Zero Temperature Coefficient Quartz Crystals for Very High Temperatures

By W. P. MASON

(*Manuscript Received Nov. 15, 1950*)

In order to determine the angles of cuts for low temperature coefficient crystals, the elastic constants of quartz have been evaluated in the temperature range from $-100°C$ to $+200°C$. This has been done by measuring a series of rotated Y-cut crystals in the thickness shear mode and a series of rotated X-cuts in the longitudinal length mode. From the measurements, low temperature coefficients AT, BT, CT, and DT type crystals can be determined which have their temperature of zero temperature coefficient at any prescribed temperature. Calculations are given for the properties of crystals to operate at 200°C. The characteristics of an AT type crystal have been investigated experimentally, and the measured results are in reasonable agreement with the calculations. It is shown that there is a maximum temperature of 190°C for which an AT type crystal can have a zero temperature coefficient.

I. Introduction

Most quartz crystals used to control the frequency of oscillators or time measuring devices are used in places where the ambient temperature does not exceed 60° to 70°C. The crystals are usually adjusted in angle so that they have a zero temperature coefficient at a temperature of about 80°C and they are temperature controlled at this temperature. However, a class of uses occurs for which the ambient temperature may be considerably higher and for these uses ordinary AT and BT crystals, for example, are not satisfactory. This is evident from Figs. 1 and 2 which show the frequency variations for these crystals over a temperature range from $-100°C$ to $+200°C$. For example, the flattest frequency temperature curve for the AT cut occurs at an angle of $+35°18'$ rotation about the X axis from the Y cut. By going to $+35°36'$ orientation about the X axis a minimum occurs at 100°C. For the BT cut shown by Fig. 2 the angle of $-49°16'$ orientation gives nearly a parabolic shape centered at 20°C. By changing the orientation to $-47°22'$ the parabola centers at 75°C.

Hence if one wishes to raise the temperature for which the zero temperature coefficient occurs he has to increase the rotation about X for the AT cut and decrease it for the BT cut. The amount needed for either orientation can best be determined by evaluating the elastic constants as a function of orientation and temperature, and that is the main purpose of this paper. The results are applied to determining the best angles of orientation for the AT, BT, CT, and DT type crystals to obtain zero temperature coeffi-

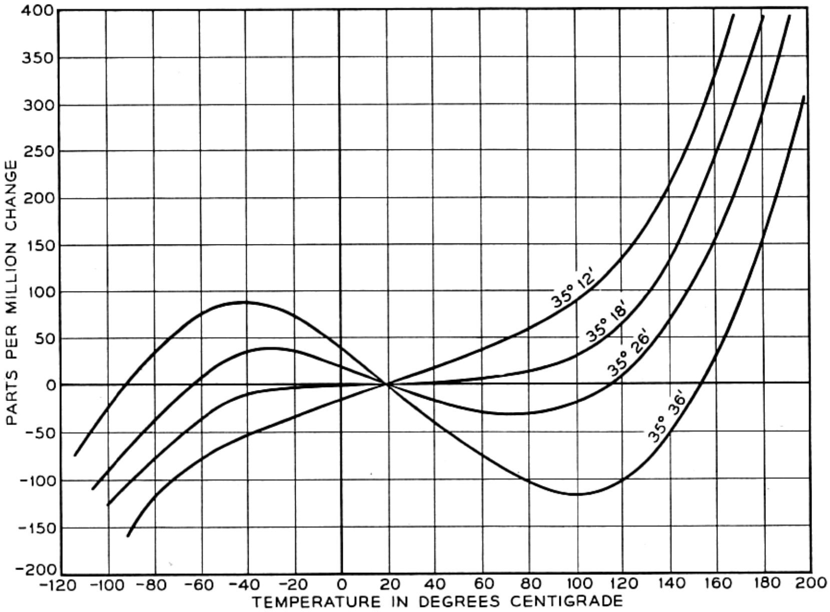

Fig. 1—Frequency temperature characteristics of AT type crystals.

