $\varphi_t' = 1 - \varphi_t$, the situation here becomes identical with that for the series graphs.

$y = 0$ and $\varphi_\tau(y) = 1$ otherwise is said to be the connection function of G. The probabilities $Pr[\varphi_\tau(y) = 1]$ and $Pr[\varphi_\tau(y) = 0]$ are called respectively the *linking* and *blocking* probabilities of G. The conditional probability $Pr[\varphi_{\tau+t}(y) = 0 \mid \varphi_\tau(y) = 0]$ is called the *retrial distribution* of G.

We note that the physical interpretation of φ_τ is as it should be; $\varphi_\tau(y) = 1$ if there is at least one idle path through G and $\varphi_\tau(y) = 0$ if otherwise. The physical interpretations of linking and blocking probabilities and retrial distribution are self-evident.

We now let x denote the vector $(x_t^{(1)}, x_t^{(2)}, \cdots, x_t^{(N)})$ and let x_t^* denote the values of x at time t. At a given time t, let $J(\delta)$, $\delta = 0$ or 1, be the set of values of x which makes $\varphi_t(x) = \delta$. Since x, as a vector, has a stationary distribution, it follows that

$$\begin{aligned} Pr[\varphi_{t_1+h}(x) = \delta_1, \cdots, \varphi_{t_n+h}(x) = \delta_n] \\ &= Pr[x_{t_1+h}^* \in J(\delta_1), \cdots, x_{t_n+h}^* \in J(\delta_n)] \\ &= Pr[x_{t_1}^* \in J(\delta_1), \cdots, x_{t_n}^* \in J(\delta_n)] \\ &= Pr[\varphi_{t_1}(x) = \delta_1, \cdots, \varphi_{t_n}(x) = \delta_n] \end{aligned}$$

Thus, we have the elementary but important fact that

Theorem 2.1. The connection function φ_t for a probability linear-graph G has a stationary distribution.

An immediate consequence of Theorem 2.1 is that the linking (or blocking) probability of G is independent of the time t and is expressible as a polynomial with integral coefficients in terms of the linking probabilities $q^{(i)}$ of the links of G. To put if differently, Theorem 2.1 asserts that in order to compute the linking (or blocking) probability of G, it suffices for us to consider the process as one consisting of repeated simultaneous tossing of N skewed coins, a process in which time does not enter. It should be remarked that this fact has long been known to telephone traffic engineers.

Another remark is in order here although a precise statement would be needlessly long. In many cases, a graph G may contain several components G_j. If these components are well-defined, it is always possible to replace them by single links whose associated random variables are $\varphi_t^{(j)}$ where $\varphi_t^{(j)}$ is the connection function of G_j. Actual computations may be greatly simplified by a repeated application of this procedure.

2.3 *An Example of a Non-Stationary Process*

In Section 2.1 we had stipulated that the link random variables of a graph have stationary probability distributions. Let us now consider a (somewhat arbitrary) example in which the stationary property is dropped.

Let ℓ_1 be a single link and let Ω be the family of functions ω associated with this link such that for the internal $0 \leqq \lambda < T$,

$$\omega(t) = 0; \qquad t < \lambda, \qquad t \geqq T$$

$$\omega(t) = 1; \qquad T > t \geqq \lambda$$

Thus, we may index functions of Ω by λ. A subset Ω_1 of Ω is said to be measurable if the corresponding indexing set Λ_1 is Borel measurable on the interval $[0, T)$ and in such cases, we identify the measure of Ω_1 with the Borel measure of Λ_1.

Let x_s be the s^{th} coordinate function of Ω; that is, $x_s(\omega) = \omega(s)$. Then,

$$Pr[x_t = 1] = \frac{t}{T}, \qquad 0 \leq t < T$$

and

$$Pr[x_t = 1] = 0 \quad \text{otherwise.}$$

Hence the probability of blocking P_1 for ℓ_1 is zero for $t < 0$, $t \geqq T$ and is t/T for $0 \leqq t < T$. That is, the blocking probability is itself a function of t.

If we restrict our attention to the time interval $[0, T)$, we may define the mean blocking $\overline{P}_1$ as the time average of P on $[0, T)$.

Then

$$\overline{P}_1 = \frac{1}{T} \int_0^T \frac{t}{T}\, dt = \tfrac{1}{2}$$

Suppose we now have two such links ℓ_1, ℓ_2, independent of each other, in parallel. Then the blocking probability of the parallel graph is

$$P = P_1^2 = \left(\frac{t}{T}\right)^2$$

The mean blocking for the parallel graph is

$$\overline{P} = \frac{1}{T} \int_0^T \left(\frac{t}{T}\right)^2 dt = \tfrac{1}{3}$$

Thus, it is erroneous here to say that the mean blocking of the parallel graph is the square of the mean blocking of each path.

Strictly speaking, the assumption of stationary distributions for link random variables is invalid in practice, since obviously a telephone central office handles more calls at noon than at midnight. However, it is the busy-hour-traffic which concerns the telephone traffic engineers, so that the assumption is in general a reasonable one.

2.4 *Linking Probability*

In this section, only linking (and blocking) probabilities of a probability linear-graph will be considered. Other characteristics of a graph (retrial distribution, connection-time distribution etc.) will be studied in a later section.

Let G be a probability linear-graph and let the random variables associated with the links of G be $x^{(1)}, x^{(2)}, \cdots$ where these random variables are no longer functions of time. Let us view our probability process as one consisting of repeated trials (or, say, tosses of batches of coins) in which, for each trial, the probabilities* of success and failure for the links are respectively

$$q^{(i)} = Pr[x^{(i)} = 1], \qquad p^{(i)} = Pr[x^{(i)} = 0] = 1 - q^{(i)}, \qquad (2.15)$$
$$i = 1, 2, \cdots$$

Let φ be the connection function for G and denote by Q and P

$$Q = Pr[\varphi(x) = 1], \qquad P = 1 - Q = Pr[\varphi(x) = 0] \qquad (2.16)$$

where x is the vector $(x^{(1)}, x^{(2)}, \cdots)$. Then Q and P are respectively the linking and blocking probabilities of G.

In the case G is a series graph of N links, we have, from Section 2.1,

$$Q = \prod_{i=1}^{N} q^{(i)}, \qquad P = 1 - Q. \qquad (2.17)$$

Similarly, when G is a parallel graph of N links,

$$Q = 1 - \prod_{i=1}^{N} (1 - q^{(i)}), \qquad P = 1 - Q. \qquad (2.18)$$

Thus for any series-parallel graph G, a combination of (2.17) and (2.18) will yield the linking and blocking probabilities of G.

