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Reduction of Skin Effect Losses by the Use of Laminated Conductors

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It has recently been discovered that it is possible to reduce skin effect losses in transmission lines by properly laminating the conductors and adjusting the velocity of transmission of the waves. The theory for such laminated transmission lines is presented in the case of planar systems for both infinitesimally thin laminae and laminae of finite thickness. A transmission line completely filled with laminated material is discussed. An analysis is given of the modes of transmission in a laminated line, and of the problem of terminating such a line.

I. Introduction

It has long been recognized that an electromagnetic wave propagating in the vicinity of an electrical conductor can penetrate only a limited distance into the interior of the material. This phenomenon is known as "skin effect" and is usually measured by a so-called "skin depth" δ. If y is measured from the surface of a conductor into its depth, the amplitude of the electromagnetic wave and the accompanying current density decreases as $e^{-y/\delta}$, provided the conductor is several times δ in thickness, so that for $y = \delta$ the amplitude has fallen to $1/e = 0.367$ times its value at the surface. The skin depth δ is given by

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} \qquad \text{(I-1)}$$

where σ is the conductivity of the material, μ is its permeability and ω is 2π times the frequency f under consideration. Throughout this paper rationalized MKS units are used.

From one point of view, skin effect serves a most useful purpose; for instance, in shielding electrical equipment or reducing crosstalk between communication circuits. On the other hand, the effect severely limits the high frequency performance of many types of electrical apparatus, including in particular the various kinds of transmission lines.

Surprisingly enough, it has been discovered that it is possible, within limits, to increase the distance to which an electromagnetic wave penetrates into a conducting material. This is done essentially by fabricating the conductor of many insulated laminae or filaments of conducting material arranged parallel to the direction of current flow. If the transverse dimensions of the laminae or filaments are small compared to the skin depth δ at the frequency under consideration, and if the velocity of the electromagnetic wave along the conductor is close to a certain critical value, the wave will penetrate into the composite conductor a distance great enough to include a thickness of conducting material many skin depths deep. Physically speaking, the lateral change of the wave through the conducting regions is very nearly cancelled by the change through the insulating regions.

In Fig. 1 there is shown a cross-section view of a coaxial cable with a

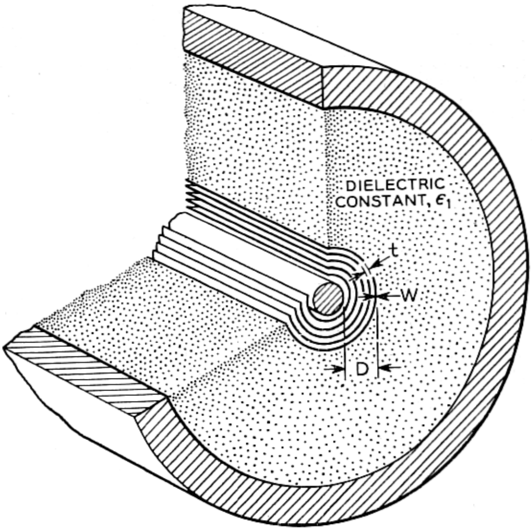

Fig. 1—Laminated transmission line.

laminated center conductor. The center conductor is formed of a non-conducting core surrounded by alternate layers of a conductor of thickness W and conductivity σ, and an insulator of thickness t and dielectric constant ϵ. The center conductor is embedded in an insulator of dielectric constant ϵ_1 which is in turn encased in the outer conductor. We will assume all the conductors and insulators to have the permeability μ_0 of free space.

We will associate with the inner laminated conductor an average dielectric constant[1] for transverse electric fields given by

$$\bar{\epsilon} = \epsilon \left(1 + \frac{W}{t}\right) \tag{I-2}$$

[1] A similar average dielectric constant has been considered by Tokio Sakurai, *Journal of Physical Society of Japan*, Vol. 5, No. 6, pp. 394–398, Nov.–Dec. 1950.

It will be shown in the following sections that the electromagnetic wave and the accompanying currents will penetrate most deeply into the center conductor if the wave travels through the line with a velocity

$$v = \frac{1}{\sqrt{\bar{\epsilon}\mu_0}} \tag{I-3}$$

One way to make the wave assume this velocity is to let the dielectric constant ϵ_1 have the value

$$\epsilon_1 = \bar{\epsilon} = \epsilon\left(1 + \frac{W}{t}\right) \tag{I-4}$$

If the depth of the stack of laminations D is small compared to the distance between the stack and the outer conductor, and if the wave travels with the velocity given in equation (I-3), it will be shown that the wave decreases with distance into the center conductor as e^{-y/δ_w} where δ_w is given by

$$\delta_w = \sqrt{3}\,(1 + t/W)(\delta/W)\delta; \quad W \ll \delta \tag{I-5}$$

Here $\delta = \dfrac{1}{\sqrt{\pi f \mu_0 \sigma}}$ is the skin depth appropriate to the material of the conducting laminae and the frequency f under consideration. Let us now also associate with the center conductor an average longitudinal conductivity given by

$$\bar{\sigma} = \sigma\,\frac{W}{W + t} \tag{I-6}$$

We will suppose for the present case that most of the attenuation of the transmission line results from the currents flowing in the inner conductor. It is easy to see that the attenuation of the line for very low frequencies will be $A/\bar{\sigma}D$ where A is a constant depending on the impedance of the line. As the frequency increases, δ_w decreases, and when δ_w becomes several times smaller than D it will be shown that the attenuation becomes $A/\bar{\sigma}\delta_w$. At still higher frequencies δ will similarly become several times smaller than W, and the attenuation then becomes $A/\sigma\delta$. From these considerations, a qualitative picture of the attenuation of the laminated line can be sketched as in Fig. 2.

For comparison, we have also sketched in Fig. 2 the attenuation that would be obtained if the laminations in Fig. 1 were replaced with solid metal. At low frequencies, the attenuation of this line would clearly be $A/\sigma D$. When the frequency becomes high enough for δ to be several times smaller than D the attenuation will be shown to become $A/\sigma\delta$.

It will be observed how the attenuation of the unlaminated line remains constant over a low range of frequencies and then rises at a rate proportional to the square root of the frequency. The laminated line has a higher initial attenuation, but remains constant to higher frequencies. At high enough frequencies the attenuation of the laminated line rises at a rate directly proportional to frequency for a while, and then eventually approaches the attenuation of the unlaminated line.

The frequency at which the attenuation of the laminated line begins to increase is greater than the corresponding frequency for the conventional line by a factor

$$\sqrt{3}\left(\frac{\sigma}{\bar{\sigma}}\right)\left(\frac{D}{W}\right)$$

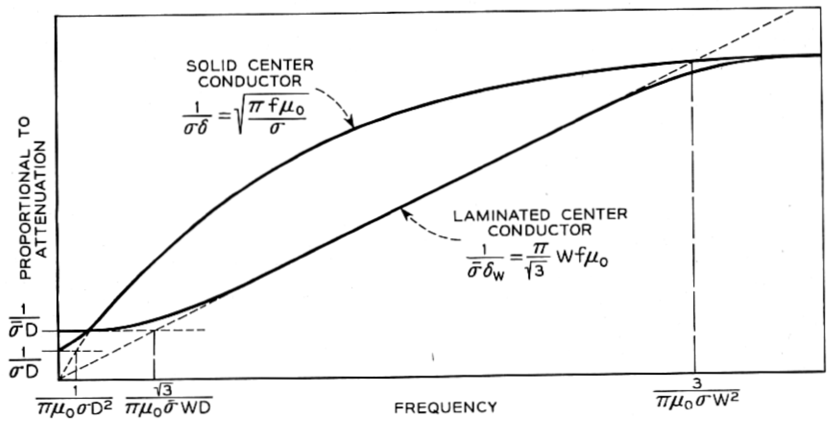

Fig. 2—Comparison of conventional and laminated transmission lines.

This is accomplished with an increase in initial attenuation by a factor

$$\sigma/\bar{\sigma} = \left(1 + \frac{t}{W}\right)$$

which we will see later will be about 3/2 in a typical case. We might make a corresponding increase in the flat range of the conventional line by decreasing σ to a new value σ_1. In that case the attenuation would be increased by a factor

$$\sqrt{3}\left(\frac{\sigma}{\bar{\sigma}}\right)\left(\frac{D}{W}\right)$$

which may be very large since $\left(\frac{\sigma}{\bar{\sigma}}\right)\left(\frac{D}{W}\right)$ is just the number of laminations of conductor or dielectric used on the center conductor.

The flat range of the conventional line might be alternatively increased to equal that of the laminated line by decreasing D to a new value D_1. In this case the attenuation would be increased by a factor

$$\sqrt{\sqrt{3}\left(\frac{\sigma}{\bar{\sigma}}\right)\left(\frac{D}{W}\right)}$$

just the square root of the factor achieved by changing σ, but still a large number.

In the frequency range in which the attenuation of the laminated line is governed by the skin depth δ_w, and is therefore increasing linearly with frequency, this attenuation is less than the attenuation of the conventional line by a factor

$$\frac{\sigma\delta}{\bar{\sigma}\delta_w} = \frac{1}{\sqrt{3}}\left(\frac{W}{\delta}\right) \tag{I-7}$$

It is interesting to note that the position of this region is governed by the conductivity of the conducting laminations, but that the attenuation is independent of the conductivity.

Considerable theoretical and experimental work has been carried out on laminated transmission lines by the author's colleagues. The following report therefore will be limited to bringing out some of the fundamental ideas in a simple way. We will for instance consider only planar systems so that the results will be only approximately applicable to real transmission lines. Other papers will more fully develop the formal theory, particularly for cylindrical systems, and discuss the practical and experimental aspects of the problem.

II. Skin Effect

We shall begin the discussion with this section by considering skin effect in various kinds of conducting media. We will first derive the skin depth equation (I-1) for an ordinary conductor like copper, and then discuss the behavior of a composite conductor made up of many thin, insulated conducting laminae. This second discussion will be first carried out for the case of infinitesimally thin laminae, and then in Section III the effects of the finite thickness of the conducting sheets will be considered.

Let us first set down and integrate Maxwell's equations in a form that will be useful in all our following discussions. Referring to the orthogonal coordinate system in Fig. 3 we shall be concerned with fields that have no variation along the z-axis and for which the z-component of electric field is zero. The only component of magnetic field is then H_z and the field equations become, in rationalized MKS units,

$$\frac{\partial H_z}{\partial y} = i\omega D_x + J_x, \tag{II-1}$$

$$-\frac{\partial H_z}{\partial x} = i\omega D_y + J_y, \tag{II-2}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\, i\omega B_z, \tag{II-3}$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = \rho \tag{II-4}$$

In these equations H, B, D, E, J and ρ all have their usual meanings. A positive time factor $e^{i\omega t}$ has been introduced.

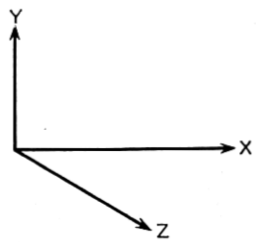

Fig. 3—Rectangular coordinate system.

