Tables of Phase Associated with a Semi-Infinite Unit Slope of Attenuation

By D. E. THOMAS

This paper presents tables of the phase associated with a semi-infinite unit slope of attenuation. The phase is given in degrees to .001 degree with an accuracy of ± .001 degree and in radians to .00001 radian with an accuracy of ± .000015 radian. The method of constructing the tables and a brief analysis of the errors are given. An appendix, which gives a detailed explanation with specific examples of the use of the tables in determining the phase associated with a given attenuation characteristic or the reactance associated with a given resistance characteristic by means of the straight line approximation method given in Bode's "Network Analysis and Feedback Amplifier Design," is included for the benefit of those who are not already acquainted with this method. The Appendix also presents an example of a non-minimum phase network[1] in which the minimum phase determined from the attenuation characteristic fails to predict the true phase of the network.

THE method described by Bode[2] for the determination of the phase associated with a given attenuation characteristic or the reactance associated with a given resistance characteristic has proved to be an extremely useful laboratory and design tool. In this method the attenuation (or real) characteristic, plotted versus the log of frequency, is approximated by a series of straight lines. The phase (or imaginary component) is then determined by summing up the individual contributions of each elementary straight line segment to the total phase (or imaginary component).

The most elementary straight line characteristic which can be used to construct a given straight line approximation is that in which the attenuation plotted against the log of frequency is constant on one side of a prescribed frequency, f_0, and has a constant slope thereafter. Such a characteristic has been called by Bode a "semi-infinite constant slope" characteristic.[3] A semi-infinite unit slope of attenuation or one in which the attenuation changes 6 *db* per octave, or 20 *db* per decade is shown in Fig. 1. The phase associated with this attenuation characteristic is plotted in Fig. 2.[4] The independent variable was chosen as f/f_0 for values of f less than f_0 and f_0/f for values of f greater than f_0 to keep it finite for all values of f and in order to show the phase plotted exactly as it is given in the tables to follow. The phase associated with a semi-infinite constant slope of

[1] For a complete discussion of *minimum phase* see Hendrik W. Bode, "Network Analysis and Feedback Amplifier Design," D. Van Nostrand Company, Inc., New York, N. Y., 1945.
[2] Ibid: Chap. XV, page 344.
[3] Ibid: Chap. XIV, page 316.
[4] Ibid: Chap. XIV page 317.

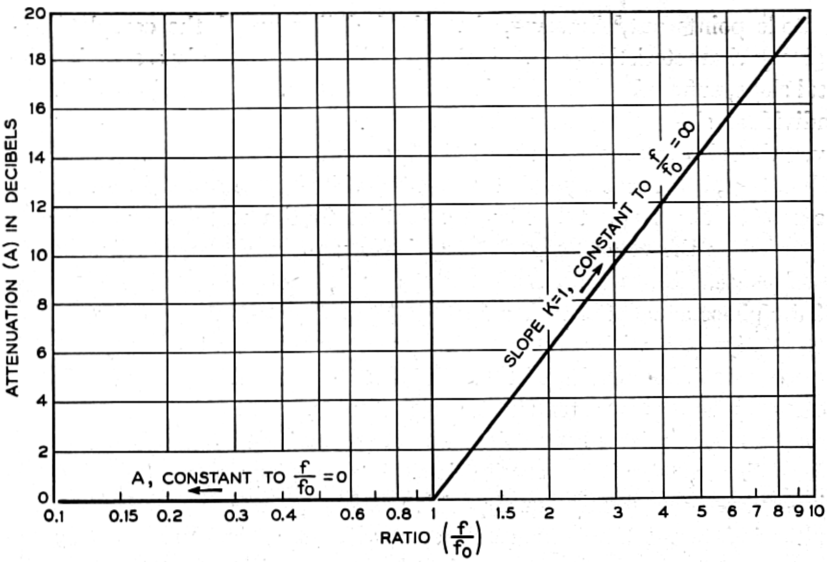

Fig. 1—Semi-infinite unit slope of attenuation.

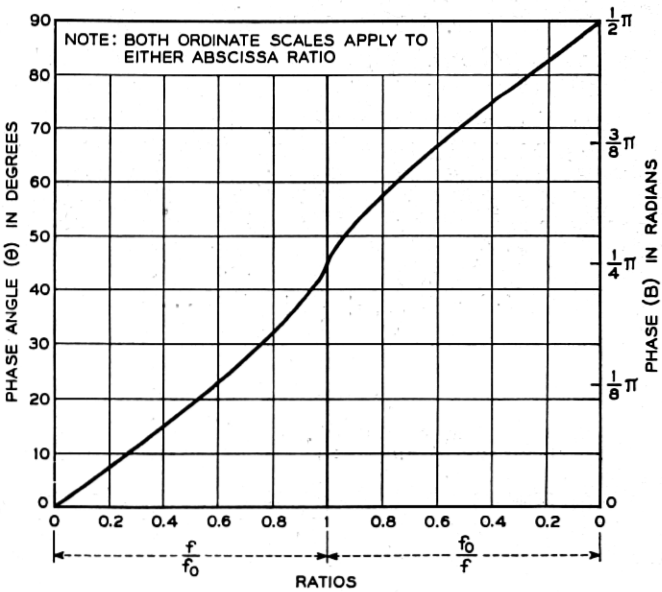

Fig. 2—Phase associated with semi-infinite unit slope of attenuation of Fig. 1.

attenuation of the same character as the semi-infinite unit slope of attenuation of Fig. 1 but of slope k, is k times the phase given in Fig. 2.

Bode points out,[5] however, that the building up of the complete imaginary characteristic from a single primitive curve, namely a semi-infinite real slope, suffers from the disadvantage that the phase contributions of the individual slopes may be rather large positive and negative quantities, even though the net phase shift is fairly small. In order to avoid this disadvantage, Bode recommends that the individual finite line segments which constitute the straight line approximation to the real characteristic be regarded as the elementary characteristics used in the summation of the total phase. He then gives a series of charts, plotted as a function of f/f_0, of the phase associated with a finite line segment having a 1 *db* change in attenuation and with a ratio of the geometric mean frequency (f_0) of the two terminal frequencies of the finite line segment to the lower terminal frequency as a parameter (ratio designated a).

However, problems have arisen where, even with the finite line segment phase charts, the phase contributions of the various elements were sufficiently large and nearly equal positive and negative quantities that difficulties in interpolation between the curves for the various values of a, given on the charts, resulted in a sufficient lack of precision that the quantity being sought was lost.

Because of the usefulness of the method in question, and with its application to a wider variety of problems, means of increasing its over-all precision and simplification of computation have constantly been sought. It had occurred to several engineers independently that a table of phase versus frequency for a semi-infinite unit slope of attenuation would prove extremely useful. The phase in radians at frequency f_c, associated with a semi-infinite unit slope of attenuation commencing at frequency f_0, is given by Bode as[6]

$$B(x_c) = \frac{2}{\pi}\left(x_c + \frac{x_c^3}{9} + \frac{x_c^5}{25} + \cdots\right) \tag{1}$$

where:

$$x_c = \frac{f_c}{f_0} = \frac{\omega_c}{\omega_0}, x_c < 1.$$

The computation time required to determine the phase at a given frequency by summation of the above series is such, that the work required to get the phase at a sufficient number of points and to a sufficient number of significant figures to prepare an adequate table proved to be sufficient to discourage this procedure.

[5] Ibid: Chap. XV page 338.
[6] Ibid: Chap. XV, page 343.