Example 2.1. Consider the series-parallel graph G shown in Fig. 2.3 where the link random variables are denoted by

$$x_1^{(1)}, \quad x_1^{(2)}, \cdots, \quad x_1^{(10)}; \quad x_2^{(1)}, \quad x_2^{(2)}, \cdots, \quad x_2^{(10)}$$

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* $p^{(i)}$ is usually called the *occupancy* of link i.

and

$$x_3^{(1)}, \quad x_3^{(2)}, \cdots, \quad x_3^{(10)}.$$

For any series path with random variables

$$x_1^{(i)}, \quad x_2^{(i)}, \quad x_3^{(i)}, \qquad i = 1, 2, \cdots, 10,$$

the linking and blocking probabilities for this path are respectively $q_1^{(i)}q_2^{(i)}q_3^{(i)}$ and $1 - q_1^{(i)}q_2^{(i)}q_3^{(i)}$ where $q_1^{(i)} = Pr[x_1^{(i)} = 1]$ and similarly for $q_2^{(i)}$ and $q_3^{(i)}$. Thus, for G,

$$P = \prod_{i=1}^{10} (1 - q_1^{(i)}\, q_2^{(i)}\, q_3^{(i)}), \qquad Q = 1 - P$$

Numerically, suppose $q_1^{(i)} = q_2^{(i)} = q_3^{(i)} = 2/3$ for all i, then

$$P = \left[1 - \left(\frac{2}{3}\right)^3\right]^{10} = \left(\frac{19}{27}\right)^{10} = 0.0298\cdots$$

This example illustrates the fact that linking and blocking probabilities of a series-parallel graph can be found in a routine manner. Although the same can be said for nonseries-parallel graphs, the computational difficulties involved are of a different nature.

Let G be a probability linear-graph and let G^* be its canonical form. Denote the link random variables of G^* by $y^{(1)}, y^{(2)}, \cdots$ and let $Y^{(i)}$ be the event $y^{(i)} = 1$. Then clearly, the linking probability Q of G is the probability of the union of events $Y^{(i)}$ or

$$Q = Pr\left[\bigcup_i Y^{(i)}\right] \tag{2.19}$$

This last probability can be found in a standard manner (cf. Reference 8, Chapter 4) since the joint probabilities $Pr[\bigcap_j Y^{(i_j)}]$ are readily obtained from the linking probabilities of the links of G.

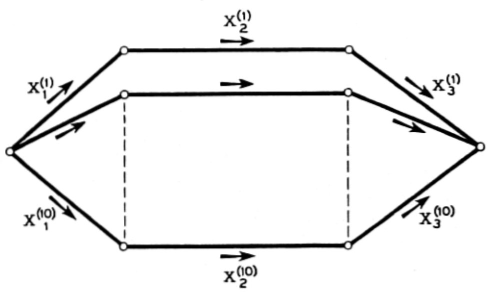

FIG. 2.3 — A series-parallel graph.

A real computational difficulty arises at this point. If the number of directed paths of G (or G^*) is not small, then the expansion of Q in terms of the aforementioned joint probabilities becomes formidable.

Example 2.2. In Figs. 2.4 and 2.5 are a non-series-parallel graph G and its canonical form G^*, respectively. In this case,

$$Q = Pr[Y^{(1)} \cup Y^{(2)} \cup Y^{(3)}] = \sum_{i=1}^{3} Pr[Y^{(i)}] - \sum_{\substack{i,j=1 \\ i<j}}^{3} Pr[Y^{(i)} \cap Y^{(j)}] + \sum_{\substack{i,j,k=1 \\ i<j<k}}^{3} Pr[Y^{(i)} \cap Y^{(j)} \cap Y^{(k)}] \tag{2.20}$$

Since $Pr[Y^{(i)} \cap Y^{(j)} \cap \cdots]$ can be found directly from G (e.g. $Pr[Y^{(1)} \cap Y^{(2)}] = q^{(1)}q^{(2)}q^{(3)}q^{(5)}$), we get

$$\begin{aligned} Q = (q^{(1)}q^{(2)} + q^{(2)}q^{(3)}q^{(5)} + q^{(3)}q^{(4)}) \\ - (q^{(1)}q^{(2)}q^{(3)}q^{(5)} + q^{(1)}q^{(2)}q^{(3)}q^{(4)} + q^{(2)}q^{(3)}q^{(4)}q^{(5)}) \\ + (q^{(1)}q^{(2)}q^{(3)}q^{(4)}q^{(5)}) \end{aligned} \tag{2.21}$$

In particular, if $q^{(1)} = q^{(2)} = q^{(3)} = q^{(4)} = q^{(5)} = q$, then

$$Q = 2q^2 + q^3 - 3q^4 + q^5 \tag{2.22}$$

2.5 *Generating Functions for Linking Probabilities.*

Let G be a graph with N links and let the links be designated by $X^{(1)}, X^{(2)}, \cdots, X^{(N)}$. Let B be the set of all directed paths of G. Then each $\beta_j \epsilon B$ is composed of links $X^{(j_1)}, X^{(j_2)}, \cdots, X^{(j_{n_j})}$ in series. Let us agree to denote the directed path β_j by the formal product

$$X^{(j_1)} \cdot X^{(j_2)} \cdot \cdots \cdot X^{(j_{n_j})}.$$

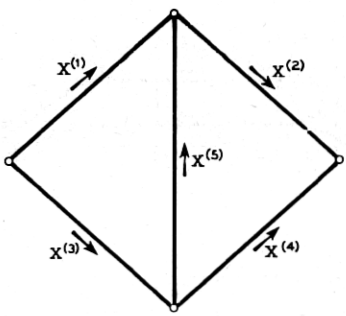

FIG. 2.4 — A nonseries-parallel graph.

In this formal product, the X's are considered as undefined real numbers and are manipulated as such *except* for the *reduction rule*

$$X^{(j)} \cdot X^{(k)} = X^{(j)} \quad \text{if} \quad j = k. \tag{2.23}$$

Otherwise, the operation $\cdot$ is ordinary multiplication.

Definition 2.4. A function $\mathfrak{Q}$ is said to be the *generating function* of G if

$$\mathfrak{Q}(s) = 1 - \prod_B^{*} (1 - X^{(j_1)} \cdot X^{(j_2)} \cdot \cdots \cdot X^{(j_{n_j})} \cdot s) \tag{2.24}$$

where Π^* denotes formal product under $\cdot$ such that *the reduction rule (2.23) is always carried out.*

Example 2.3. Let us go back to Fig. 2.4. In this case, there are three directed paths $X^{(1)} \cdot X^{(2)}$, $X^{(3)} \cdot X^{(5)} \cdot X^{(2)}$ and $X^{(3)} \cdot X^{(4)}$ so that

$$\begin{aligned} \mathfrak{Q}(s) = {} & (X^{(1)} \cdot X^{(2)} + X^{(3)} \cdot X^{(4)} + X^{(2)} \cdot X^{(3)} \cdot X^{(5)})s \\ & - (X^{(1)} \cdot X^{(2)} \cdot X^{(3)} \cdot X^{(5)} + X^{(1)} \cdot X^{(2)} \cdot X^{(3)} \cdot X^{(4)} \\ & + X^{(2)} \cdot X^{(3)} \cdot X^{(4)} \cdot X^{(5)})s^2 + (X^{(1)} \cdot X^{(2)} \cdot X^{(3)} \cdot X^{(4)} \cdot X^{(5)})s^3 \end{aligned} \tag{2.25}$$