Let us for the moment suppose that we are dealing with an anisotropic medium such that the following relations exist:

$$J_x = \sigma_x E_x; \qquad J_y = \sigma_y E_y \tag{II-5}$$

$$D_x = \epsilon_x E_x; \qquad D_y = \epsilon_y E_y \tag{II-6}$$

$$B_z = \mu_0 H_z \tag{II-7}$$

Here the σ's are conductivities, the ϵ's dielectric constants, and μ_0 is the permeability of free space. Suppose also now that the fields all vary with x according to a factor e^{-ikx}. If k has a positive real part we will be dealing with a wave moving along the x-axis in a positive direction, and a negative imaginary part will indicate that this wave is attenuated.

Using the above relations, one can easily find the following equations:

$$\frac{\partial^2 H_z}{\partial y^2} = \frac{i\omega\epsilon_x + \sigma_x}{i\omega\epsilon_y + \sigma_y}\,[i\omega\mu_0\,\sigma_y - \omega^2\mu_0\,\epsilon_y + k^2]H_z \tag{II-8}$$

$$E_x = \frac{1}{i\omega\epsilon_x + \sigma_x}\,\frac{\partial H_z}{\partial y} \tag{II-9}$$

$$E_y = \frac{ik}{i\omega\epsilon_y + \sigma_y}\,H_z \tag{II-10}$$

Let us imagine that we have a semi-infinite volume of material arranged as shown in Fig. 4 where the z-axis is pointing out of the paper. If H_{z0} is the value of H_z at $y = 0$, it is clear from equation (16) that H_z must depend upon y according to

$$H_z = H_{z0}e^{-\alpha y} \tag{II-11}$$

where

$$\alpha = \pm \sqrt{\frac{i\omega\epsilon_x + \sigma_x}{i\omega\epsilon_y + \sigma_y}[i\omega\mu_0\sigma_y - \omega^2\mu_0\epsilon_y + k^2]} \tag{II-12}$$

and the sign is chosen so that the real part of α is positive.

We can now consider the case when the material under consideration is an ordinary conductor such as copper or silver. In this case we must let

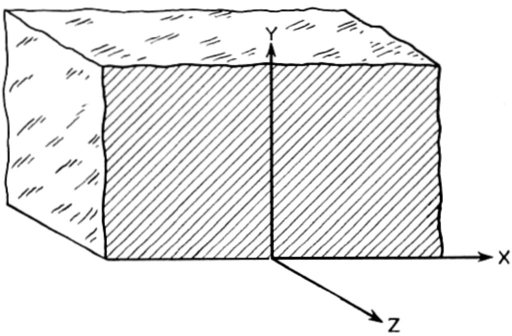

Fig. 4—Orientation of solid conductor.

$\sigma_x = \sigma_y = \sigma$ and $\epsilon_x = \epsilon_y = \epsilon$. Then α becomes (the subscript S stands for solid)

$$\alpha_S = \pm \sqrt{i\omega\mu_0\sigma - \omega^2\mu_0\epsilon + k^2} \tag{II-13}$$

Now, under any practical circumstances the propagation constant k will certainly not be more than a factor 100 larger than the propagation constant of free space $k = \sqrt{\omega^2\mu_0\epsilon_0}$. This applies also to the factor $\sqrt{\omega^2\mu_0\epsilon}$.

Consider then the ratio $\dfrac{\omega\mu_0\sigma}{\omega^2\mu_0\epsilon_0} = \dfrac{\sigma}{\omega\epsilon_0}$. For the metal copper, for instance,

$\sigma = 5.80 \times 10^7$ mhos/meter and the dielectric constant of free space $\epsilon_0 = .885 \times 10^{-11}$. If we consider frequencies as high as 10,000 megacycles the ratio is still as great as 10^8. Thus, the second two factors under the square root sign in equation (II-13) are entirely negligible and we have

$$\alpha_S = \pm \sqrt{i\omega\mu_0\sigma} \tag{II-14}$$

$$= (1 + i) \sqrt{\frac{\omega \mu_0 \sigma}{2}} \tag{II-15}$$

Finally we have

$$\text{Real part } (\alpha_S) = \frac{1}{\delta} = \sqrt{\frac{\omega \mu_0 \sigma}{2}} \tag{II-16}$$

We can now turn our attention to the stack of laminations shown in Fig. 5. There are shown a series of conducting sheets of conductivity σ and thickness W, separated by a series of insulating sheets of thickness t and dielectric constant ϵ. Suppose we let W and t approach zero while maintaining a constant ratio to obtain a homogeneous but anisotropic material

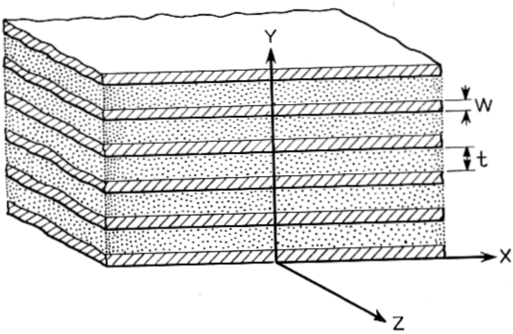

Fig. 5—Orientation of laminated conductor.

to which we can apply equation (II-12). In order to obtain ϵ_y, σ_y, ϵ_x and σ_x we can write

$$\frac{1}{\sigma_y + i\omega\epsilon_y} = \frac{1}{W + t}\left[\frac{W}{\sigma} + \frac{t}{i\omega\epsilon}\right] \tag{II-17}$$

and

$$\sigma_x + i\omega\epsilon_x = \frac{1}{W + t}\left[W\sigma + t\, i\omega\epsilon\right] \tag{II-18}$$

One then has approximately, letting $\epsilon_y = \bar{\epsilon}$ and $\sigma_x = \bar{\sigma}$

$$\bar{\epsilon} = \epsilon\left(1 + \frac{W}{t}\right) \tag{II-19}$$

$$\bar{\sigma} = \sigma\left(\frac{W}{W + t}\right) \tag{II-20}$$

$$\epsilon_x = \epsilon\left(\frac{t}{t + W}\right) \tag{II-21}$$

$$\sigma_y = \sigma \left(\frac{\omega\epsilon}{\sigma}\right)^2 \frac{W}{t}\left(\frac{W}{t} + 1\right) \tag{II-22}$$

As before, $\omega\epsilon_x$ is completely negligible compared to $\bar{\sigma}$, and furthermore σ_y is negligible compared to $\omega\bar{\epsilon}$. We have therefore for α, (the subscript 0 stands for zero thickness)

$$\alpha_0 = \pm \sqrt{\frac{\bar{\sigma}}{i\omega\bar{\epsilon}}\left[k^2 - \omega^2\mu_0\bar{\epsilon} + i\omega\mu_0\,\sigma\left(\frac{\omega\epsilon}{\sigma}\right)^2 \frac{W}{t}\left(\frac{W}{t} + 1\right)\right]} \tag{II-23}$$

This situation is rather surprising. Let us suppose conditions are such that most of the energy of the wave is flowing in the region outside the stack of laminations. If this region is filled with an insulator of dielectric constant ϵ_1, k^2 will be very nearly equal to $\omega^2\mu_0\epsilon_1$. Then α_0 will be given by

$$\alpha_0 = \pm \sqrt{\frac{1}{i}\,\bar{\sigma}\omega\mu_0\left[\left(\frac{\epsilon_1}{\bar{\epsilon}} - 1\right) + i\left(\frac{\omega\epsilon}{\sigma}\right)\left(\frac{W}{t}\right)\right]} \tag{II-24}$$

If ϵ_1 is made equal to $\bar{\epsilon}$, α_0 becomes

$$\alpha_0 = \frac{W}{W + t}\,k \tag{II-25}$$

$$= \frac{1}{1 + t/W}\,\frac{2\pi}{\lambda} \tag{II-26}$$

where λ is the longitudinal wavelength. Thus, under these conditions the wave will penetrate into the laminations to a depth $\frac{\lambda}{2\pi}\left(1 + \frac{t}{W}\right)$ before it has decreased by a factor $1/e$. This distance is of course enormous compared to ordinary skin depth.

We will see in the next section that the finite thickness of the laminae limits the penetration of the waves for $\epsilon_1 = \bar{\epsilon}$ to a distance much smaller than that implied in equation (II-26) but still large compared to conventional skin depths. In any case, we see in this simple way the suggestion of a method for obtaining great penetrations and consequently considerably reducing the attenuation of a transmission line.

The analysis of this section, carried out by assuming the medium to be anisotropic but homogeneous, can be given more physical meaning by examining a little more closely how the fields vary through the laminations shown in Fig. 5. From equations (II-8) and (II-9) one finds for a general case

$$\frac{\partial E_x}{\partial y} = \frac{1}{i\omega\epsilon_y + \sigma_y}\left[i\omega\mu_0\,\sigma_y - \omega^2\mu_0\epsilon_y + k^2\right]H_z \tag{II-27}$$

Let the value of H_z at the interface of a conducting lamina and a dielectric lamina be $(H_z)_0$. From equation (II-27), one finds just within the conducting lamina

$$\frac{\partial E_x}{\partial y} = i\omega\mu_0(H_z)_0, \tag{II-28}$$

while just with the adjacent dielectric lamina

$$\frac{\partial E_x}{\partial y} = i\omega\mu_0\left[1 - \frac{\epsilon_1}{\epsilon}\right](H_z)_0. \tag{II-29}$$

Thus, if the laminae are very thin, the change in E_x across the conducting lamina is

$$(\Delta E_x)_m = i\omega\mu_0(H_z)_0 W \tag{II-30}$$

and the change in E_x across the dielectric lamina is

$$(\Delta E_x)_d = i\omega\mu_0(H_z)_0\left(1 - \frac{\epsilon_1}{\epsilon}\right)t \tag{II-31}$$

Therefore, when $\epsilon_1 = \epsilon\left(1 + \dfrac{W}{t}\right)$, the change in E_x across the conducting lamina is just balanced by the change in E_x across the dielectric lamina. This is the basic reason for the deep penetration of the fields into the laminated structure. When $\epsilon_1 = \epsilon$, there is no change in E_x across the dielectric lamina. In this case we note from equation (II-24) that $\alpha_0 = \dfrac{W}{W + t}\alpha_S$, and we see that the waves will penetrate into the laminae an increased distance that is just accounted for by the spacing of the laminae. Thus, for $\epsilon_1 = \epsilon$, the attenuation of a laminated line will be unchanged if the laminae are replaced by solid metal.