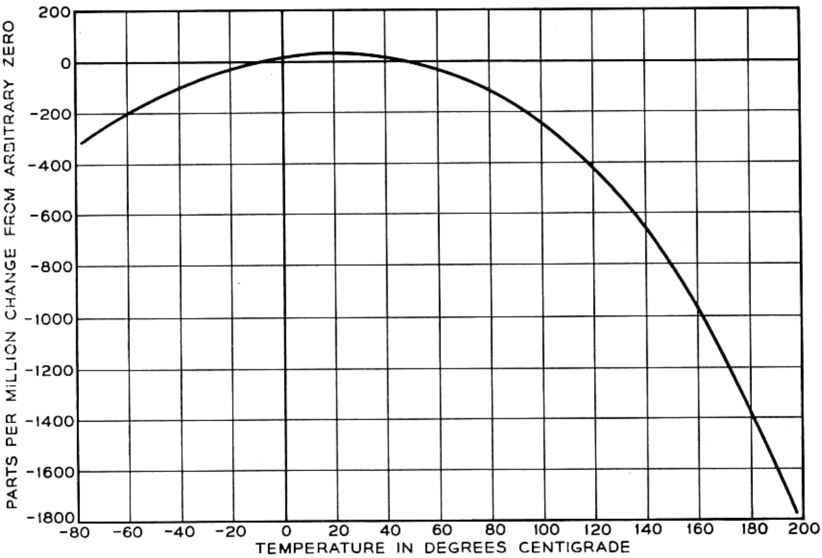

Fig. 2—Frequency temperature characteristics of BT type crystals.

cients for any arbitrary temperature. These calculated values have been checked experimentally for the AT type crystal and the angles and properties are approximated by the calculations. It is shown that there is a critical angle of +36°26′ which results in the highest temperature of 190°C for which it is possible to obtain a zero temperature coefficient AT type crystal.

II. Evaluation of the Elastic Constants as a Function of Temperature

A simple method for taking account of the temperature terms is to expand the frequencies for the known cuts in powers of the temperature around some reference temperature. Since the data of Figs. 1 and 2 run from −100°C to +200°C, a convenient temperature is 50°C. Then

$$f = f_m\,[1 + a_1(T - T_0) + a_2(T - T_0)^2 + a_3(T - T_0)^3 + \cdots] \quad (1)$$

Over this temperature range the frequencies measured can be accurately represented by terms including the cubic as the highest. If T_0 is taken as 50°C, equation (1) can be solved for the constants a_1, a_2, and a_3 and we find

$$a_1 = \frac{-\frac{1}{3}\,[f_{200^\circ} - f_{-100^\circ}] + \frac{8}{3}\,[f_{125^\circ} - f_{-25^\circ}]}{300\, f_{50^\circ}}$$

$$a_2 = \frac{\dfrac{f_{-100^\circ} + f_{200^\circ}}{2} - f_{50^\circ}}{22{,}500\, f_{50^\circ}} \quad (2)$$

$$a_3 = \frac{(f_{200^\circ} - f_{-100^\circ}) - 2[f_{125^\circ} - f_{-25^\circ}]}{5{,}062{,}500\, f_{50^\circ}}$$

where the subscripts refer to the temperatures for which the frequencies are measured. If we apply these equations to the AT crystal cut at 35°18′ and the BT at −49°16′, we find, for the frequencies, the equations

$$\begin{aligned} f_{AT} &= 1.661 \times 10^5[1 + .22 \times 10^{-6}(T - 50^\circ) \\ &\quad + 8.9 \times 10^{-9}(T - 50)^2 + 82 \times 10^{-12}(T - 50)^3 + \cdots] \\ f_{BT} &= 2.547 \times 10^5[1 - 2.2 \times 10^{-6}(T - 50) \\ &\quad - 55.5 \times 10^{-9}(T - 50)^2 - 73 \times 10^{-12}(T - 50)^3 + \cdots] \end{aligned} \quad (3)$$

In order to obtain the frequency and the variation of frequency with angle, use is made of the equation for a thickness shear vibration

$$f = \frac{1}{2t}\sqrt{\frac{c'^E_{66}}{\rho}} = \frac{1}{2t}\sqrt{\frac{c^E_{66}\cos^2\theta + c^E_{44}\sin^2\theta - 2c^E_{14}\sin\theta\cos\theta}{\rho}} \quad (4)$$

where t is the thickness of the crystal, ρ the density, θ the angle of the normal of the plate measured from the Y axis and c_{14}^{E}, c_{44}^{E} and c_{66}^{E} three of the seven elastic constants of quartz measured at constant electric field. This equation is valid for an infinite plate but is also a good approximation for a crystal whose cross-sectional dimensions are 30 to 40 times the thickness dimensions. Since there are three constants, the two measurements for the AT and BT cuts will give only two relations and we need a measurement for another angle. As discussed in Chapter X, Section 10.2 of "Piezoelectric Crystals and Their Application to Ultrasonics,"[1] the remaining cut can be obtained by measuring the thickness shear mode of a Y cut plate or the face shear mode of a Y cut crystal. The latter mode is considerably easier to dimension in order to obtain a frequency corresponding to the shear mode. Table I shows measurements for the frequency constant of a Y face shear mode for a crystal having the following dimensions: Length along the X axis is 36.86 mm, width along the Z axis = 7.625 mm; thickness along the Y axis is 0.990 mm. High harmonics were used and the frequency constant was obtained by dividing the frequency by the overtone order. This frequency is controlled by the c_{44}^{E} elastic constant according to the equation

$$f = \frac{1}{2l_z}\sqrt{\frac{c_{44}^{E}}{\rho}} \text{ or } c_{44}^{E} = 4\, f_m\, 2l_z\, 2_\rho \tag{5}$$