Note that if, in (2.25), each $X^{(j)}$ is replaced by the real number $q^{(j)}$ and s is set equal to 1, the resulting value of the generating function $\mathfrak{Q}$[denoted by $\mathfrak{Q}^*(1)$] is

$$\begin{aligned} \mathfrak{Q}^*(1) = {} & (q^{(1)}q^{(2)} + q^{(2)}q^{(3)}q^{(5)} + q^{(3)}q^{(4)}) \\ & - (q^{(1)}q^{(2)}q^{(3)}q^{(5)} + q^{(1)}q^{(2)}q^{(3)}q^{(4)} + q^{(2)}q^{(3)}q^{(4)}q^{(5)}) \\ & + (q^{(1)}q^{(2)}q^{(3)}q^{(4)}q^{(5)}) \end{aligned} \tag{2.26}$$

which is precisely the linking probability Q for the graph found previously (Example 2.2). In fact, this relation holds true in general.

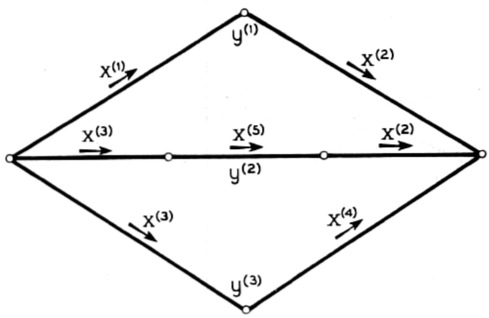

Fig. 2.5 — Canonical form of graph shown in Figure 2.4.

Theorem 2.2. Let G be a probability linear-graph and $\mathfrak{Q}$ its generating function. Then $\mathfrak{Q}^*(1)$ is the linking probability Q of G.

Proof. Let B be the set of all directed paths of G and let $y^{(i)}$ be the random variables associated with these directed paths. Let $Y^{(i)}$ be the event $y^{(i)} = 1$. Then the coefficient of s^r in $\mathfrak{Q}^*(s)$ is

$$\sum_{\substack{i_1, i_2, \ldots, i_r \\ i_1 < i_2 < \ldots < i_r}} Pr(Y^{(i_1)} \cap Y^{(i_2)} \cap \ldots \cap Y^{(i_r)})$$

except for a possible change in sign. Thus

$$\mathfrak{Q}^*(1) = \sum_i Pr(Y^{(i)}) - \sum_{\substack{i_1, i_2 \\ i_1 < i_2}} Pr(Y^{(i_1)} \cap Y^{(i_2)}) + \ldots$$
$$\pm \sum_{\substack{i_1, i_2, \ldots, i_n \\ i_1 < i_2 < \ldots < i_n}} Pr(Y^{(i_1)} \cap Y^{(i_2)} \cap \ldots \cap Y^{(i_n)}) = Q \tag{2.27}$$

This completes the proof.

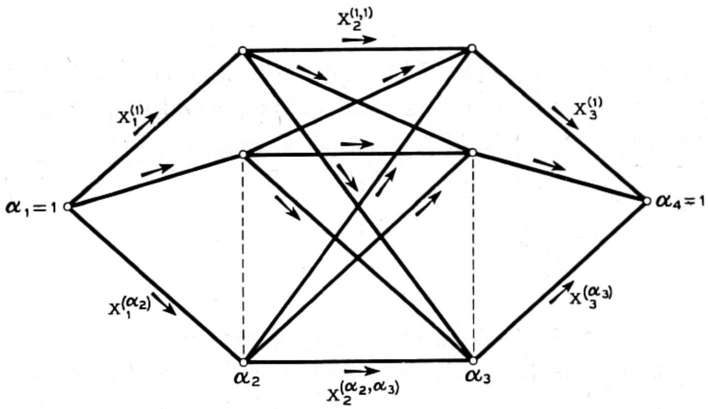

FIG. 2.6 — A general four-stage graph.

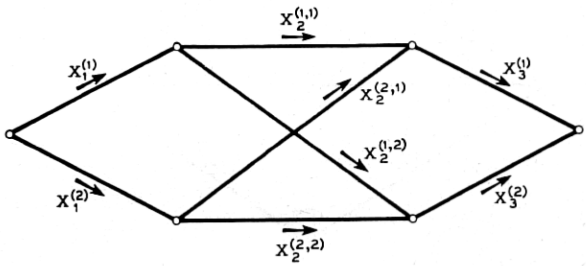

FIG. 2.7 — A simple four-stage graph.

2.6 *Some 4-Stage Graphs*

By a four-stage graph is meant a graph of the form shown in Fig. 2.6 where the graph has four stages of nodes with $\alpha_1 = 1$, α_2, α_3 and $\alpha_4 = 1$ nodes in each stage. It is clear from Fig. 2.6 that there are α_2 left links, $\alpha_2\alpha_3$ middle links and α_3 right links.

The linking probability of a four-stage graph can theoretically be found by a direct application of Theorem 2.2. The computation is simplified, however, if additional information is given. For instance, consider

Example 2.4. Let G be the four-stage graph shown in Fig. 2.7 with $\alpha_2 = \alpha_3 = 2$ where

$$\left.\begin{aligned} q_1^{(1)} &= q_1^{(2)} = q_1 \\ q_2^{(1,1)} &= q_2^{(1,2)} = q_2^{(2,1)} = q_2^{(2,2)} = q_2 \\ q_3^{(1)} &= q_3^{(2)} = q_3 \end{aligned}\right\} \tag{2.28}$$

Then we may write

$$\mathcal{Q}(s) = 1 - (1 - X_1^{(1)}\cdot X_3^{(1)}\cdot q_2 s)\cdot(1 - X_1^{(1)}\cdot X_3^{(2)}\cdot q_2 s) \cdot(1 - X_1^{(2)}\cdot X_3^{(1)}\cdot q_2 s)\cdot(1 - X_1^{(2)}\cdot X_3^{(2)}\cdot q_2 s) \tag{2.29}$$

so that

$$Q = \mathcal{Q}^*(1) = 4q_1q_2q_3 - 2q_1q_2^2q_3(q_1 + q_1q_3 + q_3) + 4q_1^2q_2^3q_3^2 - q_1^2q_2^4q_3^2$$

The same procedure can be applied to all such 4-stage graphs.

Example 2.5. In the special cases where the occupancies of the left, middle and right links are equal respectively to p_1, p_2 and p_3 (Example 2.4 is one of these), the following expression* has been obtained by D. H. Evans for the blocking probability P of a 4-stage graph:

$$P = \sum_{k=0}^{a2} \binom{\alpha_2}{k} (1 - p_1)^k p_1^{(\alpha_2 - k)}[(1 - p_3)p_2^k + p_3]^{\alpha_3}$$

This formula is of considerable value in a study of blocking in six-stage switching networks by the author and D. H. Evans (unpublished).