III. Laminations of Finite Thickness

Let us refer again to Fig. 5 where a stack of conducting laminae of thickness W and conductivity σ are shown separated by insulating laminae of thickness t and dielectric constant ϵ. First we shall inquire as to how the fields change across the conducting laminations. According to equations (II-8) and (II-9) one has

$$\frac{\partial^2 H_z}{\partial y^2} = i\omega\mu_0\sigma H_z \tag{III-1}$$

$$E_x = \frac{1}{\sigma}\frac{\partial H_z}{\partial y} \tag{III-2}$$

where $\omega\epsilon$ has been neglected against σ as before. Now, letting $\eta = \sqrt{i\omega\mu_0\sigma}$, we can write within any conducting lamina

$$H_z = Ae^{\eta y} + Be^{-\eta y} \tag{III-3}$$

$$E_x = \frac{\eta}{\sigma}[Ae^{\eta y} - Be^{-\eta y}] \tag{III-4}$$

If H_0 and E_0 are the values of H_z and E_x at the lower surface of a particular lamination, and if H_1 and E_1 are their values at the upper surface of the lamination, one can find from equations (III-3) and (III-4)

$$H_1 = H_0 \cosh \eta W + E_0 \frac{\sigma}{\eta} \sinh \eta W \tag{III-5}$$

$$E_1 = H_0 \frac{\eta}{\sigma} \sinh \eta W + E_0 \cosh \eta W \tag{III-6}$$

If we wish, this can be expressed as a matrix equation

$$\begin{Bmatrix} H_1 \\ E_1 \end{Bmatrix} = \begin{Bmatrix} \cosh \eta W & \frac{\sigma}{\eta} \sinh \eta W \\ \frac{\eta}{\sigma} \sinh \eta W & \cosh \eta W \end{Bmatrix} \begin{Bmatrix} H_0 \\ E_0 \end{Bmatrix} \tag{III-7}$$

For the dielectric laminae, equations (II-8) and (II-9) become

$$\frac{\partial^2 H_z}{\partial y^2} = (\mathrm{k}^2 - \omega^2\mu_0\epsilon)H_z \tag{III-8}$$

$$E_x = \frac{1}{i\omega\epsilon}\frac{\partial H_z}{\partial y} \tag{III-9}$$

Just as for the conducting laminae, let H_1 and E_1 be the values of H_z and E_x at the lower surface of a dielectric lamination and let H_2 and E_2 be these values at the top surface. Then, if $\xi = \sqrt{k^2 - \omega^2\mu_0\epsilon}$, one has

$$\begin{Bmatrix} H_2 \\ E_2 \end{Bmatrix} = \begin{Bmatrix} \cosh \xi l & \frac{i\omega\epsilon}{\xi} \sinh \xi l \\ \frac{\xi}{i\omega\epsilon} \sinh \xi l & \cosh \xi l \end{Bmatrix} \begin{Bmatrix} H_1 \\ E_1 \end{Bmatrix} \tag{III-10}$$

From equations (III-7) and (III-10), we can find the variation of H_z and E_x from the bottom surface of a conducting lamination designated as point zero, to the top surface of the adjacent dielectric lamination designated as point two. Thus,

$$\begin{Bmatrix} H_2 \\ E_2 \end{Bmatrix} = \begin{Bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{Bmatrix} \begin{Bmatrix} H_0 \\ E_0 \end{Bmatrix} \tag{III-11}$$

where the T's are given below

$$T_{11} = \cosh \eta W \cosh \xi t + \frac{i\omega\epsilon}{\sigma} \frac{\eta}{\xi} \sinh \eta W \sinh \xi t \tag{III-12}$$

$$T_{12} = \frac{\sigma}{\eta} \sinh \eta W \cosh \xi t + \frac{i\omega\epsilon}{\xi} \cosh \eta W \sinh \xi t \tag{III-13}$$

$$T_{21} = \frac{\xi}{i\omega\epsilon} \cosh \eta W \sinh \xi t + \frac{\eta}{\sigma} \sinh \eta W \cosh \xi t \tag{III-14}$$

$$T_{22} = \frac{\sigma}{i\omega\epsilon} \frac{\xi}{\eta} \sinh \eta W \sinh \xi t + \cosh \eta W \cosh \xi t \tag{III-15}$$

It is easy to verify from the above that

$$T_{11}T_{22} - T_{12}T_{21} = 1 \tag{III-16}$$

If we now designate the lower surface of each conducting lamination successively as points 0, 1, 2, 3, · · ·, we can write down the following simultaneous difference equations

$$H_{n+1} = T_{11}H_n + T_{12}E_n \tag{III-17}$$

$$E_{n+1} = T_{21}H_n + T_{22}E_n \tag{III-18}$$

The solutions of these difference equations are

$$H_n = A\beta^n + B\beta^{-n} \tag{III-19}$$

$$E_n = A \frac{\beta - T_{11}}{T_{12}} \beta^n + B \frac{1/\beta - T_{11}}{T_{12}} \beta^{-n} \tag{III-20}$$

where

$$\beta = \left(\frac{T_{11} + T_{22}}{2}\right) + \sqrt{\left(\frac{T_{11} + T_{22}}{2}\right)^2 - 1} \tag{III-21}$$

Let us now proceed to determine the skin depth to be associated with the stack of laminae in Fig. 5. Since we have assumed the stack to be very deep, A must be taken zero in equations (III-19) and (III-20), and the fields vary into the stack according to a factor β^{-n}, so that

$$H_n = H_0\beta^{-n} \tag{III-22}$$

If we now define

$$y_n = (W + t)n \tag{III-23}$$

one has

$$\begin{aligned} H_n &= H_0\beta^{-(y_n/(W+t))} \\ &= H_0 e^{-(\ln\beta/(W+t))y_n} \\ &= H_0 e^{-\alpha_w y_n} \end{aligned} \tag{III-24}$$

where (the subscript w indicates a thickness W for the conducting laminae)

$$\alpha_w = \frac{\cosh^{-1}\left(\dfrac{T_{11} + T_{22}}{2}\right)}{(W + t)} \tag{III-25}$$

From equations (III-12) and (III-15) one has

$$\left(\frac{T_{11} + T_{22}}{2}\right) = \cosh \eta W \cosh \xi t + \frac{1}{2}\left(\frac{i\omega\epsilon}{\sigma}\frac{\eta}{\xi} + \frac{\sigma}{i\omega\epsilon}\frac{\xi}{\eta}\right) \sinh \eta W \sinh \xi t \tag{III-26}$$

As a practical matter, only rarely will k^2 be greater than ten times $\omega^2\mu_0\epsilon$ and ϵ greater than 10 ϵ_0. Hence, ξ will be no larger than 10 $\sqrt{\omega^2\mu_0\epsilon_0}$. Furthermore, we will see that t should not be very different in magnitude from W, which must be smaller than $\sqrt{\dfrac{2}{\omega\mu_0\sigma}}$. Thus, we can be sure that ξt is smaller than 100 $\sqrt{2\dfrac{\omega\epsilon_0}{\sigma}}$, a quantity which is, as before, much smaller than unity. Under these conditions, equation (III-26) becomes approximately

$$\left(\frac{T_{11} + T_{22}}{2}\right) = \cosh \eta W + \frac{\xi t}{2}\left(\frac{i\omega\epsilon}{\sigma}\frac{\eta}{\xi} + \frac{\sigma}{i\omega\epsilon}\frac{\xi}{\eta}\right) \sinh \eta W \tag{III-27}$$

$$= \cosh \eta W + \frac{1}{2}\left(\frac{t}{W}\right)\left[\frac{i\omega\epsilon}{\sigma} + \left(1 - \frac{\epsilon_1}{\epsilon}\right)\right] \eta W \sinh \eta W \tag{III-28}$$

where we have again let $k^2 = \omega^2\mu_0\epsilon_1$ as in equation (II-24).

Let us set

$$P = \frac{1}{2}\left(\frac{t}{W}\right)\left(1 - \frac{\epsilon_1}{\epsilon}\right) \tag{III-29}$$

$$= \frac{1}{2}\left[-1 + \left(1 + \frac{t}{W}\right)\left(1 - \frac{\epsilon_1}{\bar{\epsilon}}\right)\right] \tag{III-30}$$

Then, again neglecting $\dfrac{\omega\epsilon}{\sigma}$, we have for α_w

$$\alpha_w = \frac{1}{W + t}\cosh^{-1}\left[\cosh \eta W + P(\eta W) \sinh \eta W\right] \tag{III-31}$$

By definition, $\eta W = (1 + i)\frac{W}{\delta}$. We can therefore write approximately,

$$\cosh \eta W \simeq \left[1 - \frac{1}{6}\left(\frac{W}{\delta}\right)^4\right] + i\left(\frac{W}{\delta}\right)^2 \tag{III-32}$$

$$(\eta W) \sinh \eta W \simeq -\frac{2}{3}\left(\frac{W}{\delta}\right)^4 + i2\left(\frac{W}{\delta}\right)^2 \tag{III-33}$$

Using expressions (III-32) and (III-33), equation (III-31) becomes

$$\alpha_w = \frac{1}{W + t}\cosh^{-1} \cdot \left(\left[1 - \frac{(1 + 4P)}{6}\left(\frac{W}{\delta}\right)^4\right] + i(1 + 2P)\left(\frac{W}{\delta}\right)^2\right) \tag{III-34}$$

Provided that $(1 + 2P)\left(\frac{W}{\delta}\right)^2 \ll 1$, equation (III-34) can be expanded in orders of magnitude. Thus, for

$$\left|1 - \frac{\epsilon_1}{\bar{\epsilon}}\right| \ll \frac{W}{W + t}\left(\frac{\delta}{W}\right)^2 \tag{III-35}$$

we find approximately

$$\alpha_w = \frac{1}{(W + t)}\left(\frac{W}{\delta}\right) \cdot \left(\sqrt{-f + \sqrt{f^2 + g^2}} \pm i\sqrt{f + \sqrt{f^2 + g^2}}\right) \tag{III-36}$$

where

$$f = \frac{1 + 4P}{6}\left(\frac{W}{\delta}\right)^2 \tag{III-37}$$

and

$$g = 1 + 2P \tag{III-38}$$

The plus sign is to be used when g is positive and the minus sign when g is negative.

Equations (III-36) for α_w and (II-24) for α_0 are very similar. With a little manipulation we can rewrite equation (II-24) as

$$\alpha_0 = \frac{1}{\delta}\sqrt{\frac{\bar{\sigma}}{\sigma}}\left(\sqrt{\left(\frac{\omega\epsilon}{\sigma}\right)\left(\frac{W}{t}\right) + \sqrt{\left(\frac{\omega\epsilon}{\sigma}\right)^2\left(\frac{W}{t}\right)^2 + \left(1 - \frac{\epsilon_1}{\bar{\epsilon}}\right)^2}} \pm i\sqrt{-\left(\frac{\omega\epsilon}{\sigma}\right)\left(\frac{W}{t}\right) + \sqrt{\left(\frac{\omega\epsilon}{\sigma}\right)^2\left(\frac{W}{t}\right)^2 + \left(1 - \frac{\epsilon_1}{\bar{\epsilon}}\right)^2}}\right) \tag{III-39}$$

Also equation (III-36) can be written

$$\alpha_w = \frac{1}{\delta}\sqrt{\frac{\bar{\sigma}}{\sigma}}\left(\sqrt{-\left(\frac{W}{W+t}f\right)+\sqrt{\left(\frac{W}{W+t}f\right)^2+\left(1-\frac{\epsilon_1}{\bar{\epsilon}}\right)^2}}\right. \pm i\left.\sqrt{\left(\frac{W}{W+t}f\right)+\sqrt{\left(\frac{W}{W+t}f\right)^2+\left(1-\frac{\epsilon_1}{\bar{\epsilon}}\right)^2}}\right) \quad \text{(III-40)}$$

Equation (III-39) is a very good approximation for α for a stack of infinitesimally thin laminae, but is inadequate for the discussion of situations where the finite value of W/δ is important. Equation (III-40) on the other hand is a good approximation to α when W/δ is appreciable, but the assumptions made in deriving (III-40) do not allow us to go correctly to the limit $W/\delta = 0$.