In calculating the c_{44}^{E} constant from the resonant frequencies measured, a correction is introduced by the temperature expansion constants of the crystal. This follows from equation (5) since l_z the frequency determining axis and ρ the density both change with temperature. From measurements quoted by Sosman[2] for the expansion along the Z axis and perpendicular to the Z axis, we find

$$\begin{aligned} l_z &= l_0[1 + 7.8 \times 10^{-6}(T - 50) + 2.8 \times 10^{-9}(T - 50)^2 \\ &\qquad - 1.5 \times 10^{-12}(T - 50)^3 - \cdots] \\ l_{x,y} &= l_0[1 + 14.6 \times 10^{-6}(T - 50) + 6.3 \times 10^{-9}(T - 50)^2 \\ &\qquad - 1.9 \times 10^{-12}(T - 50)^3 + \cdots] \end{aligned} \tag{6}$$

Multiplying these together the volume expansion is

$$\begin{aligned} V = l_x l_y l_z = l_0^3[1 + 37 \times 10^{-6}(T - 50) + 15.8 \times 10^{-9}(T - 50)^2 \\ - 5 \times 10^{-12}(T - 50)^3 + \cdots] \end{aligned} \tag{7}$$

[1] Pizoelecteric Crystals and Their Application to Ultrasonics, W. P. Mason, D. van Nostrand Co. 1950, Section 10.2, page 204.
[2] Sosman, The Properties of Silica, Chemical Catalogue Co., 1927, pp. 386, 387.

Since the density is the inverse of the volume, the square of the frequency constant for a crystal whose dimension is measured at 50°C must be multiplied by the factor

$$\frac{l_z}{l_x l_y l_z} = \frac{l_z}{l_x l_y}$$

in order to correct for the effect of temperature expansion on the elastic constant. This correction is shown by the third column of Table I. The fourth column is then the value of c_{44}^E for the various temperatures. The fifth column shows the values of the a_1, a_2 and a_3 constants for the temperature variation of c_{44}^E.

Table I evaluates one of the elastic constants of the frequency equation (4). To evaluate the other two constants, use is made of the frequency

TABLE I

Temperature °C	Frequency Constant Kilocycle Centimeters	Correction for temperature expansion	c_{44}^E dynes/cm² x10⁻¹⁰	Constants for c_{44}^E equation
−100	237.22	1.0029	59.82	$a_1 = -171 \times 10^{-6}$
−25	236.26	1.0016	59.26	$a_2 = -212 \times 10^{-9}$
50	235.07	1.000	58.58	$a_3 = -65 \times 10^{-12}$
+125	233.56	0.9984	57.75	
200	231.78	0.9965	56.78	

constants for the AT and BT cuts given by equation (3). Over a temperature range the thickness t is given by the equation

$$t = t_0(l_y^0 \cos^2\theta + l_z^0 \sin^2\theta) \tag{8}$$

where l_y^0 and l_z^0 are the values of unit lengths along the Y and Z axis expressed as a function of temperature. Inserting the values of (6) and (7) in equation (3), the elastic shear constants for the AT and BT cuts become

$$\begin{aligned} c_{66}^{\prime E}(\text{AT}) &= 2.924 \times 10^{11}[1 - 12 \times 10^{-6}(T - 50) \\ &\quad + 12.8 \times 10^{-9}(T - 50)^2 + 172 \times 10^{-12}(T - 50)^3 + \cdots] \\ c_{66}^{\prime E}(\text{BT}) &= 6.877 \times 10^{11}[1 - 20 \times 10^{-6}(T - 50) \\ &\quad - 176 \times 10^{-9}(T - 50)^2 - 238 \times 10^{-12}(T - 50)^3 + \cdots] \end{aligned} \tag{9}$$