In general, if the number of nodes of a four-stage graph is not small, it would be unrealistic timewise to expand $\mathcal{Q}(s)$ to yield the linking probability Q of the graph. In such cases no good theoretical method has been found which will conveniently yield the values of Q. It is possible, however, with the aid of Theorem 2.1, to devise an experiment which will yield good approximations to Q without undue labor. In fact, this remark applies to all probability linear-graphs.

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* This expression is derived from a counting procedure different from the approach outlined here. Using still another approach, an extension of this result to a more general class of 4-stage networks has been obtained by M. Goldman.

We shall now proceed to describe switching networks as probability linear-graphs.

3. SWITCHING NETWORKS AS PROBABILITY LINEAR-GRAPHS

3.1 *Probability of Blocking.*

By a switching network is meant a crosspoint network in which each input of the network can be connected to any output of the network by the operation of appropriate sequences of crosspoints. A block diagram representation of such networks is shown in Fig. 3.1. The first stage of crosspoint switches has L_0 inputs and L_1 outputs which are inputs to the second stage of switches. Thus these L_1 leads link stages 1 and 2 and are therefore called L_1-links. Similarly, the L_i-links $0 < i < s$ link stages L_i and L_{i+1} and the L_s links are the outputs of the network which has s stages.

Let L be a switching network with inputs and outputs indexed by i and j; $i = 1, 2, \cdots, L_0$, $j = 1, 2, \cdots, L_s$. Denote by $P(i, j)$ the probability of all paths from input i to output j busy. Then the blocking probability P of L is defined here as*

$$P = \frac{1}{L_0 L_s} \sum_{i=1}^{L_0} \sum_{j=1}^{L_s} P(i, j) \tag{3.1}$$

Similarly the linking probability Q of L is

$$Q = 1 - P \tag{3.2}$$

For each pair (i, j), $P(i, j)$ is found as the blocking probability of a corresponding probability linear-graph.

In this section, we assume $P(i, j)$ to be independent of i and j and each crosspoint switch (switch with arrays of crosspoints) in the network to be of nonblocking type.† There is no essential loss of generality in the first condition; although without it, the evaluation of P would become more tedious.

It remains for us to present the correspondence between a switching network L and a probability linear-graph G. This description is best given by examples. Choose an input i and an output j of L. Then the paths from i to j would involve crosspoint switches and links between

* This is one of several possible definitions of blocking probability of a network.

† Here we restrict our attention to networks consisting of switches of nonblocking type (e.g., square switches or switches with more outputs than inputs). Thus, "concentration" switches are excluded. The analysis of networks with concentrating stages involves deeper insight and will not be considered in this paper.

pairs of these switches. Represent the switches by nodes and the links by directed branches. The resulting graph G is the graph corresponding to L.

We consider G to be the model of L from which pertinent characteristics of L can be extracted. Thus, we are identifying such quantities as blocking probability, mean retrial time etc., of L with those of G.

Example 3.1. The 4-stage switching distribution network L shown in Fig. 3.2 is common (with modifications) in telephone central offices. Its corresponding probability linear-graph is shown in Fig. 3.3, with all link occupancies p. The blocking probability P of L is therefore (see Example 2.1).

$$P = [1 - (1 - p)^3]^{10} \tag{3.3}$$

Equation (3.3) is the simplest one of several formulas sometimes known as Kittredge-Molina formulas for crosspoint networks.

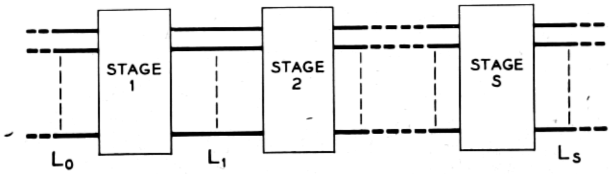

FIG. 3.1 — Block diagram of a switching network.

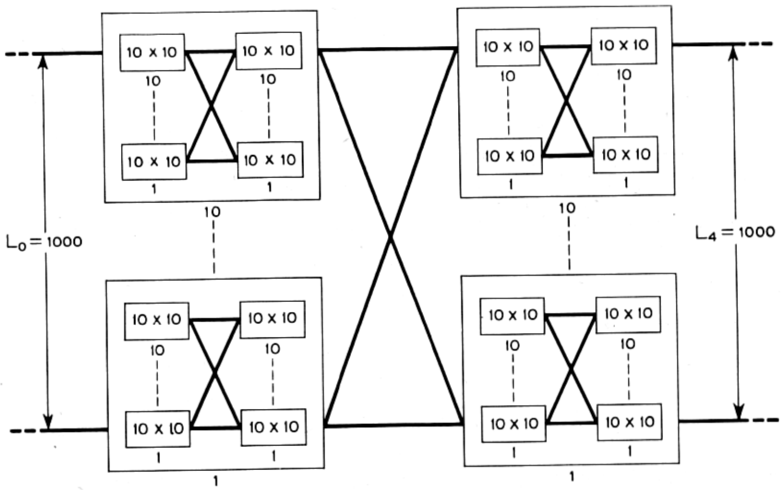

FIG. 3.2 — A four-stage switching network.

Example 3.2. Consider the partially-equipped, four-stage distribution network, first suggested by C. A. Lovell, (Fig. 3.4) which would grow with relative ease into the full distribution network of Example 3.1. In this example, we prefer to overlook the fact that the third stage switches are not of the non-blocking type. The corresponding graph is shown in Fig. 3.5 and the blocking probability is

$$P_1 = [1 - (1 - p_1)^2(1 - p_2)]^{10} \tag{3.4}$$

To calculate the link occupancies, let A be the offered traffic (in erlangs) in Example 3.1 and for the purpose of comparison, let 0.4A be the offered traffic in Example 3.2. Then

$$p = \frac{A(1 - P)}{1000}, \qquad p_1 = \frac{0.4A(1 - P_1)}{400}, \qquad p_2 = \frac{0.4A(1 - P_1)}{1000} \tag{3.5}$$

where in (3.5), the link occupancies depend themselves on the blocking

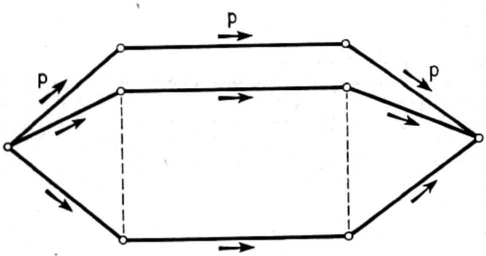

FIG. 3.3 — Graph of network shown in Figure 3.2.