To estimate how small W must be before equation (III-40) fails, let us set $\left(-\frac{W}{W+t}f\right)$ equal to $\left(\frac{\omega\epsilon}{\sigma}\right)\left(\frac{W}{t}\right)$. We will also set $\epsilon_1 = \bar{\epsilon}$, since it is only then that these terms are important. In this case,

$$W = \frac{1}{\sigma}\sqrt{12\,\frac{\bar{\epsilon}}{\mu_0}} \quad \text{(III-41)}$$

If we take $\bar{\epsilon}$ to be 5 times the dielectric constant ϵ_0 of free space, and take $\sigma = 5.8 \times 10^7$ mhos/meter for copper, we obtain $W = 3.5 \times 10^{-8}$ cm. Thus, for any practical purpose we can ignore the failure of equation (III-40) to have the proper limit for $W/\delta = 0$.

By definition, the fields decrease into the stack according to $e^{-\alpha_w y}$. Let us define the distance at which the fields have decreased by $1/e$ to be the effective skin depth δ_w. Then we have

$$\frac{1}{\delta_w} = \text{Real part } (\alpha_w)$$

$$= \frac{1}{\delta}\sqrt{\frac{\bar{\sigma}}{\sigma}}\sqrt{-\left(\frac{W}{W+t}f\right)+\sqrt{\left(\frac{W}{W+t}f\right)^2+\left(1-\frac{\epsilon_1}{\bar{\epsilon}}\right)^2}} \quad \text{(III-42)}$$

From equations (III-30) and (III-37) we find

$$-\left(\frac{W}{W+t}f\right) = \frac{1}{6}\left[\frac{W}{W+t} + 2\left(\frac{\epsilon_1}{\bar{\epsilon}} - 1\right)\right]\left(\frac{W}{\delta}\right)^2 \quad \text{(III-43)}$$

For $\epsilon_1 = \bar{\epsilon}$ equations (III-43) and (III-42) give us finally

$$\delta_w(\epsilon_1 = \bar{\epsilon}) = \sqrt{3}\left(1 + \frac{t}{W}\right)\left(\frac{\delta}{W}\right)\delta \quad \text{(III-44)}$$

With the results obtained in Sections II and III, we can compare the curves of attenuation as a function of frequency for conventional and laminated lines. Let us consider a transmission line such as that shown in Fig. 1, where we may imagine the center conductor to be either laminated as shown or made of solid metal. Let us suppose, as in the introduction, that most of the power loss is in the center conductor and that the distance between the stack and outer conductor is large compared to the depth of the stack. Clearly, the attenuation of the line will be proportional to the power per unit area flowing into the center conductor for a given power flow in the line. If H_z is the transverse magnetic field and E_x is the longitudinal electric field, the power flow into the center conductor per unit area will be given by

$$\tfrac{1}{2} \mid H_z \mid \cdot \mid E_x \mid \cos \Phi = \tfrac{1}{2} \text{ Real part } (H_z \cdot E_x^*) \qquad \text{(III-45)}$$

where Φ is the phase angle between H_z and E_x and (*) indicates the conjugate of a complex quantity. If C is the circumference of the center conductor, and Z is the characteristic impedance of the line, the attenuation of the line will be given by

$$\gamma = \frac{1}{2CZ \mid H_z \mid^2} \text{ Real part } (H_z \cdot E_x^*) \qquad \text{(III-46)}$$

First, let us suppose that the inner conductor is solid and that the frequency is very low. In that case, the uniform current density in the metal will be H_z/D, and therefore E_x will be $H_z/\sigma D$. Hence, for this case the attenuation will be

$$\gamma_s(f \text{ small}) = \frac{1}{2CZ\sigma D} \qquad \text{(III-47)}$$

In a similar manner, the attenuation of the line when the center conductor is laminated will be for very low frequencies

$$\gamma_w(f \text{ small}) = \frac{1}{2CZ\bar{\sigma}D} \qquad \text{(III-48)}$$

Next, let us consider the solid conductor again but for frequencies where $\delta \ll D$. Then we have from equations (II-9) and (II-15)

$$E_x = -\frac{1+i}{\sigma\delta} H_z \qquad \text{(III-49)}$$

Hence,

$$\gamma_s = \frac{1}{2CZ\sigma\delta} \qquad \text{(III-50)}$$

Finally, we desire the attenuation γ_w of the laminated line at elevated frequencies. Therefore we need the relation between E_x and H_z at the first surface of the stack which, for definiteness, we can take to be a metal lamina. For a sufficiently deep stack, this ratio is the same at each successive corresponding point. Referring to equations (III-7) and (III-10) we wish to find

$$R = \frac{E_0}{H_0} = \frac{E_2}{H_2} \tag{III-51}$$

By eliminating among these equations and using the same approximations made previously, we obtain

$$R = \frac{1}{\sigma W} \frac{\eta W}{\sinh \eta W} \frac{1 - \beta \cosh \eta W}{\beta} \tag{III-52}$$

If use is now made of the relation $\beta = e^{\alpha_w(W+t)}$, one finds to first order

$$R = -\frac{\alpha_w}{\bar{\sigma}} \tag{III-53}$$

Therefore, for the attenuation one has,

$$\begin{aligned} \gamma_w &= \frac{1}{2CZ\bar{\sigma}} \text{ Real part } (\alpha_w^*) \\ &= \frac{1}{2CZ\bar{\sigma}\delta_w} \end{aligned} \tag{III-54}$$

which is equivalent to an expression used in the introduction with 1/2CZ being the value of the constant A.

We have compared the attenuation of conventional and laminated lines as a function of frequency for an insulator between inner and outer conductors having the critical dielectric constant $\epsilon_1 = \bar{\epsilon}$. It is also of interest to draw the comparison as a function of dielectric constant ϵ_1 at a fixed frequency. The ratio of the two attenuations will be

$$\frac{\gamma_w}{\gamma_s} = \frac{\sigma\delta}{\bar{\sigma}\delta_w} \tag{III-55}$$

This curve is drawn in Fig. 6 for $W/\delta = 1/3$ and $t/W = 1$.

For $\epsilon_1 = \bar{\epsilon}$ we have,

$$\frac{\gamma_w}{\gamma_s} = \frac{1}{\sqrt{3}}\left(\frac{W}{\delta}\right) \tag{III-56}$$

It will also be observed that for $\epsilon_1 = \epsilon$

$$\frac{\gamma_w}{\gamma_s} = 1 \tag{III-57}$$

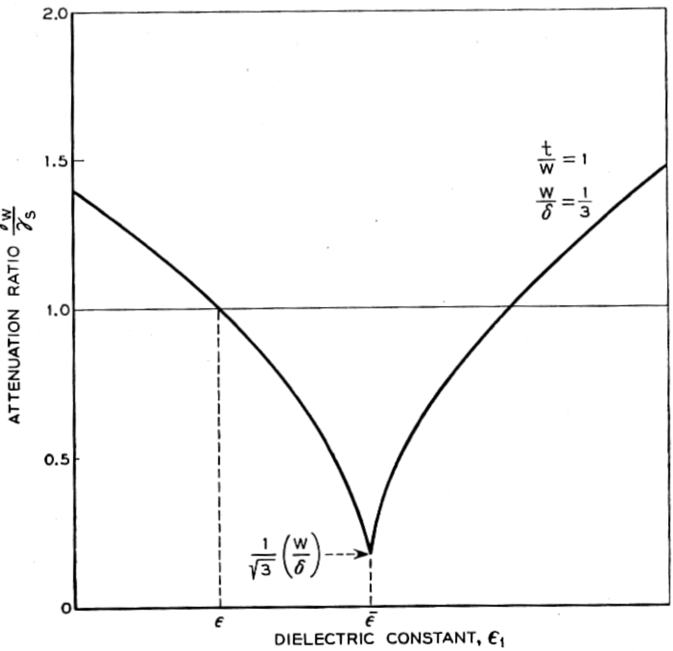

Fig. 6—Relative attenuation as a function of dielectric constant of material between stack and outer conductor.

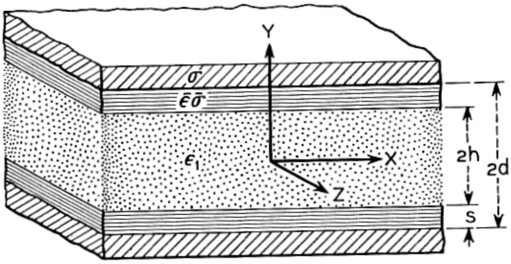

Fig. 7—Plane parallel transmission line with laminated conductors.

IV. Transmission Line with Laminated Conductors

In the preceding sections, we have considered the case of a transmission line with a depth of laminations small compared to the spacing of the conductors yet large compared to the effective skin depth δ_w. In the following sections, we will deal with several situations in which the stacks are much smaller in depth than δ_w. The size of the stack, then, reflected in the imaginary part of the propagation constant k has more effect on α than anything else and we may consider W/δ and, all the more, σ_y to be zero.

Under these conditions, we shall calculate the attenuation of the parallel plane transmission line shown in Fig. 7. In that figure we have two parallel plates or shields of conductivity σ separated a distance $2d$. Inside each plate there is a thickness s of laminated conductor of average conductivity $\bar{\sigma}$ and average transverse dielectric constant $\bar{\epsilon}$. The interior of the line is filled with a dielectric of thickness $2h = 2(d - s)$ and having a dielectric constant ϵ_1. The calculations to be made will be valid down to some low frequency at which the skin depth in the outer shields becomes equal to their thickness.

With reference to equations (II-8) and (II-9), we can write down the following expressions for the fields in the various parts of the line:

In shield:
$$H_z = Ae^{-\eta(y-d)} \tag{IV-1}$$

$$E_x = -A\frac{\eta}{\sigma}e^{-\eta(y-d)} \tag{IV-2}$$

$$\eta = \sqrt{i\omega\mu_0\sigma} \tag{IV-3}$$

In laminae:
$$H_z = B\cosh\zeta(y-d) + C\sinh\zeta(y-d) \tag{IV-4}$$

$$E_x = \frac{\zeta}{\bar{\sigma}}[B\sinh\zeta(y-d) + C\cosh\zeta(y-d)] \tag{IV-5}$$

$$\zeta = \sqrt{\frac{\bar{\sigma}}{i\omega\bar{\epsilon}}(k^2 - \omega^2\mu_0\bar{\epsilon})} \tag{IV-6}$$

In dielectric:
$$H_z = \cosh\xi y \tag{IV-7}$$

$$E_x = \frac{\xi}{i\omega\epsilon_1}\sinh\xi y \tag{IV-8}$$

$$\xi = \sqrt{k^2 - \omega^2\mu_0\epsilon_1} \tag{IV-9}$$

where A, B and C are constants.