The derivative of (1) above, however, proves to be quite simple and easy to evaluate. It is given by Bode as:

$$\frac{dB}{dx_c} = \frac{1}{\pi x_c} \log \left| \frac{1 + x_c}{1 - x_c} \right| \tag{2}$$

$$= \frac{2}{\pi} \left(1 + \frac{x_c^2}{3} + \frac{x_c^4}{5} + \cdots \right), x_c < 1. \tag{2a}$$

It therefore seemed that since the phase had already been computed by the Mathematical Research Group of the Bell Telephone Laboratories, Inc., at a

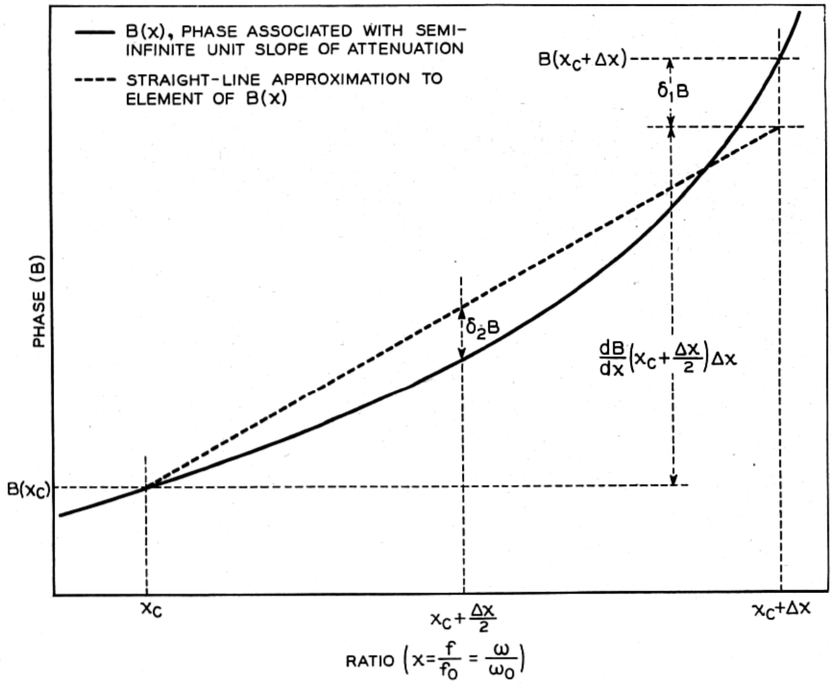

Fig. 3—Element of Fig. 2 for $f/f_0 < 1$ expanded qualitatively.

considerable number of points, using the infinite series expansion of (1) above the function in the regions between known values of phase could be constructed by assuming the intervening curve of phase as a function of $x = \dfrac{f}{f_0}$ to be a series of straight lines having the slope given by (2) above over intervals Δx of x made sufficiently small that the resultant straight line approximation would approach the true phase curve to the desired degree of accuracy for the table contemplated.

In order to evaluate the errors involved in such a procedure let us refer to Fig. 3 where a segment of the desired phase function to be constructed is qualitatively represented on a large scale. It is assumed that the phase at x_c, $B(x_c)$, is known and that it is desired to determine the error $\delta_1 B$ in phase computed for $x_c + \Delta x$ when it is assumed that the phase curve is a straight line from $B(x_c)$ at x_c, to $x_c + \Delta x$ having a slope, $\frac{dB}{dx}\left(x_c + \frac{\Delta x}{2}\right)$, the slope of the true phase curve at $x = x_c + \frac{\Delta x}{2}$.

Then:

$$\delta_1 B = B(x_c + \Delta x) - B(x_c) - \frac{dB}{dx}\left(x_c + \frac{\Delta x}{2}\right)\Delta x \tag{3}$$

where:

$$B(x_c) = \frac{2}{\pi}\left[x_c + \frac{x_c^3}{9} + \frac{x_c^5}{25} + \cdots\right]$$

$$B(x_c + \Delta x) = \frac{2}{\pi}\left[(x_c + \Delta x) + \tfrac{1}{9}(x_c^3 + 3x_c^2\Delta x + 3x_c\Delta x^2 + \Delta x^3) + \tfrac{1}{25}(x_c^5 + 5x_c^4\Delta x + 10x_c^3\Delta x^2 + 10x_c^2\Delta x^3 + 5x_c\Delta x^4 + \Delta x^5) + \cdots\right]$$

$$B(x_c + \Delta x) - B(x_c) = \frac{2}{\pi}\left[\Delta x + \frac{x_c^2\Delta x}{3} + \frac{x_c\Delta x^2}{3} + \frac{\Delta x^3}{9} + \frac{x_c^4\Delta x}{5} + \frac{2x_c^3\Delta x^2}{5} + \frac{2x_c^2\Delta x^3}{5} + \frac{x_c\Delta x^4}{5} + \frac{\Delta x^5}{25} + \cdots\right]$$

$$= \frac{2}{\pi}\left[\Delta x \sum_{n=1}^{n=\infty}\frac{x_c^{2n-2}}{2n-1} + \Delta x^2 \sum_{n=1}^{n=\infty}\frac{n x_c^{2n-1}}{2n+1} + \Delta x^3 \sum_{n=1}^{n=\infty}\frac{n(2n-1)x_c^{2n-2}}{3(2n+1)} + \cdots\right]$$

$$\frac{dB}{dx}\left(x_c + \frac{\Delta x}{2}\right)\Delta x = \frac{2}{\pi}\Delta x\left(1 + \frac{1}{3}\left[x_c^2 + 2x_c\left(\frac{\Delta x}{2}\right) + \left(\frac{\Delta x}{2}\right)^2\right] + \frac{1}{5}\left[x_c^4 + 4x_c^3\left(\frac{\Delta x}{2}\right) + 6x_c^2\left(\frac{\Delta x}{2}\right)^2 + 4x_c\left(\frac{\Delta x}{2}\right)^3 + \left(\frac{\Delta x}{2}\right)^4\right] + \cdots\right)$$

$$= \frac{2}{\pi}\left[\Delta x + \frac{x_c^2\Delta x}{3} + \frac{x_c\Delta x^2}{3} + \frac{\Delta x^3}{12} + \frac{x_c^4\Delta x}{5} + \frac{2x_c^3\Delta x^2}{5} + \frac{3x_c^2\Delta x^3}{10} + \frac{x_c\Delta x^4}{10} + \frac{\Delta x^5}{80} + \cdots\right]$$

$$= \frac{2}{\pi}\left[\Delta x \sum_{n=1}^{n=\infty}\frac{x_c^{2n-2}}{2n-1} + \Delta x^2 \sum_{n=1}^{n=\infty}\frac{n x_c^{2n-1}}{2n+1} + \Delta x^3 \sum_{n=1}^{n=\infty}\frac{n(2n-1)x_c^{2n-2}}{4(2n+1)} + \cdots\right].$$

Since Δx will be small compared to unity and since an error function is being computed it is permissible to take only the 1st term of the difference between the true phase and the computed phase, i.e. the Δx^3 term, and drop all higher order terms of Δx.
Then:

$$\delta_1 B \doteq \frac{2}{\pi}\left[\Delta x^3 \sum_{n=1}^{n=\infty} \frac{n(2n-1)x_c^{2n-2}}{3(2n+1)} - \Delta x^3 \sum_{n=1}^{n=\infty} \frac{n(2n-1)x_c^{2n-2}}{4(2n+1)}\right]$$
$$= \frac{\Delta x^3}{6\pi} \sum_{n=1}^{n=\infty} \frac{n(2n-1)x_c^{2n-2}}{2n+1}. \tag{4}$$

The equation (4) above for $\delta_1 B$ gives only the error for a single increment Δx of $x = f/f_0$. If the phase is known at $x = x_a$ and $x = x_b$ and it is desired to determine the phase at points between $x = x_a$ and $x = x_b$ then since $\delta_1 B$ always has the same sign the errors due to successive increments of x will be cumulative and the total error at $x = x_b$ will be n times the average of the $\delta_1 B$ errors of each increment of Δx between x_a and x_b where n is the total number of equi-increments of x taken between x_a and x_b. However, since the individual $\delta_1 B$ errors decrease as the cube of Δx, the individual errors will decrease as the cube of the number of increments taken between the two frequencies at which the phase is known, whereas the cumulative $\delta_1 B$ error will increase only in proportion to the first power of n. Therefore, the net result will be a vanishing of the cumulative error inversely as the square of the number of frequency increments taken to approximate the curve in the interval in question. It therefore follows that the accuracy of the proposed method of building up the function, in so far as the phase at the terminals of the straight line segments is concerned, is limited only by the number of increments of frequency selected for the summation.