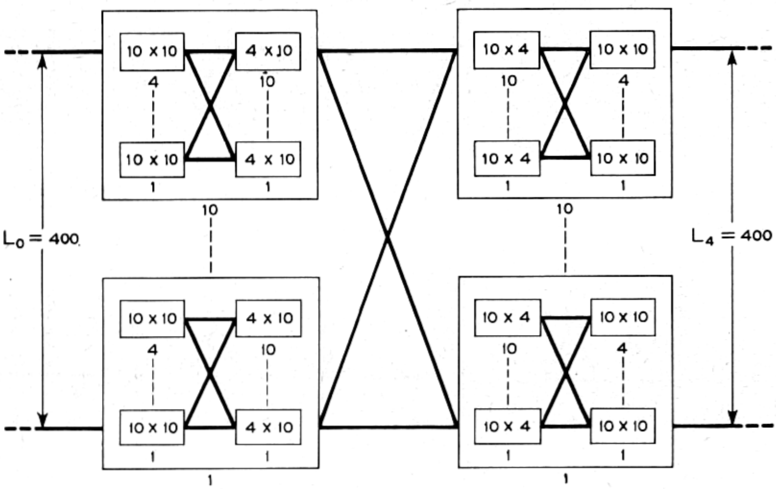

FIG. 3.4 — A partially equipped four-stage network.

probabilities. As a first approximation, the factors $(1 - P)$ and $(1 - P_1)$ may be neglected. A comparison of (3.3) and (3.4) shows then that the blocking of a partially equipped network is less than that of a fully equipped network.

Example 3.3. The switching network shown in Fig. 3.6 is a particular case of a class of non-blocking networks discovered by C. Clos.[10] Its corresponding graph is shown in Fig. 3.7 with blocking probability

$$P = [1 - (1 - p)^2]^{39} \tag{3.6}$$

Since it is known a priori that the actual blocking probability for the network is identically zero, it is interesting to compare P given by (3.6) with zero. We arrive at the following table:

p	P
0.1	7.44×10^{-29}
0.3	3.94×10^{-12}
0.5	1.34×10^{-5}
0.7	0.025
0.9	0.676
0.95	0.907
1.0	1.0

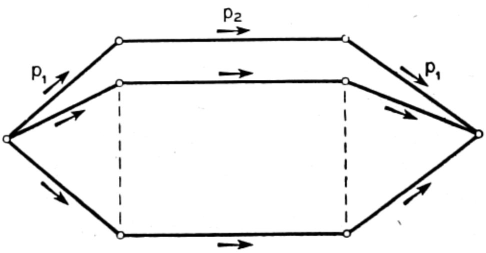

FIG. 3.5 — Graph of network shown in Figure 3.4.

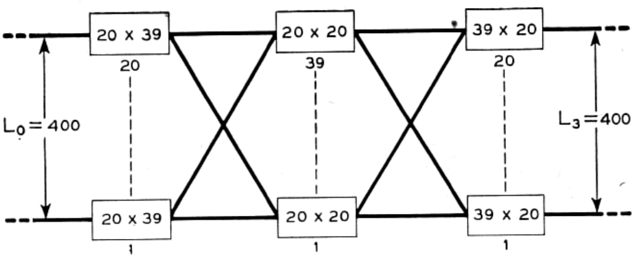

FIG. 3.6 — A three-stage non-blocking network.

This example illustrates the effect of neglecting link dependence; Equation (3.6) gives very high blocking for large p. In practice, however, a value of p greater than 0.5 is rarely encountered so that even in this case the approximation is not unreasonable. It should be remarked that more suitable formulas for computing blocking in three-stage networks have been developed by M. Karnaugh.

3.2 *Applications to Equivalent Crosspoint Minimization.*

As an application of the concepts developed here, we shall consider the problem of equivalent crosspoint minimization for switching networks.

The case in mind is a four-stage network in which we assume:

A-1. The network is symmetrical with respect to its inputs and outputs; (i.e., the network configuration remains fixed when the inputs and outputs of the network are interchanged).

A-2. Switches within each stage of the network are of identical size.

A-3. Stages 1 and 2 are combined into β identical frames (called primary frames) no two of which are interconnected.

A-4. The number of outputs in a switch in the first stage is equal to the number of switches in each frame in the second stage, with each output connected to a distinct switch in that frame.

A-5. The number of outputs in a switch in the second stage is β, with each output connected to a distinct (secondary) frame and with outputs on distinct switches in the same primary frame connected to distinct switches in the same (secondary) frame.

Under these assumptions, the network has the configuration shown in Fig. 3.8. The problem is, given the offered traffic, the maximum allowable blocking P' and the number of inputs L_0, determine x, y, α and β so that the number of equivalent crosspoints C_0 is a minimum where C_0 is given by

$$C_0 = C + k_0 L_0 + 2k_1 L_1 + 2k_2 L_2 \qquad (3.7)$$

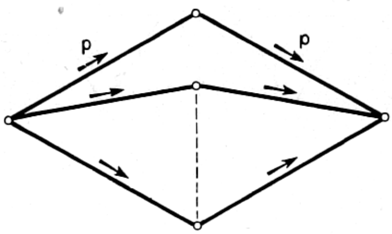

FIG. 3.7 — Graph of the three-stage non-blocking network.

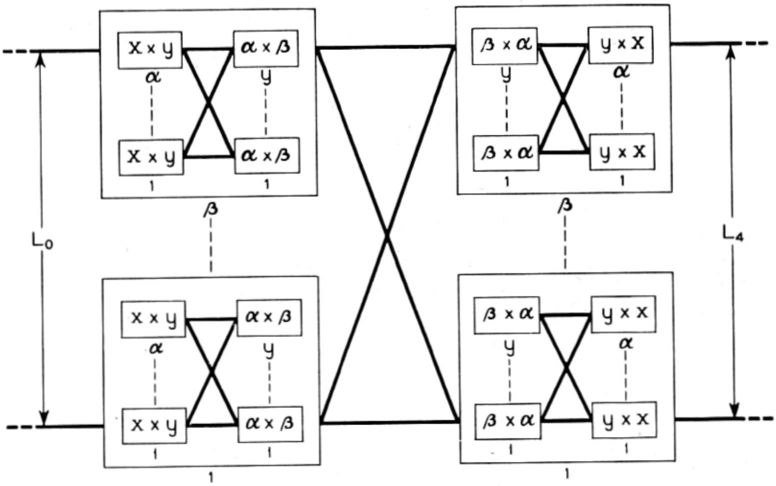

FIG. 3.8 — A general four-stage switching network.

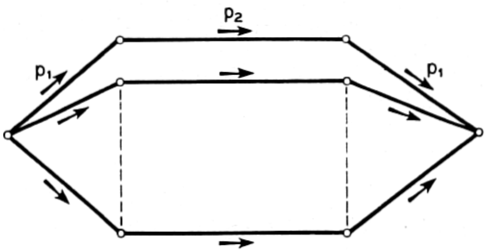

FIG. 3.9 — Graph of the general four-stage switching network.

in which k_0, k_1 and k_2 are given constants which evaluate relative network cost effects and C is the number of crosspoints in the network.