The fields H_z and E_x must match at the boundaries $y = h$ and $y = d$. Imposing these conditions, we find the characteristic equation for determining k to be

$$\tanh\zeta s = -\frac{1 + \left(\frac{\sigma}{i\omega\epsilon_1}\right)\frac{\xi}{\eta}\tanh\xi h}{\left(\frac{\sigma}{i\omega\epsilon_1}\right)\left(\frac{\bar{\sigma}}{\sigma}\right)\frac{\xi}{\zeta}\tanh\xi h + \left(\frac{\sigma}{\bar{\sigma}}\right)\frac{\zeta}{\eta}} \tag{IV-10}$$

The constants are also determined to be

$$B = \frac{\cosh \xi h}{\cosh \zeta s + \dfrac{\bar{\sigma}}{\sigma} \dfrac{\eta}{\zeta} \sinh \zeta s} \tag{IV-11}$$

$$A = B$$

$$C = -\frac{\bar{\sigma}}{\sigma} \frac{\eta}{\zeta} B \tag{IV-12}$$

Just as before, it is obvious from equation (IV-6) that k must be nearly equal to $\omega^2 \mu_0 \bar{\epsilon}$ if the fields are to penetrate deeply into the laminations. Let us guess on physical grounds that this can be accomplished by setting $\epsilon_1 = \bar{\epsilon}$. Under these conditions from equation (IV-9) and (IV-6)

$$\xi = \sqrt{\frac{i\omega\bar{\epsilon}}{\bar{\sigma}}}\, \zeta \tag{IV-13}$$

and

$$\xi h = \sqrt{\frac{i\omega\bar{\epsilon}}{\bar{\sigma}}} \left(\frac{h}{s}\right) (\zeta s)$$

Thus, if (ζs) is not to be very large and $\left(\frac{h}{s}\right)$ is not greater than 100, ξh will be a very small number. We can therefore set $\tanh \xi h = \xi h$ and rewrite equation (IV-10) as

$$(\zeta s) \tanh (\zeta s) + \frac{\left(\dfrac{h}{s}\right) (\zeta s)^2 + \left(\dfrac{\bar{\sigma}}{\sigma}\right) \eta s}{1 + \left(\dfrac{\bar{\sigma}}{\sigma}\right) \eta h} = 0 \tag{IV-14}$$

From equation (IV-3), $\eta h = (1 + i)\frac{h}{\delta}$. We shall imagine that h is many times greater than the skin depth in the outer shield and may therefore reduce equation (IV-14) to

$$\theta \tanh \theta + \frac{1 - i}{2} \left(\frac{\delta}{s}\right) \left(\frac{\sigma}{\bar{\sigma}}\right) \theta^2 + \frac{s}{h} = 0 \tag{IV-15}$$

where (ζs) has been set equal to θ. Now, if s is also many times the skin depth in the outer conductor, the second term in equation (IV-15) may be neglected compared to the first term. We have then, finally, for the characteristic equation

$$\theta \tanh \theta + \frac{s}{h} = 0 \tag{IV-16}$$

For s much smaller than h, approximate solutions of equation (IV-16) may be written

$$\theta_0^2 = -\frac{s}{h} \tag{IV-17}$$

and

$$\theta_n = in\pi\left[1 + \frac{s}{h}\frac{1}{(n\pi)^2}\right] \qquad n = 1, 2, 3, \cdots \tag{IV-18}$$

The fundamental solution (IV-17) obviously agrees with our assumption that (ζs) is not large. Similarly, the higher order solutions (IV-18) are consistent with that assumption if n is not taken too large.

We may now return to equation (IV-6) and obtain approximate expressions for k.

$$k_0 = \sqrt{\omega^2\mu_0\bar{\epsilon}}\left[1 + \frac{1}{i\omega\mu_0\bar{\sigma}}\frac{1}{2sh}\right] \tag{IV-19}$$

$$k_n = \sqrt{\omega^2\mu_0\bar{\epsilon}}\left(1 + \frac{1}{i\omega\mu_0\bar{\sigma}}\left[\frac{1}{sh} + \frac{1}{2}\left(\frac{n\pi}{s}\right)^2\right]\right) \qquad n = 1, 2, 3 \cdots \tag{IV-20}$$

We see that k^2 is indeed approximately equal to $\omega^2\mu_0\bar{\epsilon}$. The imaginary parts of (IV-19) and (IV-20) are negative and give us the desired attenuation for the fundamental and higher modes of the line in nepers per meter.

$$I(k_0) = -\frac{1}{2sh}\frac{1}{\bar{\sigma}}\sqrt{\frac{\bar{\epsilon}}{\mu_0}} \tag{IV-21}$$

$$I(k_n) = -\frac{1}{2sh}\left[2 + \frac{h}{s}(n\pi)^2\right]\frac{1}{\bar{\sigma}}\sqrt{\frac{\bar{\epsilon}}{\mu_0}} \qquad n = 1, 2, 3, \cdots \tag{IV-22}$$

where we have assumed $\delta \ll s \ll h$ and $\epsilon_1 = \bar{\epsilon}$.

Let us first comment on the fact that there exist several modes of transmission in this line. The fundamental mode with propagation constant k_0 corresponds to the ordinary mode of transmission that would exist between a pair of parallel plates such as shown in Fig. 8. The higher modes are waves that are confined almost entirely to the laminations and are not encountered in an ordinary transmission line. These modes will be more fully discussed in Section VI.

For comparison with equations (IV-21) and (IV-22), we shall next calculate the attenuation of the parallel plate transmission line shown in Fig. 8.

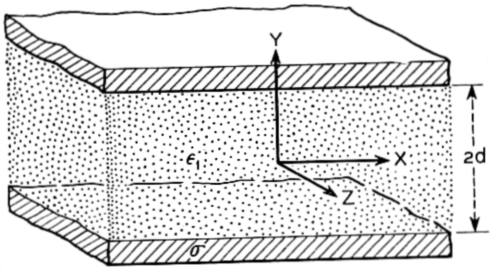

Fig. 8—Plane parallel transmission line with solid conductors.

For the fields we have again

In shield:

$$H_z = Ae^{-\eta(y-d)} \tag{IV-23}$$

$$E_x = -A\,\frac{\eta}{\sigma}\,e^{-\eta(y-d)} \tag{IV-24}$$

$$\eta = \sqrt{i\omega\mu_0\sigma} \tag{IV-25}$$

In dielectric:

$$H_z = \cosh \xi y \tag{IV-26}$$

$$E_x = \frac{\xi}{i\omega\epsilon_1}\sinh \xi y \tag{IV-27}$$

$$\xi = \sqrt{k^2 - \omega^2\mu_0\epsilon_1} \tag{IV-28}$$

By matching the fields at $y = d$ we readily obtain the characteristic equation

$$(\xi d)\tanh(\xi d) + \frac{i\omega\epsilon_1}{\sigma}\,\eta d = 0 \tag{IV-29}$$

which gives approximately

$$(\xi d)^2 = -\frac{i\omega\epsilon_1}{\sigma}\,\eta d \tag{IV-30}$$

We also determine the constant A to be unity. Proceeding as before, we find for the propagation constant

$$k = \sqrt{\omega^2\mu_0\epsilon_1}\left[1 + \frac{1}{2d\sqrt{i\omega\mu_0\sigma}}\right] \tag{IV-31}$$

and for the attenuation

$$I(k) = -\frac{1}{d}\sqrt{\frac{1}{8}\frac{\omega\epsilon_1}{\sigma}} \qquad \text{(IV-32)}$$

It is observed at once that the attenuation expressed in equation (IV-21) is independent of frequency, while that given in equation (IV-32) increases

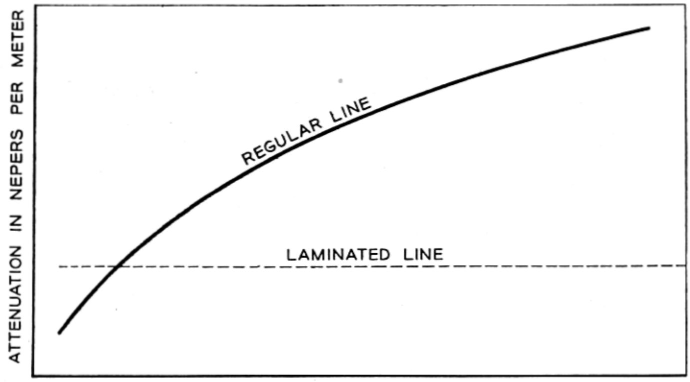

Fig. 9—Comparison of conventional and laminated lines.

as the square root of the frequency. If we take the ratio of equation (IV-32) to equation (IV-21) we obtain

$$\frac{\text{attenuation of regular line}}{\text{attenuation of laminated line}} = \left(\frac{h}{d}\right)\left(\frac{\bar{\sigma}}{\sigma}\right) s \sqrt{\frac{\omega\mu_0\sigma}{2}} \qquad \text{(IV-33)}$$

$$= \left(\frac{h}{d}\right)\left(\frac{\bar{\sigma}}{\sigma}\right)\left(\frac{s}{\delta}\right) \qquad \text{(IV-34)}$$

which is of course a thoroughly reasonable result. A sketch of the attenuation curves for the two lines is shown in Fig. 9.

As a final point, we observe that the attenuation given in equation (IV-21) is proportional to $\frac{1}{\bar{\sigma}}\sqrt{\bar{\epsilon}}$. Therefore, for stacks in which the laminations have the dimensions shown in Fig. 5, the attenuation is seen from equations (II-19) and (II-20) to be proportional to

$$\left(1 + \frac{t}{W}\right)\sqrt{1 + \frac{W}{t}}$$

This expression has a minimum for $\frac{W}{t} = 2$. Thus, the optimum arrangement is for the conductor to be twice the thickness of the dielectric. This rule holds, of course, only as long as the effective skin depth defined in equation (III-42) is greater than s. If use of the ratio $t/W = 1/2$ finds δ_w smaller than s, the best thing to do is to increase t/W until the effective skin depth δ_w becomes equal to s.

V. Transmission Line Filled with Laminations

In the last section, we have calculated the attenuation of the transmission line shown in Fig. 7. By reference to equation (IV-21), it is seen that this attenuation decreases as s increases. Since we have assumed in deducing equation (IV-21) that $s \ll h$, we cannot use that equation to find the attenuation for $s = d$. Nevertheless, the suggestion is evident that this case may be particularly interesting.

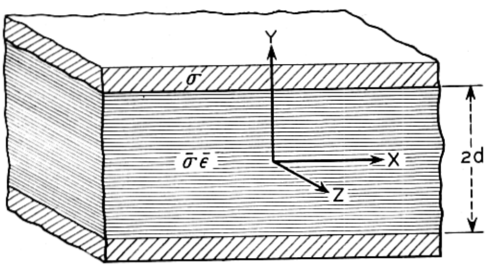

Fig. 10—Plane parallel transmission line filled with laminations.

Accordingly, let us consider the transmission line shown in Fig. 10, where the space between the outer shields has been completely filled with laminations. As before, we can write down the following fields:

In shield:

$$H_z = Ae^{-\eta(y-d)} \tag{V-1}$$

$$E_x = -A\frac{\eta}{\sigma}e^{-\eta(y-d)} \tag{V-2}$$

$$\eta = \sqrt{i\omega\mu_0\sigma} \tag{V-3}$$

In laminae:

$$H_z = \cosh \zeta y \tag{V-4}$$

$$E_x = \frac{\zeta}{\bar{\sigma}}\sinh \zeta y \tag{V-5}$$

where A is some constant. Matching fields at $y = d$, we obtain the characteristic equation

$$(\zeta d) \tanh (\zeta d) = -\frac{\bar{\sigma}}{\sigma} \eta d \tag{V-6}$$

We might also have obtained this equation by placing $h = 0$ and $s = d$ in equation (IV-10).