In order to determine the actual magnitude of errors to be expected $\delta_1 B$ was computed for $x_c = .4$ and $\Delta x = .02$ and found to be only .000015 degree. Since the total number of .02 intervals needed to be used between previously computed values of B is 5, the total cumulative error in this region for increments of this magnitude will not be greater than .0001 degree, which is entirely satisfactory, since the accuracy being sought is $\pm$ .0005 degree in B. For $x_c = .9$ and $\Delta x = .005$ the $\delta_1 B$ error proves to be only .00001 degree and since in this region the value of B has already been determined at .01 intervals by the more accurate series expansion technique referred to above, only two increments are necessary between known values of B and therefore the $\delta_1 B$ error is sufficiently small.

Having determined the order of magnitude of intervals necessary to keep $\delta_1 B$ errors small, let us examine the errors due to the departure of the straight line approximation from the true curve in the interval between x_c and $x_c + \Delta x$. Since $\delta_1 B$ will be very small it is anticipated that the maximum value of $\delta_2 B$ (see Fig. 3) will occur in the vicinity of $x_c + \frac{\Delta x}{2}$. $\delta_2 B$ at this point may be determined as shown below.

$$\delta_2 B = B\left(x_c + \frac{\Delta x}{2}\right) - B(x_c) - \frac{dB}{dx}\left(x_c + \frac{\Delta x}{2}\right)\frac{\Delta x}{2} \tag{5}$$

where:

$$B\left(x_c + \frac{\Delta x}{2}\right) - B(x_c) = \frac{2}{\pi}\left[\Delta x \sum_{n=1}^{n=\infty} \frac{x_c^{2n-2}}{2(2n-1)} + \Delta x^2 \sum_{n=1}^{n=\infty} \frac{n x_c^{2n-1}}{4(2n+1)} + \cdots\right]$$

$$\frac{dB}{dx}\left(x_c + \frac{\Delta x}{2}\right)\frac{\Delta x}{2} = \frac{2}{\pi}\left[\Delta x \sum_{n=1}^{n=\infty} \frac{x_c^{2n-2}}{2(2n-1)} + \Delta x^2 \sum_{n=1}^{n=\infty} \frac{n x_c^{2n-1}}{2(2n+1)} + \cdots\right].$$

Again retaining only the first term of the error function and dropping all higher order terms of Δx

$$\begin{aligned}\delta_2 B &\doteq \frac{2}{\pi}\left[\Delta x^2 \sum_{n=1}^{n=\infty} \frac{n x_c^{2n-1}}{4(2n+1)} - \Delta x^2 \sum_{n=1}^{n=\infty} \frac{n x_c^{2n-1}}{2(2n+1)}\right] \\ &= -\frac{\Delta x^2}{2\pi} \sum_{n=1}^{n=\infty} \frac{n x_c^{2n-1}}{2n+1}.\end{aligned} \tag{6}$$

$\delta_2 B$ proves to be negative and considerably larger than $\delta_1 B$ for the same magnitude of interval. Therefore the computed B will always exceed the true phase in the interval x_c to $x_c + \Delta x$ except above a value of x very near to $x_c + \Delta x$ where the straight line approximation crosses the true phase curve. When $x_c = .35$ and $\Delta x = .02$, $\delta_2 B$ is found to be $-.0005$ degree from (6) above, and for $x_c = .91$ and $\Delta x = .005$, $\delta_2 B$ is also found to be $-.0005$ degree. The $\delta_2 B$ errors are therefore found to be much more important than the $\delta_1 B$ errors. $\delta_2 B$ errors are not accumulative, however, and therefore increments of Δx of the above order of magnitude prove to be sufficiently small to give the accuracy being sought, namely $\pm$ .0005 degree in B.

An evaluation of the $\delta_1 B$ and $\delta_2 B$ errors for values of x_c greater than .9 is difficult due to the slowness of convergence of the series giving these errors. For values of x_c between .9 and unity, however, the frequency of known values of B determined from (1) above and available as check points is sufficient to check the adequacy of intervals insofar as $\delta_1 B$ errors are concerned. Furthermore an analysis similar to that given above for the determination of the $\delta_1 B$ and $\delta_2 B$ errors shows that an interpolation of the slopes computed for construction of the tables in question, to give the intervening slopes necessary to cut the increments of Δx in half will give check points at $x_c + \frac{\Delta x}{2}$ frequencies, with a $\delta_1 B$ error ($x_c + \frac{\Delta x}{2}$ is then the termination of a straight line segment since the Δx interval has been halved) of comparable order of magnitude to the $\delta_1 B$ error for the original interval selected and therefore small in comparison to the $\delta_2 B$ error for the original Δx interval. This technique was therefore used in checking the adequacy of the intervals in so far as $\delta_2 B$ errors are concerned in the region $x_c = .9$ to $x_c = 1.0$.

Using the procedure outlined above the phase associated with the semi-infinite unit slope of attenuation of Fig. 1 was computed for values of f less than f_0 and is given as a function of f/f_0 in Table I in degrees and in Table III in radians. For values of f greater than f_0 the phase was computed as a function of f_0/f utilizing the odd symmetry behavior of the phase characteristic of Fig. 2 on opposite sides of $f/f_0 = 1$, and this phase is tabulated in Table II in degrees and in Table IV in radians. For the other type of semi-infinite unit slope of attenuation in which the attenuation slope is constant and equal to unity at all frequencies below f_0 and the attenuation is constant for all frequencies above f_0 (with the constant slope of attenuation intersecting the f_0 axis at the same point as the constant attenuation line) the same tables can be used by reading the values of phase for $f/f_0 < 1$ from the f_0/f tables and the values of phase for $f_0/f < 1$ from the f/f_0 tables.

The intervals over which the straight line approximation to the true phase was assumed are given below:

Δx		x_c		
.02	from	.00	to	.40
.01	"	.40	"	.70
.005	"	.70	"	.92
.002	"	.92	"	.98
.001	"	.98	"	.996
.0005	"	.996	"	.998
.0002	"	.998	"	.999
.0001	"	.999	"	.9998
.00005	"	.9998	"	1.0000

The points at which the cumulative sum of the straight line increments of phase was corrected to the phase as determined from (1) above are listed below:

Every	.1	from	.00	to	.40
"	.05	"	.40	"	.80
"	.02	"	.80	"	.90
"	.01	"	.90	"	.99

and at .996, .998, .999, and 1.000

A study of the errors based on the error analysis discussed above indicates that the computed values of B in degrees are accurate to $\pm$ .0005 degree and since there is an additional possibility of $\pm$ .0005 degree error in dropping all figures beyond the third decimal place, the over-all reliability of the degree tables is $\pm$ .001 degree. Similarly the computed values of B in radians are accurate to $\pm$ .00001 radian and since there is an additional possibility of $\pm$ .000005 radian error in dropping all figures beyond the fifth decimal place, the over-all reliability of the radian tables is $\pm$ .000015 radian. Since the function tabulated was constructed by a series of straight line approximations to the true phase, interpolation to get the phase for values of f/f_0 or f_0/f between those given in the tables in problems where this is necessary, will result in the same accuracy as that given for the tabulated values.

Murlan S. Corrington[7] of Radio Corporation of America has computed the phase in radians for the semi-infinite unit slope of attenuation of Fig. 1 for approximately 100 values of f/f_0 using equations 15-9 and 15-11 of Bode's "Network Analysis and Feedback Amplifier Design" and has given a table of these values to five decimal places. Where the values of Table III differ from Corrington's values, his value is given as a superscript. Since his approach is the more exact one, it is assumed that where a difference exists, his value is correct. The differences have a maximum value of one figure in the fifth decimal place which is consistent with the accuracy of $\pm$ .000015 radian given for Table III. However, linear interpolation of Corrington's values to get the function to three figures in f/f_0, which precision in f/f_0 is really needed to utilize five figure accuracy in B, will result in errors considerably larger than those of Table III for the higher values of f/f_0.