From equation (4), the frequency equation, we have

$$\begin{aligned} c_{66}^{\prime E}(\text{BT}) &= 0.4258 c_{66}^E + 0.5742\, c_{44}^E + 0.9890\, c_{14}^E \\ c_{66}^{\prime E}(\text{AT}) &= 0.6661\, c_{66}^E + 0.3339\, c_{44}^E - 0.934\, c_{14}^E \end{aligned} \tag{10}$$

Since c_{44}^E is already known, tke two equations can be solved for c_{66}^E and c_{14}^E and we find

$$\begin{aligned} c_{66}^E &= 0.8892\, c_{66}'^E(\mathrm{BT}) + 0.9328\, c_{66}'^E(\mathrm{AT}) - 0.8221\, c_{44}^E \\ c_{14}^E &= 0.6282\, c_{66}'^E(\mathrm{BT}) - 0.4016\, c_{66}'^E(\mathrm{AT}) - 0.2264\, c_{44}^E \end{aligned} \tag{11}$$

Inserting the values from Table I and equation (9) the three elastic constants become

$$\begin{aligned} c_{44}^E &= 58.58 \times 10^{10}[1 - 171 \times 10^{-6}(T - 50) \\ &\quad - 212 \times 10^{-9}(T - 50)^2 - 65 \times 10^{-12}(T - 50)^3 + \cdots] \\ c_{66}^E &= 40.26 \times 10^{10}[1 + 168 \times 10^{-6}(T - 50) \\ &\quad - 5 \times 10^{-9}(T - 50)^2 - 167 \times 10^{-12}(T - 50)^3 + \cdots] \\ c_{14}^E &= 18.20 \times 10^{10}[1 + 90 \times 10^{-6}(T - 50) \\ &\quad - 270 \times 10^{-9}(T - 50)^2 - 630 \times 10^{-12}(T - 50)^3 + \cdots] \end{aligned} \tag{12}$$

To determine the frequency and temperature coefficients for any angle, one substitutes the values of the elastic constants and the temperature expansion coefficients in the frequency equation (4), which results in the expression

$$\begin{aligned} f^2 &= 10^{10}[(3.802 \cos^2\theta + 5.526 \sin^2\theta - 3.426 \sin\theta\cos\theta) + 10^{-6}(T - 50) \\ &\quad [668 \cos^2\theta - 828 \sin^2\theta - 336 \sin\theta\cos\theta - 13.5 \sin^2\theta\cos^2\theta - 46 \\ &\quad \sin^3\theta\cos\theta] + 10^{-9}(T - 50)^2[395 \cos^2\theta - 1160 \sin^2\theta + 354 \\ &\quad \sin\theta\cos\theta - 12 \sin^2\theta\cos^2\theta - 24 \sin^3\theta\cos\theta] + 10^{-12}(T - 50)^3 \\ &\quad [310 \cos^2\theta - 884 \sin^2\theta + 1130 \sin\theta\cos\theta] + \text{—}] \end{aligned} \tag{13}$$

III. Properties of AT and BT Cut Crystals Having Zero Temperature Coefficients at High Temperatures

The process for obtaining high frequencies cuts of the AT and BT type that will have zero temperature coefficients at a high temperature—for example 200°C—is to substitute for T in equation (13) the value

$$T = 200^\circ + \Delta T \tag{14}$$

Inserting this value in (13) and collecting the results in powers of ΔT, we find

$$\begin{aligned} f^2 \times 10^{-10} &= [3.913 \cos^2\theta + 5.373 \sin^2\theta - 3.465 \sin\theta\cos\theta - 0.0023 \\ &\quad \sin^2\theta\cos^2\theta - 0.0075 \sin^3\theta\cos\theta] + (\Delta T) \times 10^{-6} \\ &\quad [807 \cos^2\theta - 1236 \sin^2\theta - 154 \sin\theta\cos\theta - 17 \sin^2\theta \\ &\quad \cos^2\theta - 53 \sin^3\theta\cos\theta] + (\Delta T)^2 \times 10^{-9}[534 \cos^2\theta \\ &\quad - 1558 \sin^2\theta + 862 \sin\theta\cos\theta - 12 \sin^2\theta\cos^2\theta - 24 \\ &\quad \sin^3\theta\cos\theta] + (\Delta T)^3 \times 10^{-12}[310 \cos^2\theta - 884 \sin^2\theta \\ &\quad + 1130 \sin\theta\cos\theta] \end{aligned} \tag{15}$$

V. Evaluation for the Remainder of the Elastic Constants over a Wide Temperature Range

In order to evaluate the remainder of the elastic constants measurements were made of the frequencies of a series of X-cut longitudinal crystals over the same temperature range from −100°C to +200°C.[3] The longitudinal crystals measured had their lengths at −30°, 0°, +30° and +60° from the Y axis. For the four crystals measured the results are shown by