The corresponding graph shown in Fig. 3.9 consists of y parallel paths so that the blocking probability for the network is

$$P = (1 - q_1^2 q_2)^y \tag{3.8}$$

As a first approximation, we have

$$q_1 = 1 - \frac{A'}{\alpha\beta y}, \qquad q_2 = 1 - \frac{A'}{\beta^2 y} \tag{3.9}$$

where

$$A' = A(1 - P') \tag{3.10}$$

and A is the offered erlangs. It then follows that

$$C_0 = L_0(k_0 + 2y) + \frac{2(A')^{3/2}}{(1 - q_1)(1 - q_2)^{1/2}y^{1/2}} + \frac{2k_1A'}{1 - q_1} + \frac{2k_2A'}{1 - q_2} \quad (3.11)$$

Our problem is to find q_1, q_2 and y which will minimize C_0 under the condition

$$(1 - q_1^2 q_2) - (P')^{1/y} = 0 \quad (3.12)$$

Using the Lagrangian multiplier method, we arrive at the following system of four equations

$$\frac{(A')^{3/2}}{(1 - q_1)^2(1 - q_2)^{1/2}y^{1/2}} + \frac{k_1A'}{(1 - q_1)^2} = \lambda q_1 q_2 \quad (3.13)$$

$$\frac{(A')^{3/2}}{(1 - q_1)(1 - q_2)^{3/2}y^{1/2}} + \frac{k_2A'}{(1 - q_2)^2} = \lambda q_1^2 \quad (3.14)$$

$$\frac{(A')^{3/2}}{(1 - q_1)(1 - q_2)^{1/2}y^{3/2}} - \lambda \frac{(P')^{1/y} \ell n P'}{y^2} = 2L_0 \quad (3.15)$$

$$1 - q_1^2 q_2 - (P')^{1/y} = 0 \quad (3.16)$$

in which the unknowns to be solved for are q_1, q_2, y and λ. These can be found numerically in specific problems.

It should be noted that in using the graph representation to find the blocking probability of a switching network, we had tacitly assumed that the switches should be of non-blocking type. But solutions to (3.13)–(3.16) do not guarantee that this will be the case. Thus, the values of x, y, α and β obtained from this minimization process should be regarded as approximations.

We now consider the following special case which is of independent interest. In equation (3.7), let $k_0 = k_1 = k_2 = 0$. This situation corresponds physically to the interpretation that the link costs are negligible as compared to the crosspoint costs. Equations (3.13)–(3.16) then become

$$\frac{(A')^{3/2}}{(1 - q_1)^2(1 - q_2)^{1/2}y^{1/2}} = \lambda q_1 q_2 \quad (3.17)$$

$$\frac{(A')^{3/2}}{(1 - q_1)(1 - q_2)^{3/2}y^{1/2}} = \lambda q_1^2 \quad (3.18)$$

$$2L_0 - \frac{(A')^{3/2}}{(1 - q_1)(1 - q_2)^{1/2}y^{3/2}} + \lambda \frac{(P')^{1/y} \ell n P'}{y^2} = 0 \quad (3.19)$$

$$1 - q_1^2 q_2 - (P')^{1/y} = 0. \quad (3.20)$$

From (3.17) and (3.18) it turns out that

$$q_1 = q_2 = q \tag{3.21}$$

so that the problem of solving four equations is now reduced to that of solving a single equation

$$\frac{(A')^{3/2}}{(1 - q)^{3/2} y^{3/2}} \left[1 - \frac{(1 - q^3) \ln (1 - q^3)}{q^2(1 - q)} \right] = 2L_0 . \tag{3.22}$$

It is interesting to note that in this case, as an immediate consequence of the minimization process, the second and third stage switches turn out to be square switches and hence are of non-blocking type. We remark also that a similar minimization process may be applied to three-stage networks.

Example 3.4. Suppose it is desired to design a minimum crosspoint, four-stage network with $A = 400$ erlangs, $L_0 = 800$ input lines and $P' = 0.01$. Solving Equation (3.22), we find

$$y = 15 \qquad q = 0.642$$

Therefore, from Equations (3.9) and (3.10), we get

$$\alpha = \beta = 8.65 \cong 9, \qquad x \cong 10.$$

For these values of α, β, x and y, we find

$$C = 46{,}200 \text{ crosspoints.}$$

4. RETRIAL AND CONNECTION-TIME DISTRIBUTIONS

4.1 *Retrial Distribution.*

Given a switching network, it is of interest to know whether a path which is blocked at some given time can be established some time later and with what degree of success. More precisely, let L be a switching network and G its corresponding graph. By the *retrial distribution* (denoted by $P_{00}(t)$) of G is meant the conditional probability*

$$P_{00}(t) = Pr[\varphi_{\tau+t} = 0 \mid \varphi_\tau = 0]$$

where φ_t is the connection function of G.

In order to find $P_{00}(t)$, we consider the following process† for the link random variables:

1. The link random variables are mutually independent with common probability distributions.

* The notation $P_{\delta_1\delta_2}(t)$ is used for conditional probabilities whereas the notation $P_t(\delta_1, \delta_2)$ is used for joint probabilities.

† This is a special case of stochastic processes discussed in Reference 1.

2. If X is a link random variable, the conditional probability of a change from $x = 1$ to $x = 0$ during $(t, t + h)$ is $\lambda h + 0(h)$, from $x = 0$ to $x = 1$ during $(t, t + h)$ is $\mu h + 0(h)$. The conditional probability of more than one change in $(t, t + h)$ is $0(h)$; (i.e. the link holding and idle time lengths are exponentially distributed with parameters μ and λ; cf. Reference 8, Chapter 17).

We then obtain the following system of four differential equations for each link random variable.

$$\begin{aligned} P_{00}'(t) &= -\mu P_{00}(t) + \lambda P_{01}(t) \cdot \\ P_{01}'(t) &= -\lambda P_{01}(t) + \mu P_{00}(t) \cdot \\ P_{10}'(t) &= -\mu P_{10}(t) + \lambda P_{11}(t) \cdot \\ P_{11}'(t) &= -\lambda P_{11}(t) + \mu P_{10}(t) \cdot \end{aligned} \tag{4.1}$$

with initial conditions

$$P_{00}(0) = P_{11}(0) = 1, \qquad P_{10}(0) = P_{01}(0) = 0$$

The solution to this system of equations is (cf. Reference 1)

$$\left.\begin{aligned} P_{00}(t) &= \frac{\lambda + \mu e^{-(\lambda+\mu)t}}{\lambda + \mu} = p + qe^{-(\lambda+\mu)t} \\ P_{01}(t) &= \frac{\mu - \mu e^{-(\lambda+\mu)t}}{\lambda + \mu} = q - qe^{-(\lambda+\mu)t} \\ P_{10}(t) &= \frac{\lambda - \lambda e^{-(\lambda+\mu)t}}{\lambda + \mu} = p - pe^{-(\lambda+\mu)t} \\ P_{11}(t) &= \frac{\mu + \lambda e^{-(\lambda+\mu)t}}{\lambda + \mu} = q + pe^{-(\lambda+\mu)t} \end{aligned}\right\} \tag{4.2}$$

where p is the occupancy of the link and $q = 1 - p$. Since

$$\left.\begin{aligned} P_t(1, 1) &= 1 - 2P(0) + P_t(0, 0) \\ P_t(0, 0) &= 2P(0) + P_t(1, 1) - 1 \\ P_t(1, 0) &= P_t(0, 1) = P(0) - P_t(0, 0) = P(1) - P_t(1, 1) \end{aligned}\right\} \tag{4.3}$$

retrial distributions for series-parallel graphs can be found directly from (4.3) or (2.7)–(2.10) and (2.11)–(2.14).