Acknowledgment

The writer wishes to thank Miss J. D. Goeltz who carried out the calculations of the basic Tables and of the illustrative examples of this paper.

APPENDIX

Use of Tables I to IV in Determining Phase from Attenuation or Reactance from Resistance

The first step in determining the phase associated with a given attenuation characteristic using the tables described in the basic paper is to plot the attenuation as a function of log frequency to a suitable scale. Such an attenuation characteristic is illustrated in Fig. 4a. The attenuation characteristic is then approximated by a series of straight lines such as are shown in dotted form. The number of straight lines used will depend upon the accuracy desired in the resultant phase. As a rule, an approximation to the attenuation which does not depart by more than $\pm$ .5 *db* will give a resultant phase which does not depart by more than $\pm$ 3° from the true phase.

If we now examine the straight line attenuation approximation of Fig. 4a,

[7] Murlan S. Corrington, "Table of the Integral $\frac{2}{\pi}\int_0^x \frac{\tanh^{-1} t}{t} dt$" *R.C.A. Review* September, 1946, page 432.

we see that it can be constructed by adding a number of semi-infinite constant slopes of attenuation as shown in Fig. 4b. The first of these will be a semi-infinite slope of magnitude k_1 commencing at the first critical frequency

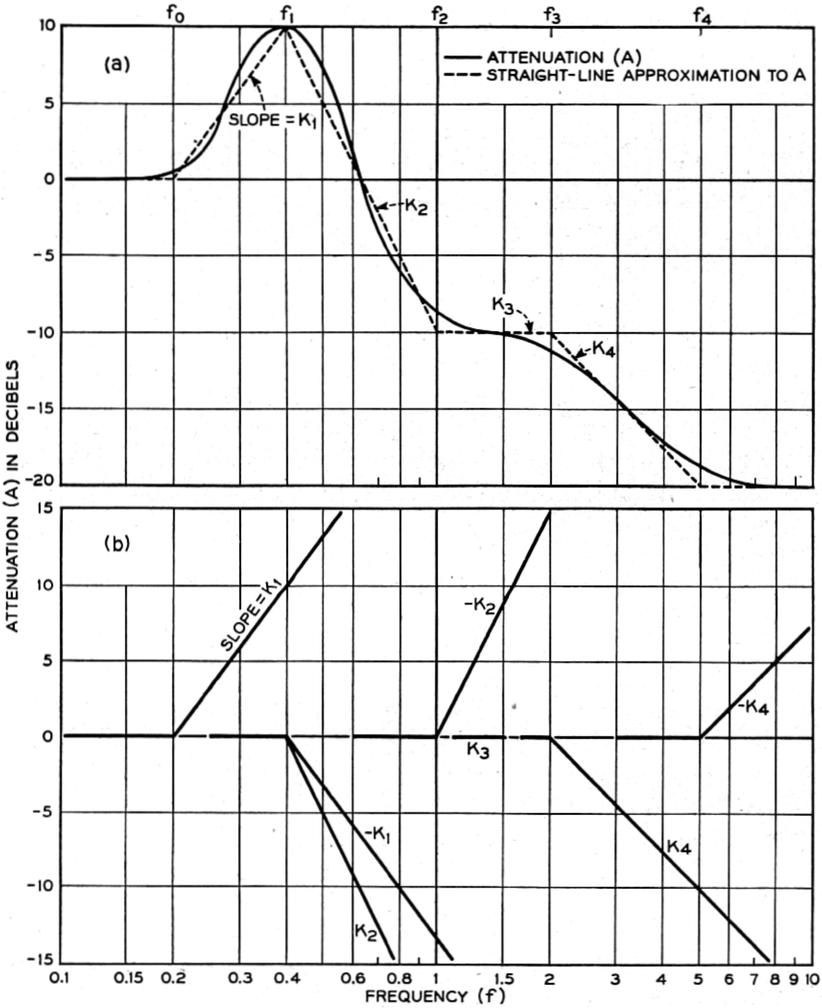

Fig. 4—(a) Straight line approximation to attenuation characteristic. (b) Individual semi-infinite constant slopes of attenuation which add to produce the straight line approximation of Fig. 4(a).

f_0. The second will be a semi-infinite slope of magnitude $-k_1$ commencing at the critical frequency f_1 which must be added to correct for the fact that the first straight line of slope $+k_1$ does not extend to infinity, but terminates at the critical frequency f_1, where the straight line approximation assumes a new slope. In order to achieve this new slope a semi-infinite slope of magnitude k_2, commencing at frequency f_1, must be added. This process is continued up the frequency scale until the entire straight line approximation is constructed.

The total phase $\theta(f)$ at a particular frequency f is then given by the sum of the phase at frequency f associated with each of the semi-infinite constant slopes of attenuation which together make up the straight line approxmation.

Thus:

$$\theta(f) = k_1\theta_0 - k_1\theta_1 + k_2\theta_1 - k_2\theta_2 + k_3\theta_2 - k_3\theta_3 + k_4\theta_3 - k_4\theta_4$$

or for the general straight line approximation having slopes

$$k_1, k_2, \cdots k_n$$

$$\theta(f) = k_1 (\theta_0 - \theta_1) + k_2 (\theta_1 - \theta_2) + \cdots k_n (\theta_{n-1} - \theta_n)$$

where:

θ_n is the phase at frequency f associated with the semi-infinite unit slope of attenuation commencing at frequency f_n and extending to $f = \infty$ and is read from Tables I or III for $f < f_n$ and Tables II or IV for $f > f_n$,

and

k_n is the slope of the straight line approximation between f_{n-1} and f_n given by:

$$k_n = \frac{A_n - A_{n-1}}{20 \log \dfrac{f_n}{f_{n-1}}}$$

where:

A_n is the attenuation at frequency f_n on the straight line approximation.

Note that in Fig. 4a the attenuation is constant from zero frequency to the first critical frequency f_0. In many problems, there is a constant slope below frequency f_1 to frequency zero. In that event, the initial critical frequency, f_0, will be zero, and θ_0 will be 90°. ($f_0/f = 0$ at all finite frequencies.) When this occurs, k_1 must be determined by choosing a finite frequency f_0' and taking the ratio of attenuation change between f_0' and f_1 to 20 log of the ratio of f_1 to f_0'. Similarly, the attenuation is constant in the illustration from the top critical frequency f_4 to infinity, whereas in many problems the attenuation will have a constant slope extending from the top critical frequency to infinity. In these cases, the top critical frequency will be infinity and the final angle θ_n will, of course, be zero. Here again the final slope k_n must be determined over a finite portion of this infinite slope.

It will also be noted that in the illustration given the characteristic is approximated, commencing at zero frequency, by a series of semi-infinite slopes, each of which is a constant times the characteristic of Fig. 1 of the basic paper, for which Tables I to IV were computed. The characteristic could have been approximated just as well with a series of semi-infinite constant slopes, commencing at $f = \infty$ and going down in frequency, each having a flat attenuation above a critical frequency f_n and constant slope at frequencies below. In summing the phase for such an approximation Tables I to IV may be used by reading the angles for f/f_n from the f_0/f tables and vice versa as indicated in the basic paper.

As an illustration of the above procedure, consider the determination of the phase associated with the characteristic given by 20 log | Z | shown in Fig. 5. The characteristic is first approximated by a series of straight lines as shown in dotted form. The critical frequencies and values of $A = 20$ log | Z | at these critical frequencies are then read from the straight line approximation[8] and the slopes of the various straight line segments determined as illustrated in Table V.