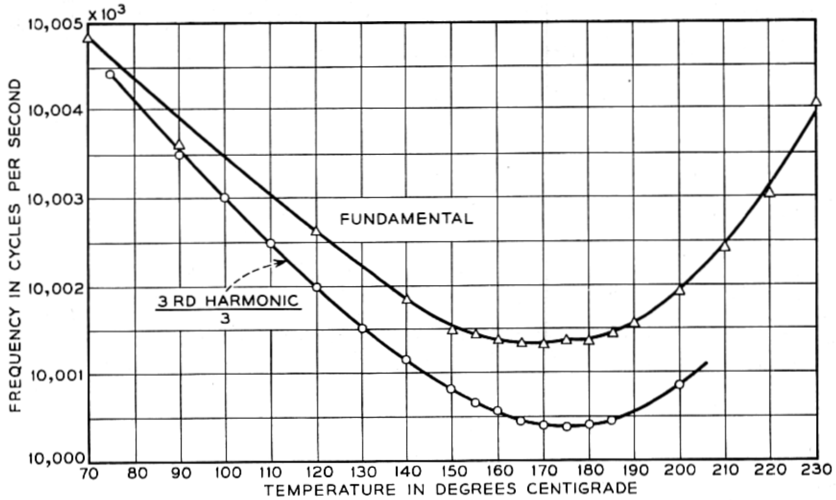

Fig. 5—Frequency temperature curves for fundamental and third overtone for a crystal cut at 36°45′ rotation.

Table II. The analysis for $f_m = f_{50°}$ and the three constants a_1, a_2 and a_3 are also shown.

To correct for the temperature expansion coefficients, the increase along l_x is given by the last equation of (16) while $l \pm 30°$ and $l \pm 60°$ are

$$\begin{aligned} l \pm 30° &= .25l_z + .75l_x = l_0[1 + 12.9 \times 10^{-6}(T - 50) \\ &\quad + 5.42 \times 10^{-9}(T - 50)^2 - 1.8 \times 10^{-12}(T - 50)^3 + \cdots] \\ l \pm 60° &= .75l_z + .25l_x = l_0[1 + 9.5 \times 10^{-6}(T - 50) \\ &\quad + 3.68 \times 10^{-9}(T - 50)^2 - 1.6 \times 10^{-12}(T - 50)^3 + \cdots] \end{aligned} \tag{19}$$

Since the frequency of a long thin bar is given by the equation

$$f = \frac{1}{2l}\frac{1}{\sqrt{\rho s_{22}^{E'}}} \quad \text{or} \quad s_{22}^{E'} = \frac{1}{4f_m^2 l^2 \rho} \tag{20}$$

introducing the length correction from (19) and the density correction from (7) one can correct for the effect of temperature expansion.

[3] These measurements were made by T. G. Kinsley.

TABLE II

0° X Cut	+30° X cut	+60° X cut	−30° X cut
l = 22.97 mm	l = 19.90 mm	l = 20.00 mm	l = 19.95 mm
w = 2.58 mm	w = 2.98 mm	w = 2.99 mm	w = 3.01 mm
t = 0.99 mm	t = 1.00 mm	t = 1.005 mm	t = 1.00 mm
f_{200} = 118,386	f_{200} = 169,177	f_{200} = 162,646	f_{200} = 128,788
f_{125} = 118,493	f_{125} = 170,177	f_{125} = 164,225	f_{125} = 129,460
f_{50} = 118,544	f_{50} = 170,828	f_{50} = 165,435	f_{50} = 130,010
f_{-25} = 118,566	f_{-25} = 171,240	f_{-25} = 166,375	f_{-25} = 130,480
f_{-100} = 118,554	f_{-100} = 171,460	f_{-100} = 167,170	f_{-100} = 130,750
f_m = 272.4 kc cm	f_m = 340.05 kc cm	f_m = 330.9 kc cm	f_m = 259.45 kc cm
$a_1 = -4.32 \times 10^{-6}$	$a_1 = -42.6 \times 10^{-6}$	$a_1 = -88.2 \times 10^{-6}$	$a_1 = -51.4 \times 10^{-6}$
$a_2 = -27.8 \times 10^{-9}$	$a_2 = -132.8 \times 10^{-9}$	$a_2 = -142 \times 10^{-9}$	$a_2 = -82.2 \times 10^{-9}$
$a_3 = -18.3 \times 10^{-12}$	$a_3 = -91.8 \times 10^{-12}$	$a_3 = -135.5 \times 10^{-12}$	$a_3 = +59.1 \times 10^{-12}$