We can verify that an approximate solution of equation (V-6) is

$$(\zeta d) = \frac{n\pi i}{2}\left[1 - \frac{\sigma}{\bar{\sigma}} \frac{1}{\eta d}\right] \qquad n = 1, 3, 5, \cdots \tag{V-7}$$

Proceeding as before, we find for the propagation constant

$$k_n = \sqrt{\omega^2 \mu_0 \bar{\epsilon}}\left[1 + \frac{1}{2i\omega\mu_0\bar{\sigma}}\left(\frac{n\pi}{2d}\right)^2\right], \tag{V-8}$$

and for the attenuation

$$I(k_n) = -\frac{1}{2\bar{\sigma}}\sqrt{\frac{\bar{\epsilon}}{\mu_0}}\left(\frac{n\pi}{2d}\right)^2 \tag{V-9}$$

If we place $n = 1$ in equation (V-9) and compare the result with equation (IV-21), we see that the attenuation of the transmission line has been indeed decreased by completely filling it with laminations, and without sacrifice of the frequency independent characteristic. Furthermore, it is no longer necessary to supply a dielectric with dielectric constant equal to $\bar{\epsilon}$. This case clearly represents something unfamiliar in the way of transmission lines. We might in fact consider the laminated material as a new kind of transmission medium.

In order to visualize this situation more completely, let us study the distribution of fields and currents inside the transmission line. We will be interested in H_z, E_y which can be obtained from equations (II-10) and (V-8), the current $J = \bar{\sigma} E_x$, the Poynting vector $P = 1/2$ Real part $(E_y H_z^*)$, the total current $I = \left|\int_0^d J dy\right|$, and the total power $W = \int_{-d}^{d} P dy$. From equation (V-7), we can take ζ equal approximately to $\frac{n\pi i}{2d}$ and obtain

$$H_z = \cos \frac{n\pi y}{2d} \tag{V-10}$$

$$E_y = \sqrt{\mu_0/\bar{\epsilon}} \cos \frac{n\pi y}{2d} \tag{V-11}$$

$$J = -\left(\frac{n\pi}{2d}\right) \sin \frac{n\pi y}{2d} \tag{V-12}$$

$$P = 1/2\sqrt{\mu_0/\bar{\epsilon}} \cos \frac{n\pi y}{2d} \tag{V-13}$$

$$I = 1 \tag{V-14}$$

$$W = 1/2 \sqrt{\mu_0/\bar{\epsilon}}\, d \tag{V-15}$$

For comparison let us write down these same quantities for the transmission line of Fig. 8. This can be done in an obvious way from equations (IV-23) to (IV-28) and by use of the characteristic equation (IV-30). We have then approximately,

$$H_s = \frac{1}{\sqrt{2}} \tag{V-16}$$

$$E_y = \frac{1}{\sqrt{2}} \sqrt{\frac{\mu_0}{\epsilon_1}} \tag{V-17}$$

$$P = \frac{1}{2\sqrt{2}} \sqrt{\frac{\mu_0}{\epsilon_1}} \tag{V-18}$$

$$W = \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_1}}\, d \tag{V-19}$$

and in the shield

$$J = -\frac{1}{\sqrt{2}} \eta e^{-\eta(y-d)} \tag{V-20}$$

$$I = \frac{1}{\sqrt{2}} \tag{V-21}$$

If we let $\epsilon_1 = \bar{\epsilon}$, the second set of equations has been normalized to the same transmitted power as the first set. A comparison of these equations is shown in Fig. 11. The decreased attenuation of the laminated line, is, if course, accounted for by its much smaller current density, even though its total current is bigger by a factor $\sqrt{2}$. Only the fundamental mode of the laminated line is considered in Fig. 11. The higher modes will be discussed in the next section.

VI. Modes of Transmission

We have seen that both the transmission line partly filled with laminations as in Section IV, and the completely filled line described in Section V, have fundamental and higher modes of transmission. Let us now examine this situation more closely.

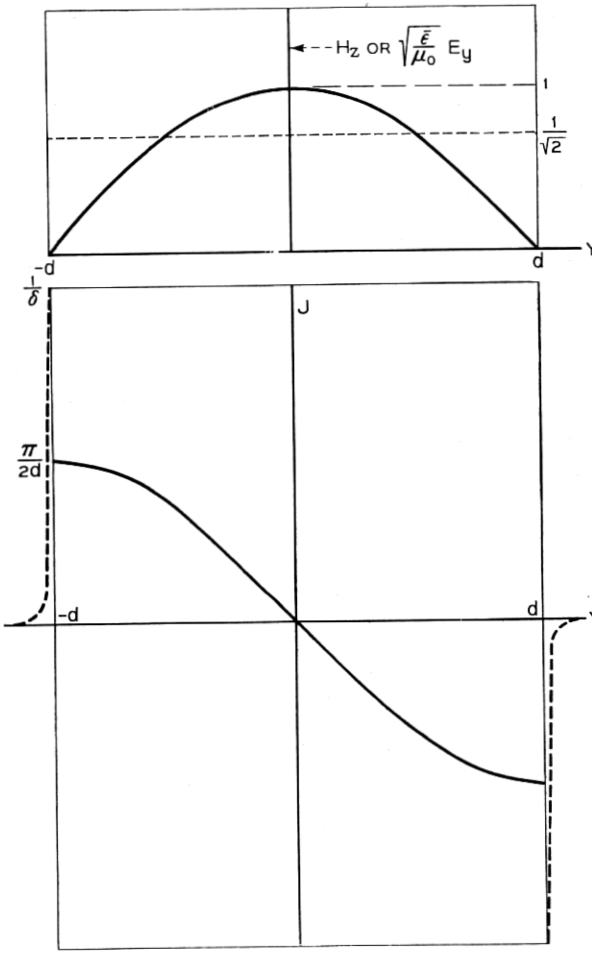

Fig. 11—Distribution of fields and current in transmission line filled with laminations.

First, in the case of the partially filled line, we can use the results of Section IV to find the following approximate expressions for the current density and magnetic field in the lamina:

$$J_0 = -\frac{1}{s} \tag{VI-1}$$

$$J_n = (-1)^{n+1}\left(\frac{h}{s}\right)(n\pi)^2\,\frac{1}{s}\cos\frac{n\pi}{s}(y-d) \tag{VI-2}$$

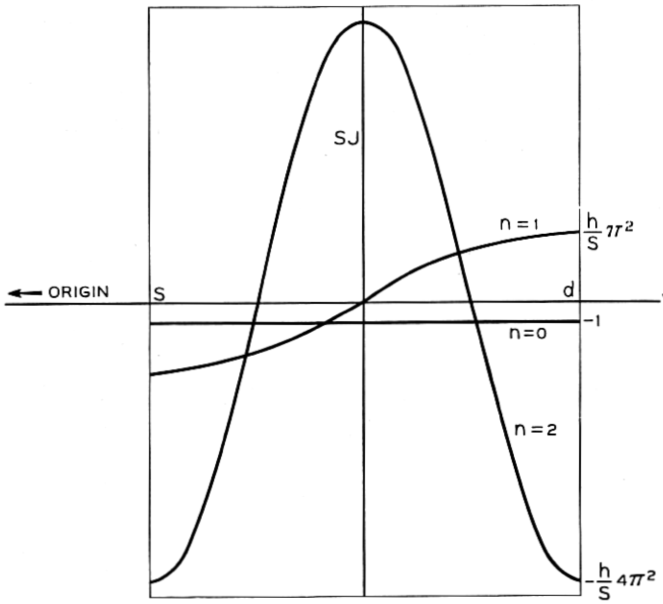

Fig. 12—Distribution of current in laminae of partially filled line for various modes.

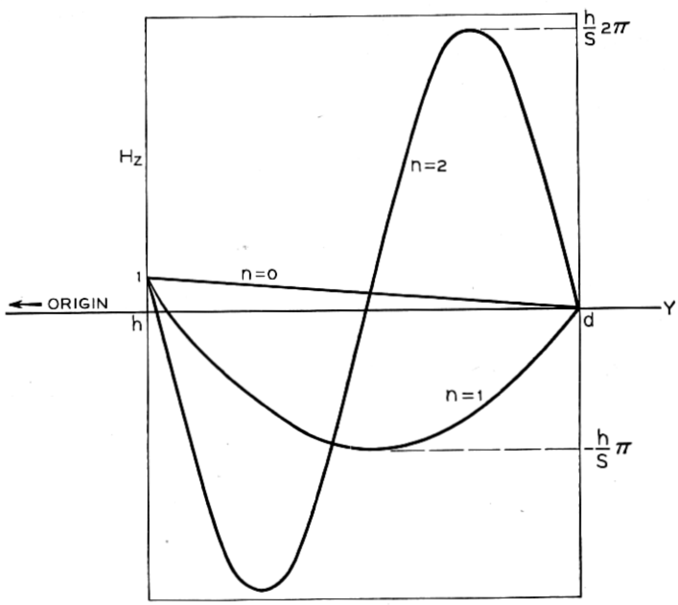

Fig. 13—Distribution of magnetic field in laminae of partially filled line for various modes.

$$(H_z)_0 = -\frac{1}{s}(y - d) \qquad \text{(VI-3)}$$

$$(H_z)_n = (-1)^{n+1}\left(\frac{h}{s}\right) n\pi \sin\left[\left(n\pi + \frac{s}{h}\frac{1}{n\pi}\right)\frac{y - d}{s}\right] \qquad \text{(VI-4)}$$

The current densities for $n = 0, 1, 2$ are shown in Fig. 12. We see that the current density for $n = 0$ is all of one phase, while for the higher modes the current density has one or more reversals of phase. For these higher

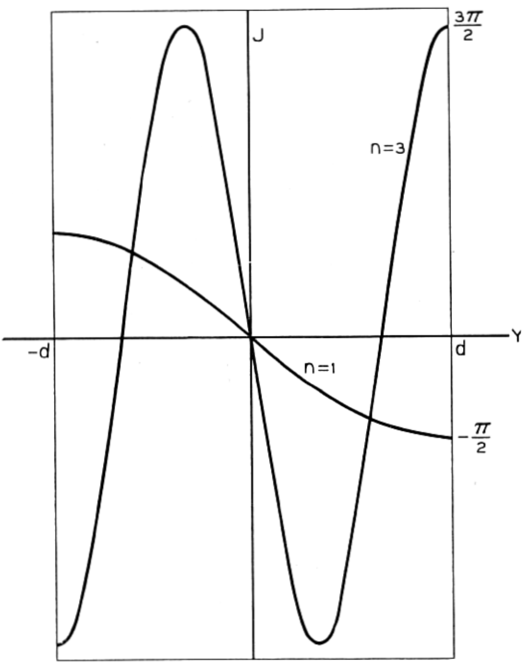

Fig. 14—Current distribution in completely filled line for various modes.

modes, the net current in the laminae is essentially zero and consequently we should expect only small fields in the interior of the line. This supposition is confirmed in Fig. 13, where the magnetic fields in the laminae are drawn for $n = 0, 1, 2$. The fields for all the modes have a common value unity at $y = s$, but for the higher modes the fields in the interior of the laminae are much greater than unity.