Example 4.1. Consider the switching network shown in Fig. 3.2 the corresponding graph of which was shown in Fig. 3.3. For this graph we find, for each path, [say path (i)]

$$P_t^{(i)}(1, 1) = P^{(i)}(1)P_{11}^{(i)}(t) = q^3(q + pe^{-(\lambda+\mu)t})^3 \tag{4.4}$$

Hence, for the entire graph, from (4.3) and (3.3)

$$P_t(0, 0) = [2(1 - q^3) + q^3(q + pe^{-(\lambda+\mu)t})^3 - 1]^{10} \quad (4.5)$$
$$= (1 - 2q^3 + [q(q + pe^{-(\lambda+\mu)t})]^3)^{10}$$

To fix ideas, let the "average holding time" $1/\mu = 200$ sec., and link occupancy $p = 1/3$. Then, since

$$\lambda q = \mu p, \quad (4.6)$$

we get

$$\lambda = \frac{1}{400}, \qquad q = \frac{2}{3} \quad (4.7)$$

For these values of μ and p, we get

$$P_t(0, 0) = \left[\frac{11}{27} + \frac{8}{27}\left(\frac{2}{3} + \frac{1}{3}\, e^{-(3/400)t}\right)^3\right]^{10} \quad (4.8)$$

4.2. *Mean Retrial Time*

Let us now consider the following problem. We begin with the hypothesis that all paths from a given input to a given output of a switching network are busy initially. If a retrial is made every x seconds until one path is found free, what is the expected number of seconds $\bar{t}_x$ for the establishment of a path?

To solve this problem, let G be the graph corresponding to the switching network with connection function φ_t with probability distributions P, Q, $P_t(0, 0)$, $P_t(0, 1)$, $P_t(1, 0)$ and $P_t(1, 1)$. We use the notation $P_{\delta_1\delta_2}(t)$ to mean the transition probability of $\varphi_t = \delta_2$ at time t given $\varphi_t = \delta_1$ initially. Let $m_x(n)$ be the probability of success at the n^{th} trial, given that all paths are busy initially. Then

$$\bar{t}_x = \sum_{k=1}^{\infty} kxm_x(k) \quad (4.9)$$

Since

$$m_x(k) = (P_{00}(x))^{k-1}P_{01}(x); \qquad k = 1, 2, \cdots \quad (4.10)$$

we find

$$\bar{t}_x = \sum_{k=1}^{\infty} kx(P_{00}(x))^{k-1}P_{01}(x) = \frac{x}{1 - P_{00}(x)} = \frac{xP}{P - P_x(0, 0)} \quad (4.11)$$

where P is the blocking probability of the network.

We shall call $\bar{t}_x$ the *mean retrial time* (in interval of x seconds). In particular, if retrials are made in arbitrarily small time interval, the limit $\lim_{x\to 0}\bar{t}_x$, when existent, is called the *limiting retrial time* $\bar{t}$. Physically, $\bar{t}$ is the expected number of seconds the first path becomes free given that all paths are busy initially. From (4.11), we find

$$\bar{t} = \lim_{x\to 0} \frac{xP}{P - P_x(0, 0)} = -\frac{P}{\left[\frac{d}{dx} P_x(0, 0)\right]_{x=0}} \tag{4.12}$$

Example 4.2. Going back to Example 4.1, $P_x(0, 0)$, which is the joint probability of the network blocked initially and also blocked x seconds later, was given by (4.8):

$$P_x(0, 0) = \left[\frac{11}{27} + \frac{8}{27}\left(\frac{2}{3} + \frac{1}{3} e^{-(3/400)x}\right)^3\right]^{10}$$

Differentiating $P_x(0, 0)$ with respect to x and setting $x = 0$, we find the limiting retrial time as given by (4.12) is 31.67 seconds. On the other hand, if a retrial is made once every second (that is, $x = 1$), the mean retrial time as given by (4.11) is 32.26 seconds.

We remark that together with the method of Section 2, the same procedure can be applied to non-series-parallel graphs such as those shown in Figures 2.4 and 2.7.

4.3 *Connection-Time Distributions*

Up to now we have not paid attention to the physical structure of the crosspoints in a switching network. If the crosspoints are made up of active elements such as gas diodes, the following problem then presents itself. It is known that it takes a certain time for a gas tube to break down after a voltage higher than the breakdown voltage is applied. In a switching network, the establishment of a path in general involves the breakdown of several such crosspoints in series and it is important that the breakdown time of a path must be reasonably short lest the system bogs down from this inherent delay.

There is experimental evidence that the breakdown time of a gas tube has roughly an exponential distribution. Thus, if X is a random variable representing the breakdown time of a tube (for a fixed applied voltage), then we postulate

$$Pr[X \leq x] = 1 - e^{-\alpha x} \tag{4.13}$$

where

$$E(X) = \frac{1}{\alpha} \tag{4.14}$$

We shall assume that the tubes behave independently and have identical breakdown time characteristics and that the voltage across each tube before breakdown remains a fixed constant regardless of the behavior of the other tubes. The reason for imposing the latter condition is clear since the mean breakdown time $1/\alpha$ depends upon the voltage applied.

Let L be a gas tube switching network and G its corresponding graph with connection function φ_t. Let ψ be a random variable representing the breakdown time of G. We define the *connection-time distribution* Φ of L to mean the joint probability.

$$\Phi = Pr[\varphi_t = 1 \text{ and } \psi \leqq t] \tag{4.15}$$

i.e., the joint probability of finding at least one idle path and the breakdown time of some idle path less than or equal to t. It is also desirable to denote by Ψ the probability

$$\Psi = 1 - \Phi = Pr[\varphi_t = 0 \text{ or } (\varphi_t = 1 \text{ and } \psi \geqq t)] \tag{4.16}$$

In a graph G, the tubes are represented by nodes of G. The number of nodes in a path in G then represents the number of tubes in series in the path.

To be concrete, let us now consider the network shown in Fig. 3.2 with its associated graph shown in Fig. 3.3. We shall study two methods of establishing paths through the network.

1. *End-Matching*. A voltage is applied to both ends of the network simultaneously (across a single input and a single output). We wish to find Φ (or Ψ) of the network.