Having determined the slopes of the various segments of the straight line approximation, the phase at any desired frequency is summed as illustrated in Table VI where the phase for $f = 1.5$ is summed.

The mesh computed value of θ for the network in question is plotted in Fig. 6 and it will be noted that the phase summation of Table VI checks the true value to within the accuracy to which the phase can be read from the curve. The identical procedure is followed in determining the phase at any other frequency. As an illustration of the accuracy of the method, the phase was determined at a considerable number of frequencies and the results shown as individual points in Fig. 6. The straight line approximation to 20 log | Z | of Fig. 5 was of the order of $\pm$ .25 *db* and, in accordance with the estimated accuracy of the method given above, the maximum departure of the phase summation from the true phase is approximately $\pm$ 1.5°.

A much simpler approximation than that of Fig. 5 may be used without a great loss in accuracy. For instance, a five-line approximation determined by the critical frequencies of Table VII will match 20 log | Z | to within approximately $\pm$ .5 *db* and therefore should give a phase summation within $\pm$ 3° of the true phase. The phase was actually summed at 12 frequencies chosen at random for this five-line approximation and the maximum departure of the summed phase from the true phase was 3.2°. With experience in use of the method, simpler approximations can be used and the phase determined more accurately than the limits of accuracy of the summation at individual frequencies by plotting the individual summations

[8] The original plot was expanded and had much greater scale detail than can be shown with clarity on a single page plate.

TABLE I—DEGREES PHASE (±.001°) FOR SEMI-INFINITE ATTENUATION SLOPE $k = 1 f < f_0$

f/f_0	0	1	2	3	4	5	6	7	8	9
.00	.000	.036	.073	.109	.146	.182	.219	.255	.292	.328
.01	.365	.401	.438	.474	.511	.547	.584	.620	.657	.693
.02	.730	.766	.803	.839	.875	.912	.948	.985	1.021	1.058
.03	1.094	1.131	1.167	1.204	1.240	1.277	1.313	1.350	1.386	1.423
.04	1.459	1.496	1.532	1.569	1.605	1.642	1.678	1.715	1.751	1.788
.05	1.824	1.861	1.897	1.934	1.970	2.007	2.043	2.080	2.116	2.153
.06	2.189	2.226	2.262	2.299	2.335	2.372	2.409	2.445	2.482	2.518
.07	2.555	2.591	2.628	2.664	2.701	2.737	2.774	2.810	2.847	2.884
.08	2.920	2.957	2.993	3.030	3.066	3.103	3.140	3.176	3.213	3.249
.09	3.286	3.322	3.359	3.396	3.432	3.469	3.505	3.542	3.578	3.615
.10	3.652	3.688	3.725	3.762	3.798	3.835	3.871	3.908	3.945	3.981
.11	4.018	4.054	4.091	4.128	4.164	4.201	4.238	4.274	4.311	4.347
.12	4.384	4.421	4.457	4.494	4.531	4.568	4.604	4.641	4.678	4.714
.13	4.751	4.788	4.824	4.861	4.898	4.934	4.971	5.008	5.044	5.081
.14	5.118	5.155	5.191	5.228	5.265	5.302	5.338	5.375	5.412	5.449
.15	5.485	5.522	5.559	5.596	5.632	5.669	5.706	5.743	5.779	5.816
.16	5.853	5.890	5.927	5.963	6.000	6.037	6.074	6.111	6.148	6.184
.17	6.221	6.258	6.295	6.332	6.369	6.405	6.442	6.479	6.516	6.553
.18	6.590	6.626	6.663	6.700	6.737	6.774	6.811	6.848	6.885	6.922
.19	6.959	6.996	7.033	7.070	7.106	7.143	7.180	7.217	7.254	7.291
.20	7.328	7.365	7.402	7.439	7.476	7.513	7.550	7.587	7.624	7.661
.21	7.698	7.735	7.772	7.809	7.846	7.883	7.920	7.957	7.994	8.032
.22	8.069	8.106	8.143	8.180	8.217	8.254	8.291	8.329	8.366	8.403
.23	8.440	8.477	8.514	8.551	8.589	8.626	8.663	8.700	8.737	8.774
.24	8.811	8.849	8.886	8.923	8.960	8.998	9.035	9.072	9.109	9.147
.25	9.184	9.221	9.259	9.296	9.333	9.370	9.408	9.445	9.482	9.519
.26	9.557	9.594	9.631	9.669	9.706	9.744	9.781	9.818	9.856	9.893
.27	9.931	9.968	10.006	10.043	10.080	10.118	10.155	10.193	10.230	10.267
.28	10.305	10.342	10.380	10.417	10.455	10.492	10.530	10.568	10.605	10.643
.29	10.680	10.718	10.755	10.793	10.830	10.868	10.906	10.943	10.981	11.018
.30	11.056	11.094	11.131	11.169	11.207	11.244	11.282	11.320	11.358	11.395
.31	11.433	11.471	11.508	11.546	11.584	11.622	11.659	11.697	11.735	11.772
.32	11.810	11.848	11.886	11.924	11.962	12.000	12.037	12.075	12.113	12.151
.33	12.189	12.227	12.265	12.303	12.341	12.379	12.416	12.454	12.492	12.530
.34	12.568	12.606	12.644	12.682	12.720	12.758	12.797	12.835	12.873	12.911
.35	12.949	12.987	13.025	13.063	13.101	13.139	13.177	13.215	13.254	13.292
.36	13.330	13.368	13.406	13.445	13.483	13.521	13.559	13.598	13.636	13.674
.37	13.713	13.751	13.789	13.827	13.866	13.904	13.942	13.981	14.019	14.057
.38	14.096	14.134	14.173	14.211	14.250	14.288	14.327	14.365	14.404	14.442
.39	14.481	14.519	14.558	14.596	14.635	14.673	14.712	14.750	14.789	14.827
.40	14.866	14.905	14.943	14.982	15.021	15.059	15.098	15.137	15.175	15.214
.41	15.253	15.292	15.330	15.369	15.408	15.447	15.486	15.525	15.563	15.602
.42	15.641	15.680	15.719	15.758	15.797	15.836	15.875	15.914	15.953	15.991
.43	16.030	16.070	16.109	16.148	16.187	16.226	16.265	16.304	16.343	16.382
.44	16.421	16.460	16.500	16.539	16.578	16.617	16.657	16.696	16.735	16.774
.45	16.813	16.853	16.892	16.931	16.971	17.010	17.050	17.089	17.128	17.168
.46	17.207	17.247	17.286	17.326	17.365	17.405	17.444	17.484	17.523	17.563
.47	17.602	17.642	17.681	17.721	17.761	17.800	17.840	17.880	17.919	17.959
.48	17.999	18.039	18.078	18.118	18.158	18.198	18.238	18.277	18.317	18.357
.49	18.397	18.437	18.477	18.517	18.557	18.597	18.637	18.677	18.717	18.757
.50	18.797	18.837	18.877	18.917	18.958	18.998	19.038	19.078	19.118	19.158
.51	19.198	19.239	19.279	19.320	19.360	19.400	19.441	19.481	19.521	19.562
.52	19.602	19.642	19.683	19.723	19.764	19.804	19.845	19.885	19.926	19.967
.53	20.007	20.048	20.088	20.129	20.170	20.211	20.251	20.292	20.333	20.373
.54	20.414	20.455	20.496	20.537	20.578	20.619	20.660	20.701	20.741	20.782