For the completely filled line, we have from equation (V-12)

$$J_n = -\frac{n\pi}{2d}\sin\frac{n\pi y}{2d} \qquad \text{(VI-5)}$$

These current densities are drawn in Fig. 14 for $n = 1$ and $n = 3$. It will be observed that in each case a net current flows in one sense on one side of the center line and in the opposite sense on the other side of the line.

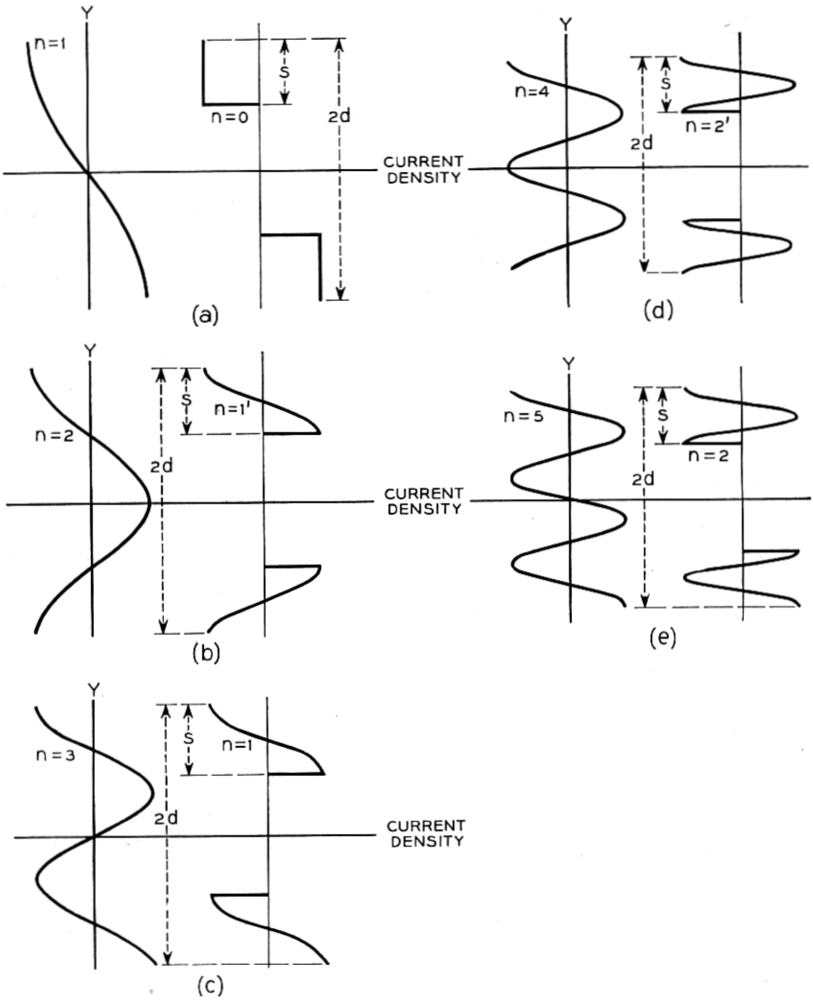

Fig. 15—Correspondence of modes between partially filled line and wholly filled line.

We can now profitably compare the modes for the two types of lines. Clearly there should be a one to one correspondence of the modes since the partially filled line can be made to approach the completely filled line continuously by adding more laminated material. This correspondence is shown schematically in Fig. 15. The modes we have discussed to this point have all been antisymmetric in current density about the center plane. It will be clear from Fig. 15 that there are in addition another set of modes symmetric in the current density. For the completely filled case these are the modes $n = 2, 4, 6 \cdots$, and for the partially filled case the modes $n = 1', 2', 3', \cdots$.

An important point can now be made. For the completely filled case, there are higher modes such as $n = 3$ where a net current flows on one side of the center plane. The corresponding mode, however, for the partially filled case with $s \ll h$ has nearly zero net current on either side of the center plane. Thus, for the partially filled case with s much smaller than h there are no modes except the fundamental with large fields in the interior of the line.

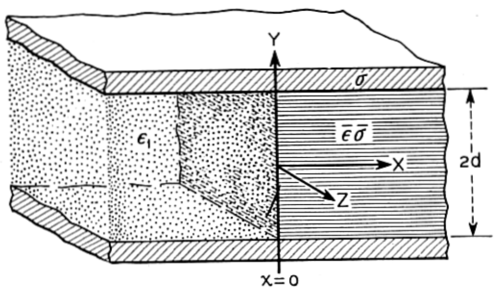

Fig. 16—Junction between two plane parallel transmission lines, one of which is filled with laminations.

VII. Termination of a Laminated Line

The discussion of modes of transmission in the last two sections enables us now to consider what occurs at the junction of an unlaminated transmission line and one partially or completely filled with laminated material. We will, for simplicity, consider mainly the case of the completely filled line as shown in Fig. 16. To the left of $x = 0$ there is an unlaminated line such as shown in Fig. 8 filled with a dielectric of dielectric constant ϵ_1. To the right of $x = 0$ there is a line of the type considered in Section V filled with laminated material of average transverse dielectric constant $\bar{\epsilon}$ and average longitudinal conductivity $\bar{\sigma}$. We shall consider separately what happens to a wave incident upon the boundary from left or right.

When expressing the fields in the unlaminated line, we shall have to include certain unpropagated modes which have not yet been discussed. These modes must attenuate to the left, and can be written

$$H_z = \cos \frac{n\pi y}{d} e^{n\pi x/d} \qquad \text{(VII-1)}$$

$$E_x = -\frac{1}{i\omega\epsilon_1}\left(\frac{n\pi}{d}\right)\sin\frac{n\pi y}{d}\,e^{n\pi x/d} \qquad n = 1, 2, 3 \cdots \tag{VII-2}$$

$$E_y = -\frac{1}{i\omega\epsilon_1}\left(\frac{n\pi}{d}\right)\cos\frac{n\pi y}{d}\,e^{n\pi x/d} \tag{VII-3}$$

It has been assumed that the wavelength in the dielectric of the frequency under consideration is much greater than d.

We shall first consider a wave incident upon the boundary from the left. From equations (V-16), (V-17), (VII-1) and (VII-3) we have for $x < 0$

$$H_z = Ae^{-ikx} + Be^{ikx} + \sum_m C_m \cos\frac{m\pi y}{d}\,e^{m\pi x/d} \tag{VII-4}$$

$$E_y = \sqrt{\frac{\mu_0}{\epsilon_1}}\,[Ae^{-ikx} - Be^{ikx}] - \frac{1}{i\omega\epsilon_1}\sum_m C_m\left(\frac{m\pi}{d}\right)\cos\frac{m\pi y}{d}\,e^{m\pi x/d} \tag{VII-5}$$

where $m = 1, 2, 3 \cdots$ and k is given by (IV-31). For $x > 0$ we have from equations (V-10) and (V-11)

$$H_z = \sum_n D_n \cos\frac{n\pi y}{2d}\,e^{-ik_n x} \tag{VII-6}$$

$$E_y = \sqrt{\frac{\mu_0}{\bar{\epsilon}}}\sum_n D_n \cos\frac{n\pi y}{2d}\,e^{-ik_n x} \tag{VII-7}$$

where $n = 1, 3, 5 \cdots$ and k_n is given by (V-8). At $x = 0$ the boundary conditions give

$$A + B + \sum_m C_m \cos\frac{m\pi y}{d} = \sum_n D_n \cos\frac{n\pi y}{2d} \tag{VII-8}$$

$$\sqrt{\frac{\mu_0}{\epsilon_1}}\,(A - B) - \frac{1}{i\omega\epsilon_1}\sum_m C_m\left(\frac{m\pi}{d}\right)\cos\frac{m\pi y}{d} = \sqrt{\frac{\mu_0}{\bar{\epsilon}}}\sum_n D_n \cos\frac{n\pi y}{2d} \tag{VII-9}$$

If $\epsilon_1 = \bar{\epsilon}$, it can be seen from (VII-8) and (VII-9) that $B = C_m = 0$. Thus there is no reflected wave and no unpropagated waves are needed. Let us consider only this case. The coefficients D_n are determined by

$$A = \sum_n D_n \cos\frac{n\pi y}{2d} \tag{VII-10}$$

which yields

$$D_n = \frac{4}{n\pi} (-1)^{(n-1)/2} A \tag{VII-11}$$

Referring to equation (V-15) we have for the power flow in the transmitted wave,

$$W = \frac{d}{2} \sqrt{\frac{\mu_0}{\bar{\epsilon}}} \sum_n D_n^2 \tag{VII-12}$$

$$= A^2 \sqrt{\frac{\mu_0}{\bar{\epsilon}}} \frac{8d}{\pi^2} \sum_n \frac{1}{n^2} \tag{VII-13}$$

Let us now find the ratio of the power transmitted in the fundamental mode $n = 1$ to the total power transmitted, which is also the total incident power as can be checked from equation (VII-12). We have

$$\frac{\text{power in fundamental}}{\text{total power}} = \frac{8}{\pi^2} \tag{VII-14}$$

Thus, in exciting the fundamental mode of the laminated line we have a power loss which can be expressed in db as

$$\begin{aligned} \text{db loss} &= 10 \log \frac{\pi^2}{8} \\ &= 0\,912 \end{aligned} \tag{VII-15}$$

Let us next consider a wave composed of the fundamental mode of the laminated line incident upon the boundary from the right. For $x < 0$ we have in this case

$$H_z = Be^{ikx} + \sum_m C_m \cos \frac{m\pi y}{d} e^{m\pi x/d} \tag{VII-16}$$

$$E_y = -B \sqrt{\frac{\mu_0}{\epsilon_1}} e^{ikx} - \frac{1}{i\omega\epsilon_1} \sum_m C_m \left(\frac{m\pi}{d}\right) \cos \frac{m\pi y}{d} e^{m\pi x/d} \tag{VII-17}$$

where again $m = 1, 2, 3, \cdots$ and k is given by (IV-31). For $x > 0$

$$H_z = Me^{ik_1x} \cos \frac{\pi y}{2d} + \sum_n N_n \cos \frac{n\pi y}{2d} e^{-ik_nx} \tag{VII-18}$$

$$E_y = \sqrt{\frac{\mu_0}{\bar{\epsilon}}} \left[-Me^{ik_1x} \cos \frac{\pi y}{2d} + \sum_n N_n \cos \frac{n\pi y}{2d} e^{-ik_nx} \right] \tag{VII-19}$$

with k_n given by (V-8) and $n = 1, 3, 5, \cdots$. At $x = 0$,

$$B + \sum_m C_m \cos \frac{m\pi y}{d} = M \cos \frac{\pi y}{2d} + \sum_n N_n \cos \frac{n\pi y}{2d} \qquad \text{(VII-20)}$$

$$\begin{aligned} -B - \frac{1}{ik} \sum_m C_m \left(\frac{m\pi}{d}\right) \cos \frac{m\pi y}{d} \\ = \sqrt{\frac{\epsilon_1}{\bar{\epsilon}}} \left[-M \cos \frac{\pi y}{2d} + \sum_n N_n \cos \frac{n\pi y}{2d} \right] \end{aligned} \qquad \text{(VII-21)}$$