First, let us digress to state a useful lemma the proof of which can be found in probability texts.

Lemma 4.1. Let $X_1, X_2, \cdots, X_n$ be mutually independent random variables with common distribution

$$Pr[X_i \leqq x] = 1 - e^{-\alpha x}; \qquad x \geqq 0 \tag{4.17}$$

then the distribution of the sum $X_1 + X_2 + \cdots + X_n$ is

$$Pr(X_1 + X_2 + \cdots + X_n \leqq t) = 1 - e^{-\alpha t} \sum_{j=0}^{n-1} \frac{(\alpha t)^j}{j!} \tag{4.18}$$

Reproducing the graph, Fig. 3.3, we have, for a single path

$$Pr[(\text{path busy}) \text{ or } (\text{path idle and breakdown time} \geqq t)]$$

$$= Pr[\text{path busy}] + Pr[\text{path idle and breakdown time} \geqq t] \tag{4.19}$$

Now a path is idle only if all three links making up the path are idle so that, from Lemma 4.1,*

$$\begin{aligned} Pr[\text{path idle and breakdown time} \geqq t] &= Pr[\text{path idle}] \\ \cdot Pr_{\text{given path idle}} [\text{breakdown time} \geqq t] &= q^3 e^{-\alpha t} \sum_{j=0}^{3} \frac{(\alpha t)^j}{j!} \end{aligned} \tag{4.20}$$

Hence, (4.19) is given by the expression

$$(1 - q^3) + q^3 e^{-\alpha t} \sum_{j=0}^{3} \frac{(\alpha t)^j}{j!} \tag{4.21}$$

For ten independent paths in parallel, it is clear

$$\begin{aligned} \Psi &= Pr[\varphi_t = 0 \text{ or } (\varphi_t = 1 \text{ and } \psi \geqq t)] \\ &= \{Pr[(\text{one path busy}) \text{ or } (\text{one path idle and breakdown time} \geqq t)]\}^{10} \qquad (4.22) \\ &= \left[(1 - q^3) + q^3 e^{-\alpha t} \sum_{j=0}^{3} \frac{(\alpha t)^j}{j!} \right]^{10} \end{aligned}$$

and

$$\Phi = 1 - \Psi = 1 - \left[(1 - q^3) + q^3 e^{-\alpha t} \sum_{j=0}^{3} \frac{(\alpha t)^j}{j!} \right]^{10} . \tag{4.23}$$

We shall defer numerical examples in order to arrive at a comparison with the next case.

2. *Center-Matching.* In this method, the marks propagate in both directions and are matched in the center links (see Fig. 4.1).

In this case, we need to consider the left side and right side breakdown times individually and then combine them. For a single path, we may write

$$Pr[\text{path idle and breakdown time} \leqq t] = \text{I} \cdot \text{II} \tag{4.24}$$

where

$$\left. \begin{aligned} \text{I} &= Pr[\text{path idle}] \\ \text{II} &= Pr_{\text{given path idle}} [\text{breakdown time} \leqq t] \end{aligned} \right\} \tag{4.25}$$

The conditional probability II can be written as a product

$$\begin{aligned} \text{II} = \{&Pr_{\text{given path idle}} [\text{breakdown time of left side} \leqq t]\} \\ &\cdot \{Pr_{\text{given path idle}} [\text{breakdown time of right side} \leqq t]\}, \end{aligned} \tag{4.26}$$

---

* We shall use the notations $Pr[B \mid A]$ and $Pr_{\text{given event } A}$ [event B] interchangeably.

so that, from Lemma 1,

$$\text{II} = \left(1 - e^{-\alpha t} \sum_{j=0}^{1} \frac{(\alpha t)^j}{j!}\right)^2 \tag{4.27}$$

Combining as before, we obtain

$$\Psi = \left[1 - q^3\left(1 - e^{-\alpha t} \sum_{j=0}^{1} \frac{(\alpha t)^j}{j!}\right)^2\right]^{10} \tag{4.28}$$

$$\Phi = 1 = \Psi \tag{4.29}$$

We shall consider the following numerical example. For a 10,000-line telephone central office, it is reasonable to assume, from the point of view of control circuits, that the connection time of a call should not exceed 5 milliseconds. Using this value for t and letting $q = 0.7$, we find for several values of mean tube breakdown time (0.5, 1, 5 and 10 milliseconds) the probabilities Ψ shown in Table 4.1.

This table shows that for the network in question, in order to meet the 5 millisecond time requirement, the tubes must have a mean breakdown time of much less than 1 millisecond to insure reasonable blocking.

4.4 *Remarks and Conclusions.*

From the discussion of the last section it is clear that, for a gas tube switching network, blocking probability alone is insufficient as a design

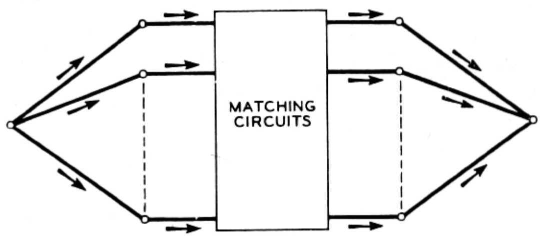

Fig. 4.1 — Graph of four-stage gas tube network with center matching.

TABLE 4.1

Mean Tube Breakdown Time in Milliseconds $1/\alpha$	αt	End-Matching Ψ	Center-Matching Ψ	Blocking Probability P
0.5	10	0.0158	0.0153	0.015
1	5	0.0548	0.0225	0.015
5	1	0.937	0.785	0.015
10	0.5	0.99	0.97	0.015

criterion, since it neglects completely the control circuit requirements. It is more natural to use the joint probability Ψ (which is the probability of either no path available or, when there are paths available, the breakdown time exceeds t) as the generalized blocking for design considerations. Table 4.1 shows how radically different Ψ can be from the blocking probability P.

We may consider the converse situation. Suppose, for the 10,000-line office considered before, in addition to restricting the connection time to not over 5 milliseconds, we further demand that the generalized blocking Ψ must not be higher than some fixed value (say 0.02). Then, from the discussion given here, it is possible to determine the maximum gas tube breakdown time allowable. Thus, generalized blocking can be used as a criterion in determining the choice of the type of gas tubes suitable for the switching network in question.

The fact that it takes time to establish a path in a gas tube network has a definite bearing on the retrial distribution discussed in Section 4.1. Because of it, the mean retrial time on blocked calls will be modified; that is, the retrial distribution of a network depends in part on the characteristics of the gas tubes used in the network.

In this paper we have attempted to study switching networks in terms of a simplified model. Because of the elementary character of the model chosen, many problems are left unsettled. For example, from the point of view of applicability, one would want to know how our results would alter if the assumption of independence among links is dropped. To go a step further, except for Section 4.3, switching networks in this paper have been considered as isolated entities by themselves. A more realistic study should consist of viewing a switching network together with its associated control circuits which entails great difficulties at this time. Any progress in this direction is, of course, highly desirable.

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