TABLE I—*Continued*

f/f_0	0	1	2	3	4	5	6	7	8	9
.55	20.823	20.864	20.906	20.947	20.988	21.029	21.070	21.111	21.152	21.193
.56	21.234	21.276	21.317	21.358	21.400	21.441	21.482	21.524	21.565	21.606
.57	21.648	21.689	21.731	21.772	21.814	21.855	21.897	21.939	21.980	22.022
.58	22.063	22.105	22.147	22.189	22.230	22.272	22.314	22.356	22.397	22.439
.59	22.481	22.523	22.565	22.607	22.649	22.691	22.733	22.775	22.817	22.859
.60	22.901	22.943	22.986	23.028	23.070	23.112	23.155	23.197	23.239	23.281
.61	23.324	23.366	23.409	23.451	23.494	23.536	23.579	23.621	23.664	23.706
.62	23.749	23.792	23.834	23.877	23.920	23.963	24.006	24.048	24.091	24.134
.63	24.177	24.220	24.263	24.306	24.349	24.392	24.435	24.478	24.521	24.564
.64	24.607	24.651	24.694	24.738	24.781	24.824	24.868	24.911	24.954	24.998
.65	25.041	25.085	25.128	25.172	25.216	25.259	25.303	25.347	25.390	25.434
.66	25.478	25.522	25.566	25.610	25.654	25.698	25.742	25.786	25.830	25.873
.67	25.917	25.962	26.006	26.050	26.095	26.139	26.183	26.228	26.272	26.316
.68	26.361	26.405	26.450	26.494	26.539	26.584	26.628	26.673	26.718	26.762
.69	26.807	26.852	26.897	26.942	26.987	27.032	27.077	27.122	27.167	27.212
.70	27.257	27.302	27.348	27.393	27.438	27.484	27.529	27.574	27.620	27.665
.71	27.711	27.757	27.802	27.848	27.894	27.939	27.985	28.031	28.077	28.123
.72	28.169	28.215	28.261	28.307	28.353	28.399	28.445	28.492	28.538	28.584
.73	28.631	28.677	28.724	28.770	28.817	28.863	28.910	28.957	29.003	29.050
.74	29.097	29.144	29.191	29.238	29.285	29.332	29.379	29.426	29.473	29.521
.75	29.568	29.615	29.663	29.710	29.757	29.805	29.853	29.900	29.948	29.996
.76	30.043	30.091	30.139	30.187	30.235	30.283	30.331	30.379	30.428	30.476
.77	30.524	30.572	30.621	30.669	30.718	30.766	30.815	30.864	30.913	30.961
.78	31.010	31.059	31.108	31.157	31.206	31.255	31.305	31.354	31.403	31.453
.79	31.502	31.551	31.601	31.651	31.700	31.750	31.800	31.850	31.900	31.950
.80	32.000	32.050	32.100	32.150	32.201	32.251	32.301	32.352	32.403	32.453
.81	32.504	32.555	32.606	32.657	32.707	32.758	32.810	32.861	32.912	32.963
.82	33.015	33.066	33.118	33.170	33.221	33.273	33.325	33.377	33.429	33.481
.83	33.533	33.586	33.638	33.690	33.743	33.795	33.848	33.901	33.954	34.006
.84	34.059	34.113	34.166	34.219	34.272	34.325	34.379	34.433	34.486	34.540
.85	34.594	34.648	34.702	34.756	34.810	34.865	34.919	34.974	35.028	35.083
.86	35.138	35.193	35.248	35.303	35.358	35.413	35.469	35.524	35.580	35.636
.87	35.691	35.747	35.804	35.860	35.916	35.972	36.029	36.086	36.142	36.199
.88	36.256	36.313	36.370	36.428	36.485	36.542	36.600	36.658	36.716	36.774
.89	36.832	36.891	36.949	37.008	37.067	37.125	37.184	37.244	37.303	37.362
.90	37.422	37.482	37.542	37.602	37.662	37.722	37.783	37.844	37.904	37.965
.91	38.026	38.088	38.149	38.211	38.273	38.334	38.397	38.459	38.522	38.584
.92	38.647	38.710	38.773	38.837	38.901	38.965	39.029	39.093	39.157	39.222
.93	39.287	39.352	39.418	39.483	39.549	39.615	39.681	39.748	39.815	39.882
.94	39.949	40.017	40.085	40.153	40.221	40.290	40.359	40.428	40.497	40.567
.95	40.638	40.708	40.779	40.850	40.921	40.993	41.066	41.138	41.211	41.285
.96	41.358	41.432	41.507	41.582	41.657	41.733	41.809	41.887	41.964	42.042
.97	42.120	42.199	42.278	42.359	42.439	42.521	42.603	42.686	42.769	42.854
.98	42.938	43.024	43.111	43.199	43.288	43.378	43.469	43.561	43.655	43.750
.99	43.846	43.945	44.045	44.148	44.253	44.361	44.473	(refer to table below)		

.9960	44.473	.9984	44.763	.9992	44.871
.9965	44.530	.9985	44.776	.9993	44.886
.9970	44.589	.9986	44.789	.9994	44.900
.9975	44.649	.9987	44.802	.9995	44.915
.9980	44.711	.9988	44.816	.9996	44.931
.9981	44.724	.9989	44.829	.9997	44.946
.9982	44.737	.9990	44.843	.9998	44.963
.9983	44.750	.9991	44.857	.9999	44.980
				1.0000	45.000