Let us again let $\epsilon_1 = \bar{\epsilon}$. By adding and subtracting we find

$$-\frac{1}{ik} \sum_m C_m \frac{m\pi}{d} \cos \frac{m\pi y}{d} = 2 \sum_n N_n \cos \frac{n\pi y}{2d} \qquad \text{(VII-22)}$$

$$2B + \frac{1}{ik} \sum_m C_m \frac{m\pi}{d} \cos \frac{m\pi y}{d} = 2M \cos \frac{\pi y}{2d} \qquad \text{(VII-23)}$$

From (VII-23) it is clear that

$$2B = \frac{1}{2d} \int_{-d}^{d} 2M \cos \frac{\pi y}{2d}\, dy \qquad \text{(VII-24)}$$

or

$$B = \frac{2}{\pi} M \qquad \text{(VII-25)}$$

Thus, we have for the transmitted power

$$\begin{aligned} W_t &= d \sqrt{\frac{\mu_0}{\epsilon_1}}\, B^2 \\ &= d \sqrt{\frac{\mu_0}{\epsilon_1}} \left(\frac{2}{\pi}\right)^2 M^2 \end{aligned} \qquad \text{(VII-26)}$$

and for the incident power

$$W_i = \frac{d}{2} \sqrt{\frac{\mu_0}{\bar{\epsilon}}}\, M^2$$

The ratio of transmitted power to incident power is therefore

$$\frac{W_t}{W_i} = \frac{8}{\pi^2} \qquad \text{(VII-27)}$$

just as in equation (VII-14). There is thus the same power loss in crossing the boundary in either direction. It is interesting to note, however, that in the second case the non-propagating modes in the unlaminated line are excited.

From equations (VII-22), (VII-23) and (VII-25) we can find

$$\sum_n N_n \cos \frac{n\pi y}{2d} = M\left(\frac{2}{\pi} - \cos \frac{\pi y}{2d}\right) \tag{VII-28}$$

and consequently

$$N_1 = M\left(\frac{8}{\pi^2} - 1\right), \tag{VII-29}$$

and

$$N_n = M\frac{8}{\pi^2}\frac{1}{n}(-1)^{(n-1)/2}, \qquad n \neq 1. \tag{VII-30}$$

The reflected power is found as before to be for the fundamental mode of the laminated line

$$W_{r1} = \frac{d}{2}\sqrt{\frac{\mu_0}{\bar{\epsilon}}}\, M^2\left(\frac{8}{\pi^2} - 1\right)^2 \tag{VII-31}$$

and for the higher modes

$$\begin{aligned}\sum_{n\neq 1} W_{rn} &= \frac{d}{2}\sqrt{\frac{\mu_0}{\bar{\epsilon}}}\, M^2 \sum_{n\neq 1}\left(\frac{8}{n\pi^2}\right)^2, \\ &= \frac{d}{2}\sqrt{\frac{\mu_0}{\bar{\epsilon}}}\, M^2 \frac{8}{\pi^2}\left(1 - \frac{8}{\pi^2}\right).\end{aligned} \tag{VII-32}$$

We can now easily check that

$$\begin{aligned}\frac{W_t}{W_i} + \frac{W_{r1}}{W_i} + \frac{1}{W_i}\sum_{n\neq 1} W_{rn} &= \frac{8}{\pi^2} \\ &+ \left(\frac{8}{\pi^2} - 1\right)^2 + \frac{8}{\pi^2}\left(1 - \frac{8}{\pi^2}\right) = 1\end{aligned} \tag{VII-33}$$

The case of the partially filled line can be studied in a manner similar to the above discussion, and will show smaller power losses for waves transmitted through the boundary. The problem is further complicated by the presence of unpropagated modes in the partially filled line similar to those in the unlaminated line.

APPENDIX A

Plane Waves

It is interesting to inquire about the waves that exist in a laminated medium of infinite extent. Let us return to equation (II-23). It is easy to show that α_0 is zero for k given approximately by

$$k = \sqrt{\omega^2\mu_0\bar{\epsilon}}\left[1 - i\frac{1}{2}\frac{W}{t}\left(\frac{\omega\epsilon}{\sigma}\right)\right] \tag{A-1}$$

For $\alpha_0 = 0$, E_x is of course zero. We thus have a plane wave propagating through the medium with a wavelength appropriate to the average dielectric constant $\bar{\epsilon}$ and with a very small attenuation proportional to the square of the frequency. Equation (A-1) can be obtained from equation (45) in Sakurai's paper.[2]

If we wish next to observe the effect of finite thickness laminations, we can require α_w to be zero in equation (III-34). In this case we obtain

$$k = \sqrt{\omega^2\mu_0\bar{\epsilon}}\left[1 - i\frac{1}{12}\frac{W}{W+t}\left(\frac{W}{\delta}\right)^2\right] \tag{A-2}$$

Again the attenuation is proportional to the square of the frequency. The attenuation given by equation (A-2) is equal to that obtained from equation (A-1) if

$$W = \frac{1}{\sigma}\sqrt{12\left(1 + \frac{W}{t}\right)\frac{\epsilon}{\mu_0}} \tag{A-3}$$

For copper, equation (A-3) requires W to be of the order of 10^{-8} cm. Under ordinary circumstances, therefore, the attenuation given in equation (A-1) is much smaller than that obtained when consideration is given to the finite thickness of the laminations.

APPENDIX B

Transmission Line Filled with Laminations of Finite Width

Let us consider the transmission line shown in Fig. 17. As before we have a set of metal laminae of width W and conductivity σ separated by insulating laminae of width t and dielectric constant ϵ. The laminae will be numbered as shown in the figure from 1 to N. Let us define r and p by the relations

$$r = \frac{\beta - T_{11}}{T_{12}}; \qquad p = \frac{1/\beta - T_{11}}{T_{12}} \tag{B-1}$$

Then, from equations (III-19) and (III-20), we can write for the z-component of magnetic field and the x-component of electric field,

$$H_n = A\beta^n + B\beta^{-n} \tag{B-2}$$

$$E_n = rA\beta^n + pB\beta^{-n} \tag{B-3}$$

[2] Tokio Sakurai, loc. cit., page 398.

From equations (II-9) and (II-15) we clearly have

$$\frac{E_N}{H_N} = -R; \qquad \frac{E_0}{H_0} = R \tag{B-4}$$

where $R = \frac{\alpha_s}{\sigma}$ and $\alpha_s = \frac{1+i}{\delta}$. It is easy now to obtain the characteristic equation

$$\begin{vmatrix} R\beta^N & R\beta^{-N} & \beta^N & \beta^{-N} \\ R & R & -1 & -1 \\ r & 0 & -1 & 0 \\ 0 & p & 0 & -1 \end{vmatrix} = 0 \tag{B-5}$$

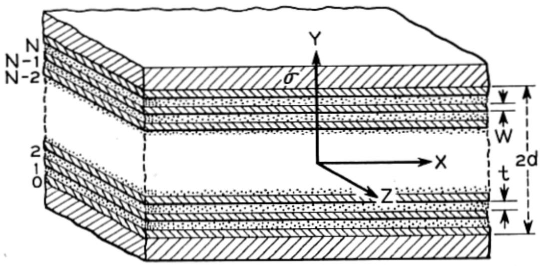

Fig. 17—Transmission line completely filled with laminations.

After expansion and use of expressions (B-1), the characteristic equation is found to take the form

$$\coth (N \ln \beta) = \frac{R^2 T_{12} + T_{21}}{R\left(\frac{1}{\beta} - \beta\right)} \tag{B-6}$$

which can be written, using the relation $\ln \beta = \alpha_w(W + t)$,

$$2 \sinh \alpha_w(W + t) \coth (2d\alpha_w) = RT_{12} + \frac{1}{R} T_{21} \tag{B-7}$$

Let us now assume, as in Section III, that $|\xi t|$ and $|\eta W|$ are much smaller than unity. Equation (B-7) can then be written

$$(2\alpha_w d) \coth (2\alpha_w d) = -\left(\frac{W}{W + t}\right)(\alpha_s d) \cdot \left(2(P + 1) + i\frac{2}{3}(3P + 1)\left(\frac{W}{\delta}\right)^2\right) \tag{B-8}$$

The quantity on the right of equation (B-8) is of the order of d/δ. We write, therefore,

$$(2\alpha_w d) \coth (2\alpha_w d) = 0\left(\frac{d}{\delta}\right) \tag{B-9}$$

and

$$(2\alpha_w d) = \pi i \left[1 + 0\left(\frac{\delta}{d}\right)\right] \tag{B-10}$$

Thus we have approximately

$$\cosh \alpha_w (W + t) = 1 - \frac{1}{2}\left(\frac{\pi(W + t)}{2d}\left[1 + 0\left(\frac{\delta}{d}\right)\right]\right)^2 \tag{B-11}$$

Furthermore, from equation (III-34)

$$\cosh \alpha_w (W + t) = 1 - \frac{1 + 4P}{6}\left(\frac{W}{\delta}\right)^4 + i\,(1 + 2P)\left(\frac{W}{\delta}\right)^2 \tag{B-12}$$

We can now equate equations (B-11) and (B-12) and after suitable manipulation obtain

$$k = \sqrt{\omega^2\mu_0\bar{\epsilon}}\left(1 - \frac{i}{2\left(1 + \frac{t}{W}\right)}\left[\frac{\pi^2}{8}\left(1 + \frac{t}{W}\right)^2\left(\frac{\delta}{d}\right)^2 + \frac{1}{6}\left(\frac{W}{\delta}\right)^2\right]\right) \tag{B-13}$$

where we have now dropped a term of order $\left(\frac{\delta}{d}\right)$ compared to unity.

Equation (B-13) is of considerable interest. It is observed that for $\left(\frac{W}{\delta}\right) \ll \left(\frac{\delta}{d}\right)$ the attenuation becomes that given in equation (V-8) for infinitely thin laminae. For $\left(\frac{W}{\delta}\right) \gg \left(\frac{\delta}{d}\right)$, on the other hand, the attenuation approaches that given in equation (A-2) for an unbounded array of finite laminae. This can be considered in two ways. Let us ask for the condition that the two terms contributing to the attenuation in (B-13) are equal. We find for this to be true that

$$d = \frac{\pi}{2}\sqrt{3}\left(1 + \frac{t}{W}\right)\left(\frac{\delta}{W}\right)\delta \tag{B-14}$$

In other words, at a given frequency, the attenuation of the line can be little reduced by making d larger than the value given in equation (B-14) which will be recognized as approximately the skin depth given in equation (III-44).

Alternatively, we see that the attenuation as a function of frequency remains constant from low frequencies up to the point where δ satisfies equation (B-14). At higher frequencies, the attenuation increases parabolically. At frequencies where $\delta \ll W$, the attenuation will clearly correspond to propagation in a parallel plate transmission line of width t and will therefore increase with the square root of the frequency. This behavior is similar to that discussed in Section I for a line with a thin stack of laminations on its inner conductor.