Applying these corrections to the frequency equations of Table II the resulting compliance constants become

$$
\begin{aligned}
s^E_{22(0^\circ X)} &= s^E_{11} = 1.271 \times 10^{-12}[1 + 16.5 \times 10^{-6}(T - 50^\circ) \\
&\quad + 58.5 \times 10^{-9}(T - 50^\circ)^2 + 33 \times 10^{-12}(T - 50)^3 + \cdots] \\
s^E_{22(+30^\circ X)} &= 0.8159 \times 10^{-12}[1 + 96.4 \times 10^{-6}(T - 50) \\
&\quad + 276.5 \times 10^{-9}(T - 50)^2 + 219.4 \times 10^{-12}(T - 50)^3 + \cdots] \\
s^E_{22(-30^\circ X)} &= 1.402 \times 10^{-12}[1 + 114.4 \times 10^{-6}(T - 50) \\
&\quad + 178 \times 10^{-9}(T - 50)^2 - 91.6 \times 10^{-12}(T - 50)^3 + \cdots] \\
s^E_{22(+60^\circ X)} &= 0.8614 \times 10^{-12}[1 + 186.4 \times 10^{-6}(T - 50) \\
&\quad + 302.2 \times 10^{-9}(T - 50)^2 + 385.3 \times 10^{-12}(T - 50)^3 + \cdots]
\end{aligned} \tag{21}
$$

The equation for the compliance constant s'^E_{22} for an X-cut crystal at an angle θ from the Y axis has been shown to be[4]

$$
\begin{aligned}
s'^E_{22} = s^E_{11} \cos^4\theta + s^E_{33} \sin^4\theta - 2s^E_{14} \cos^3\theta \sin\theta \\
+ (2s^E_{13} + s^E_{44}) \sin^2\theta \cos^2\theta
\end{aligned} \tag{22}
$$

Solving for the constants in terms of the compliances for the four angles measured we find

$$
\begin{aligned}
&s^E_{11} = s^E_{22(0^\circ X)};\ s^E_{14} = \frac{s^E_{22(+30^\circ)} - s^E_{22(-30^\circ)}}{1.3};\ s^E_{33} = s^E_{22(0^\circ)} + 2s^E_{22(+60^\circ)} \\
&- \frac{4}{3} s^E_{22(30^\circ)} - \frac{2}{3} s^E_{22(-30^\circ)};\ (2s^E_{13} + s^E_{44}) = -\frac{10}{3} s^E_{22(0^\circ)} - \frac{2}{3} s^E_{22(60^\circ)} \\
&\qquad + \frac{28}{9} s^E_{22(+30^\circ)} + \frac{26}{9} s^E_{22(-30^\circ)}
\end{aligned} \tag{23}
$$

Hence adding the results we find

$$
\begin{aligned}
s^E_{11} &= 1.271 \times 10^{-12}[1 + 16.5 \times 10^{-6}(T - 50^\circ) \\
&\quad + 58.5 \times 10^{-9}(T - 50)^2 + 33 \times 10^{-12}(T - 50)^3 + \cdots] \\
s^E_{33} &= 0.971 \times 10^{-12}[1 + 134.5 \times 10^{-6}(T - 50) \\
&\quad + 144 \times 10^{-9}(T - 50)^2 + 570 \times 10^{-12}(T - 50)^3 + \cdots] \\
s^E_{14} &= -0.4506 \times 10^{-12}[1 + 139.5 \times 10^{-6}(T - 50) \\
&\quad + 40 \times 10^{-9}(T - 50)^2 - 54 \times 10^{-12}(T - 50)^3 + \cdots] \\
(2s^E_{13} + s^E_{44}) &= 1.785 \times 10^{-12}[1 + 300 \times 10^{-6}(T - 50) \\
&\quad + 460 \times 10^{-9}(T - 50)^2 - 98 \times 10^{-12}(T - 50)^3 + \cdots]
\end{aligned} \tag{24}
$$

[4] See "Piezoelectric Crystals and Their Application to Ultrasonics," page 204, equation 10.26.