TABLE II—DEGREES PHASE (±.001°) FOR SEMI-INFINITE ATTENUATION SLOPE $k = 1$

$f > f_0$

f_0/f	0	1	2	3	4	5	6	7	8	9
.00	90.000	89.964	89.927	89.891	89.854	89.818	89.781	89.745	89.708	89.672
.01	89.635	89.599	89.562	89.526	89.489	89.453	89.416	89.380	89.343	89.307
.02	89.270	89.234	89.197	89.161	89.125	89.088	89.052	89.015	88.979	88.942
.03	88.906	88.869	88.833	88.796	88.760	88.723	88.687	88.650	88.614	88.577
.04	88.541	88.504	88.468	88.431	88.395	88.358	88.322	88.285	88.249	88.212
.05	88.176	88.139	88.103	88.066	88.030	87.993	87.957	87.920	87.884	87.847
.06	87.811	87.774	87.738	87.701	87.665	87.628	87.591	87.555	87.518	87.482
.07	87.445	87.409	87.372	87.336	87.299	87.263	87.226	87.190	87.153	87.116
.08	87.080	87.043	87.007	86.970	86.934	86.897	86.860	86.824	86.787	86.751
.09	86.714	86.678	86.641	86.604	86.568	86.531	86.495	86.458	86.422	86.385
.10	86.348	86.312	86.275	86.238	86.202	86.165	86.129	86.092	86.055	86.019
.11	85.982	85.946	85.909	85.872	85.836	85.799	85.762	85.726	85.689	85.653
.12	85.616	85.579	85.543	85.506	85.469	85.432	85.396	85.359	85.322	85.286
.13	85.249	85.212	85.176	85.139	85.102	85.066	85.029	84.992	84.956	84.919
.14	84.882	84.845	84.809	84.772	84.735	84.698	84.662	84.625	84.588	84.551
.15	84.515	84.478	84.441	84.404	84.368	84.331	84.294	84.257	84.221	84.184
.16	84.147	84.110	84.073	84.037	84.000	83.963	83.926	83.889	83.852	83.816
.17	83.779	83.742	83.705	83.668	83.631	83.595	83.558	83.521	83.484	83.447
.18	83.410	83.374	83.337	83.300	83.263	83.226	83.189	83.152	83.115	83.078
.19	83.041	83.004	82.967	82.930	82.894	82.857	82.820	82.783	82.746	82.709
.20	82.672	82.635	82.598	82.561	82.524	82.487	82.450	82.413	82.376	82.339
.21	82.302	82.265	82.228	82.191	82.154	82.117	82.080	82.043	82.006	81.968
.22	81.931	81.894	81.857	81.820	81.783	81.746	81.709	81.671	81.634	81.597
.23	81.560	81.523	81.486	81.449	81.411	81.374	81.337	81.300	81.263	81.226
.24	81.189	81.151	81.114	81.077	81.040	81.002	80.965	80.928	80.891	80.853
.25	80.816	80.779	80.741	80.704	80.667	80.630	80.592	80.555	80.518	80.481
.26	80.443	80.406	80.369	80.331	80.294	80.256	80.219	80.182	80.144	80.107
.27	80.069	80.032	79.994	79.957	79.920	79.882	79.845	79.807	79.770	79.733
.28	79.695	79.658	79.620	79.583	79.545	79.508	79.470	79.432	79.395	79.357
.29	79.320	79.282	79.245	79.207	79.170	79.132	79.094	79.057	79.019	78.982
.30	78.944	78.906	78.869	78.831	78.793	78.756	78.718	78.680	78.642	78.605
.31	78.567	78.529	78.492	78.454	78.416	78.378	78.341	78.303	78.265	78.228
.32	78.190	78.152	78.114	78.076	78.038	78.000	77.963	77.925	77.887	77.849
.33	77.811	77.773	77.735	77.697	77.659	77.621	77.584	77.546	77.508	77.470
.34	77.432	77.394	77.356	77.318	77.280	77.242	77.203	77.165	77.127	77.089
.35	77.051	77.013	76.975	76.937	76.899	76.861	76.823	76.785	76.746	76.708
.36	76.670	76.632	76.594	76.555	76.517	76.479	76.441	76.402	76.364	76.326
.37	76.287	76.249	76.211	76.173	76.134	76.096	76.058	76.019	75.981	75.943
.38	75.904	75.866	75.827	75.789	75.750	75.712	75.673	75.635	75.596	75.558
.39	75.519	75.481	75.442	75.404	75.365	75.327	75.288	75.250	75.211	75.173
.40	75.134	75.095	75.057	75.018	74.979	74.941	74.902	74.863	74.825	74.786
.41	74.747	74.708	74.670	74.631	74.592	74.553	74.514	74.475	74.437	74.398
.42	74.359	74.320	74.281	74.242	74.203	74.164	74.125	74.086	74.047	74.009
.43	73.970	73.930	73.891	73.852	73.813	73.774	73.735	73.696	73.657	73.618
.44	73.579	73.540	73.500	73.461	73.422	73.383	73.343	73.304	73.265	73.226
.45	73.187	73.147	73.108	73.069	73.029	72.990	72.950	72.911	72.872	72.832
.46	72.793	72.753	72.714	72.674	72.635	72.595	72.556	72.516	72.477	72.437
.47	72.398	72.358	72.319	72.279	72.239	72.200	72.160	72.120	72.081	72.041
.48	72.001	71.961	71.922	71.882	71.842	71.802	71.762	71.723	71.683	71.643
.49	71.603	71.563	71.523	71.483	71.443	71.403	71.363	71.323	71.283	71.243
.50	71.203	71.163	71.123	71.083	71.042	71.002	70.962	70.922	70.882	70.842
.51	70.802	70.761	70.721	70.680	70.640	70.600	70.559	70.519	70.479	70.438
.52	70.398	70.358	70.317	70.277	70.236	70.196	70.155	70.115	70.074	70.033
.53	69.993	69.952	69.912	69.871	69.830	69.789	69.749	69.708	69.667	69.627
.54	69.586	69.545	69.504	69.463	69.422	69.381	69.340	69.299	69.259	69.218

f_0/f	0	1	2	3	4	5	6	7	8	9
.55	69.177	69.136	69.094	69.053	69.012	68.971	68.930	68.889	68.848	68.807
.56	68.766	68.724	68.683	68.642	68.600	68.559	68.518	68.476	68.435	68.394
.57	68.352	68.311	68.269	68.228	68.186	68.145	68.103	68.061	68.020	67.978
.58	67.937	67.895	67.853	67.811	67.770	67.728	67.686	67.644	67.603	67.561
.59	67.519	67.477	67.435	67.393	67.351	67.309	67.267	67.225	67.183	67.141
.60	67.099	67.057	67.014	66.972	66.930	66.888	66.845	66.803	66.761	66.719
.61	66.676	66.634	66.591	66.549	66.506	66.464	66.421	66.379	66.336	66.294
.62	66.251	66.208	66.166	66.123	66.080	66.037	65.994	65.952	65.909	65.866
.63	65.823	65.780	65.737	65.694	65.651	65.608	65.565	65.522	65.479	65.436
.64	65.393	65.349	65.306	65.262	65.219	65.176	65.132	65.089	65.046	65.002
.65	64.959	64.915	64.872	64.828	64.784	64.741	64.697	64.653	64.610	64.566
.66	64.522	64.478	64.434	64.390	64.346	64.302	64.258	64.214	64.170	64.127
.67	64.083	64.038	63.994	63.950	63.905	63.861	63.817	63.772	63.728	63.684
.68	63.639	63.595	63.550	63.506	63.461	63.416	63.372	63.327	63.282	63.238
.69	63.193	63.148	63.103	63.058	63.013	62.968	62.923	62.878	62.833	62.788
.70	62.743	62.698	62.652	62.607	62.562	62.516	62.471	62.426	62.380	62.335
.71	62.289	62.243	62.198	62.152	62.106	62.061	62.015	61.969	61.923	61.877
.72	61.831	61.785	61.739	61.693	61.647	61.601	61.555	61.508	61.462	61.416
.73	61.369	61.323	61.276	61.230	61.183	61.137	61.090	61.043	60.997	60.950
.74	60.903	60.856	60.809	60.762	60.715	60.668	60.621	60.574	60.527	60.479
.75	60.432	60.385	60.337	60.290	60.243	60.195	60.147	60.100	60.052	60.004
.76	59.957	59.909	59.861	59.813	59.765	59.717	59.669	59.621	59.572	59.524
.77	59.476	59.428	59.379	59.331	59.282	59.234	59.185	59.136	59.087	59.039
.78	58.990	58.941	58.892	58.843	58.794	58.745	58.695	58.646	58.597	58.547
.79	58.498	58.449	58.399	58.349	58.300	58.250	58.200	58.150	58.100	58.050
.80	58.000	57.950	57.900	57.850	57.799	57.749	57.699	57.648	57.597	57.547
.81	57.496	57.445	57.394	57.343	57.293	57.242	57.190	57.139	57.088	57.037
.82	56.985	56.934	56.882	56.830	56.779	56.727	56.675	56.623	56.571	56.519
.83	56.467	56.414	56.362	56.310	56.257	56.205	56.152	56.099	56.046	55.994
.84	55.941	55.887	55.834	55.781	55.728	55.675	55.621	55.567	55.514	55.460
.85	55.406	55.352	55.298	55.244	55.190	55.135	55.081	55.026	54.972	54.917
.86	54.862	54.807	54.752	54.697	54.642	54.587	54.531	54.476	54.420	54.364
.87	54.309	54.253	54.196	54.140	54.084	54.028	53.971	53.914	53.858	53.801
.88	53.744	53.687	53.630	53.572	53.515	53.458	53.400	53.342	53.284	53.226
.89	53.168	53.109	53.051	52.992	52.933	52.875	52.816	52.756	52.697	52.638
.90	52.578	52.518	52.458	52.398	52.338	52.278	52.217	52.156	52.096	52.035
.91	51.974	51.912	51.851	51.789	51.727	51.666	51.603	51.541	51.478	51.416
.92	51.353	51.290	51.227	51.163	51.099	51.035	50.971	50.907	50.843	50.778
.93	50.713	50.648	50.582	50.517	50.451	50.385	50.319	50.252	50.185	50.118
.94	50.051	49.983	49.915	49.847	49.779	49.710	49.641	49.572	49.503	49.433
.95	49.362	49.292	49.221	49.150	49.079	49.007	48.934	48.862	48.789	48.715
.96	48.642	48.568	48.493	48.418	48.343	48.267	48.191	48.113	48.036	47.958
.97	47.880	47.801	47.722	47.641	47.561	47.479	47.397	47.314	47.231	47.146
.98	47.062	46.976	46.889	46.801	46.712	46.622	46.531	46.439	46.345	46.250
.99	46.154	46.055	45.955	45.852	45.747	45.639	45.527	(refer to table below)		