All the compliance constants are now determined except s_{44}^E, s_{13}^E and s_{12}^E. From the relations for a crystal in the quartz class[5]

$$s_{44}^E = \frac{c_{66}^E}{c_{44}^E c_{66}^E - c_{14}^{E2}}; \quad s_{66}^E = 2(s_{11}^E - s_{12}^E) = \frac{c_{44}^E}{c_{44}^E c_{66}^E - c_{14}^{E2}} \tag{25}$$

the remaining constants can be obtained. Inserting the values of c_{44}^E, c_{66}^E and c_{14}^E from (12), we find

$$\begin{aligned} s_{44}^E &= 1.986 \times 10^{-12}[1 + 201 \times 10^{-6}(T - 50°) \\ &\quad + 200 \times 10^{-9}(T - 50)^2 - 26 \times 10^{-12}(T - 50)^3 + \cdots] \\ s_{66}^E &= 2.89 \times 10^{-12}[1 - 138 \times 10^{-6}(T - 50°) \\ &\quad - 18 \times 10^{-9}(T - 50)^2 + 3 \times 10^{-12}(T - 50)^3 + \cdots] \\ s_{13}^E &= -0.1005 \times 10^{-12}[1 - 678 \times 10^{-6}(T - 50°) \\ &\quad - 2110 \times 10^{-9}(T - 50)^2 + 610 \times 10^{-12}(T - 50)^3 + \cdots] \\ s_{12}^E &= -0.174 \times 10^{-12}[1 - 1270 \times 10^{-6}(T - 50°) \\ &\quad - 575 \times 10^{-9}(T - 50)^2 - 215 \times 10^{-12}(T - 50)^3 + \cdots] \end{aligned} \tag{26}$$

It is sometimes desirable to use the c values as a function of temperature. The remaining values can be obtained from the relations valid for quartz[5]

$$2c_{11}^E = \frac{s_{33}^E}{\alpha} + \frac{s_{44}^E}{\beta}; \quad 2c_{12}^E = \frac{s_{33}^E}{\alpha} - \frac{s_{44}^E}{\beta}; \quad c_{13}^E = -\frac{s_{13}^E}{\alpha}; \quad c_{33}^E = \frac{s_{11}^E + s_{12}^E}{\alpha} \tag{27}$$

where

$$\alpha = s_{33}^E(s_{11}^E + s_{12}^E) - 2s_{13}^{E2}; \qquad \beta = s_{44}^E(s_{11}^E - s_{12}^E) - 2s_{14}^{E2}$$

$$\begin{aligned} c_{33}^E &= 104.8 \times 10^{+10}[1 - 165 \times 10^{-6}(T - 50) \\ &\quad - 187 \times 10^{-9}(T - 50)^2 - 410 \times 10^{-12}(T - 50)^3 + \cdots] \\ c_{13}^E &= 9.6 \times 10^{+10}[1 - 510 \times 10^{-6}(T - 50) \\ &\quad - 2000 \times 10^{-9}(T - 50)^2 + 600 \times 10^{-12}(T - 50)^3 + \cdots] \\ c_{11}^E &= 86.75 \times 10^{+10}[1 - 53.5 \times 10^{-6}(T - 50) \\ &\quad - 75 \times 10^{-9}(T - 50)^2 - 15 \times 10^{-12}(T - 50)^3 + \cdots] \\ c_{12}^E &= 6.15 \times 10^{10}[1 - 3030 \times 10^{-6}(T - 50) \\ &\quad - 1500 \times 10^{-9}(T - 50)^2 + 1910 \times 10^{-12}(T - 50)^3 + \cdots] \end{aligned} \tag{28}$$

[5] See Piezoelectric Crystals and Their Application to Ultrasonics, page 207.

VI. Predicted Angles for CT and DT Face Shear Crystals

The other two cuts of primary interest for frequency controlled oscillators are the CT and DT low frequency face shear modes. An exact solution for the frequency vibration of a face shear mode has not yet been obtained, but Hight and Willard[6,7] have pointed out an empirical relation that agrees with the measured frequencies over the entire range of angles of rotated Y cut crystals. This relation is for a square crystal

$$f = \frac{1.23}{l}\sqrt{\frac{1}{\rho s_{55}^{E'}}} \tag{29}$$

where l is one edge dimension and $s_{55}^{E'}$ the shear elastic constant pertaining to the face shear mode. In terms of the orientation angle[6]

$$s_{55}^{E'} = s_{44}^{E}\cos^2\theta + s_{66}^{E}\sin^2\theta + 4s_{14}^{E}\sin\theta\cos\theta \tag{30}$$

Introducing the values of s_{44}^{E}, s_{66}^{E} and s_{14}^{E} from equations (24) and (26) the frequency becomes

$$f_m^2 \times 10^{-10} = \frac{14.27}{\begin{gathered}[(1.986\cos^2\theta + 2.89\sin^2\theta - 1.802\sin\theta\cos\theta) \\ + (399\cos^2\theta - 398\sin^2\theta - 251.5\sin\theta\cos\theta) \\ \times 10^{-6}(T-50) + (397\cos^2\theta - 52\sin^2\theta \\ - 72\sin\theta\cos\theta) \times 10^{-9}(T-50)^2 + (-52\cos^2\theta \\ + 8.7\sin^2\theta + 98\sin\theta\cos\theta) \times 10^{-12}(T-50)^3 + \cdots\end{gathered}} \tag{31}$$

Since the formula is very approximate the small correction due to temperature expansions has been neglected. With this equation the indicated angles for zero temperature coefficient—which are obtained by setting the

[6] A Simplified Circuit for Frequency Substandards Employing a New Type of Low Frequency Zero-Temperature-Coefficient Quartz Crystal, S. C. Hight and G. W. Willard, *Proc. I.R.E.*, Vol. 25, No. 5, pp. 549–563, May 1937. The factor 1.23 agrees better with experiment than the value 1.25 given in the paper.

[7] Since this paper was written a much more nearly exact solution of a face shear mode vibration has been obtained by R. D. Mindlin and H. T. O'Neil. This solution is an extension of the thickness shear vibration of a crystal published by Mindlin (Journal of Applied Physics, probably March issue 1951). For a square plate there are two solutions which are very close in frequency. For case A which corresponds to l of equation (29) lying along the X direction the empirical factor F becomes

$$F = 1.2718 - .03471g - .03727g^2 \quad \text{where } g = \frac{\pi^2}{12}\frac{s_{11}'}{s_{55}'}$$

and s_{11}' and s_{55}' are the elastic compliances corresponding to the rotated cuts. For the B case which corresponds to l lying along z' the same formula holds but $g = \frac{\pi^2}{12}\frac{s_{33}'}{s_{55}'}$.

multiplier of $(T - 50)$ equal to zero and solving for the rotation angles θ_1 and θ_2—are

$$\theta_1 = +36°20' \text{ and } \theta_2 = -53°50' \tag{32}$$

as compared to the experimental values of $+38°20'$ and $-52°$, which represents a shift of about $+2°$ orientation for both angles. At these calculated angles the frequencies are within about 1.5 per cent of the experimental values and the curvature constants agree approximately with the measured values.

To obtain the angles for any other temperatures, for example 200°C, we substitute

$$T = 200 + \Delta T \tag{33}$$

and obtain the expansion in powers of ΔT. For 200°C this results in

$$f_m^2 \times 10^{-10} = \frac{14.27}{\begin{aligned}&[2.055 \cos^2\theta + 2.829 \sin^2\theta - 1.840 \sin\theta\cos\theta] \\ &+ [514 \cos^2\theta - 413 \sin^2\theta - 266 \sin\theta\cos\theta] \times 10^{-6}\Delta T \\ &+ [373 \cos^2\theta - 48 \sin^2\theta - 28 \sin\theta\cos\theta] \times 10^{-9}(\Delta T)^2 \\ &+ [-52 \cos^2\theta + 8.7 \sin^2\theta + 98 \sin\theta\cos\theta] \\ &\qquad \times 10^{-12}(\Delta T)^3)\end{aligned}} \tag{34}$$

The zero temperature coefficient angles are obtained by setting the coefficients of ΔT equal to zero giving

$$514 \cos^2\theta - 413 \sin^2\theta - 266 \sin\theta\cos\theta = 0 \tag{35}$$

Solving for θ we find

$$\theta = +39°50' \text{ and } -56° \tag{36}$$

If we add 2° to each of these in order to correct for the difference between the formula and the measured results at 50°C, the most probable angles for zero coefficients at 200°C are

$$\theta = +41°50' \text{ and } -54° \tag{37}$$

At these angles the indicated frequencies and variations of frequencies with temperature should be

$$\begin{aligned}&\theta = 41°51'; \\ &f = \frac{3.12 \times 10^5}{l}[1 - 63 \times 10^{-9}(\Delta T)^2 - 8 \times 10^{-12}(\Delta T)^3 + \cdots] \\ &\theta = -54°; \\ &f = \frac{2.04 \times 10^5}{l}[1 - 14 \times 10^{-9}(\Delta T)^2 + 8 \times 10^{-12}(\Delta T)^3 + \cdots]\end{aligned} \tag{38}$$

While these results are probably not very exact on account of the lack of an exact solution for the frequency of a face shear plate, they indicate the angles and approximate variations with temperature for high temperature plates. So far no experimental results have been obtained for crystals of this type.