.9960	45.527	.9984	45.237	.9992	45.129
.9965	45.470	.9985	45.224	.9993	45.114
.9970	45.411	.9986	45.211	.9994	45.100
.9975	45.351	.9987	45.198	.9995	45.085
.9980	45.289	.9988	45.184	.9996	45.069
.9981	45.276	.9989	45.171	.9997	45.054
.9982	45.263	.9990	45.157	.9998	45.037
.9983	45.250	.9991	45.143	.9999	45.020
				1.0000	45.000

TABLE III—RADIANS PHASE (±.000015) FOR SEMI-INFINITE ATTENUATION SLOPE $k = 1\ f < f_0$

f/f_0	0	1	2	3	4	5	6	7	8	9
.00	0.00000	0.00064	0.00127	0.00191	0.00255	0.00318	0.00382	0.00446	0.00509	0.00573
.01	0.00637	0.00700	0.00764	0.00828	0.00891	0.00955	0.01019	0.01082	0.01146	0.01210
.02	0.01273	0.01337	0.01401	0.01464	0.01528	0.01592	0.01655	0.01719	0.01783	0.01846
.03	0.01910	0.01974	0.02037	0.02101	0.02165	0.02228	0.02292	0.02356	0.02419	0.02483
.04	0.02547	0.02611	0.02674	0.02738	0.02802	0.02865	0.02929	0.02993	0.03057	0.03120
.05	0.03184	0.03248	0.03311	0.03375	0.03439	0.03503	0.03566	0.03630	0.03694	0.03757
.06	0.03821	0.03885	0.03949	0.04012	0.04076	0.04140	0.04204	0.04267	0.04331	0.04395
.07	0.04459	0.04523	0.04586	0.04650	0.04714	0.04778	0.04841	0.04905	0.04969	0.05033
.08	0.05097	0.05160	0.05224	0.05288	0.05352	0.05416	0.05479	0.05543	0.05607	0.05671
.09	0.05735	0.05799	0.05862	0.05926	0.05990	0.06054	0.06118	0.06182	0.06245	0.06309
.10	0.06373	0.06437	0.06501	0.06565	0.06629	0.06693	0.06757	0.06821	0.06885	0.06949
.11	0.07013^{2}	0.07076	0.07140	0.07204	0.07268	0.07332	0.07396	0.07460	0.07524	0.07588
.12	0.07652	0.07716	0.07780	0.07844	0.07908	0.07972	0.08036	0.08100	0.08164	0.08228
.13	0.08292	0.08356	0.08420	0.08484	0.08548	0.08612	0.08676	0.08740	0.08804	0.08868
.14	0.08932	0.08996	0.09061	0.09125	0.09189	0.09253	0.09317	0.09381	0.09445	0.09509
.15	0.09574^{3}	0.09638	0.09702	0.09766	0.09830	0.09894	0.09959	0.10023	0.10087	0.10151
.16	0.10215	0.10279	0.10344	0.10408	0.10472	0.10537	0.10601	0.10665	0.10729	0.10794
.17	0.10858	0.10922	0.10987	0.11051	0.11115	0.11179	0.11244	0.11308	0.11372	0.11437
.18	0.11501	0.11565	0.11630	0.11694	0.11759	0.11823	0.11888	0.11952	0.12016	0.12081
.19	0.12145	0.12210	0.12274	0.12339	0.12403	0.12468	0.12532	0.12596	0.12661	0.12725
.20	0.12790	0.12854	0.12919	0.12984	0.13048	0.13113	0.13178	0.13242	0.13307	0.13371
.21	0.13436	0.13501	0.13565	0.13630	0.13695	0.13759	0.13824	0.13888	0.13953	0.14018
.22	0.14082	0.14147	0.14212	0.14277	0.14342	0.14406	0.14471	0.14536	0.14601	0.14666
.23	0.14730	0.14795	0.14860	0.14925	0.14990	0.15055	0.15119	0.15184	0.15249	0.15314
.24	0.15379	0.15444	0.15509	0.15574	0.15639	0.15704	0.15769	0.15834	0.15899	0.15964
.25	0.16029	0.16094	0.16159	0.16224	0.16289	0.16354	0.16419	0.16484	0.16549	0.16615
.26	0.16680	0.16745	0.16810	0.16875	0.16941	0.17006	0.17071	0.17137	0.17202	0.17267
.27	0.17332	0.17398	0.17463	0.17528	0.17593	0.17659	0.17724	0.17789	0.17855	0.17920
.28	0.17985	0.18051	0.18116	0.18182	0.18247	0.18313	0.18378	0.18444	0.18509	0.18575
.29	0.18641^{0}	0.18706	0.18772	0.18837	0.18903	0.18968	0.19034	0.19099	0.19165	0.19230
.30	0.19296	0.19362	0.19428	0.19493	0.19559	0.19625	0.19691	0.19757	0.19823	0.19888
.31	0.19954	0.20020	0.20086	0.20152	0.20218	0.20283	0.20349	0.20415	0.20481	0.20547
.32	0.20613	0.20679	0.20745	0.20811	0.20877	0.20943	0.21009	0.21076	0.21142	0.21208
.33	0.21274^{3}	0.21340	0.21406	0.21472	0.21538	0.21605	0.21671	0.21737	0.21803	0.21869
.34	0.21935	0.22002	0.22068	0.22135	0.22201	0.22268	0.22334	0.22401	0.22467	0.22534
.35	0.226^{5}0^{9}0^{9}	0.22666	0.22733	0.22799	0.22866	0.22932	0.22999	0.23065	0.23132	0.23198
.36	0.23265	0.23332	0.23398	0.23465	0.23532	0.23599	0.23666	0.23732	0.23799	0.23866
.37	0.23933^{2}	0.24000	0.24067	0.24134	0.24200	0.24267	0.24334	0.24401	0.24468	0.24534
.38	0.24601	0.24669	0.24736	0.24803	0.24870	0.24937	0.25005	0.25072	0.25139	0.25206
.39	0.25274^{3}	0.25341	0.25408	0.25475	0.25542	0.25610	0.25677	0.25744	0.25811	0.25879
.40	0.25946	0.26013	0.26081	0.26148	0.26216	0.26284	0.26351	0.26419	0.26486	0.26554
.41	0.26621	0.26689	0.26757	0.26824	0.26892	0.26960	0.27028	0.27095	0.27163	0.27231
.42	0.27299	0.27367	0.27435	0.27503	0.27571	0.27639	0.27706	0.27774	0.27842	0.27910
.43	0.27978	0.28047	0.28115	0.28183	0.28251	0.28319	0.28388	0.28456	0.28524	0.28592
.44	0.28660^{1}	0.28729	0.28797	0.28866	0.28934	0.29003	0.29071	0.29140	0.29208	0.29276
.45	0.29345	0.29414	0.29482	0.29551	0.29620	0.29688	0.29757	0.29826	0.29894	0.29963
.46	0.30032	0.30101	0.30170	0.30239	0.30308	0.30377	0.30446	0.30515	0.30584	0.30652
.47	0.30721^{2}	0.30791	0.30860	0.30929	0.30998	0.31068	0.31137	0.31206	0.31275	0.31344
.48	0.31414	0.31483	0.31553	0.31622	0.31692	0.31761	0.31831	0.31900	0.31970	0.32039
.49	0.32109	0.32179	0.32248	0.32318	0.32388	0.32458	0.32527	0.32597	0.32667	0.32737
.50	0.32807	0.32877	0.32947	0.33017	0.33087	0.33157	0.33227	0.33297	0.33368	0.33438
.51	0.33508	0.33578	0.33648	0.33719	0.33789	0.33860	0.33930	0.34000	0.34071	0.34141
.52	0.34212	0.34282	0.34353	0.34424	0.34495	0.34565	0.34636	0.34707	0.34777	0.34848
.53	0.34919	0.34990	0.35061	0.35132	0.35203	0.35274	0.35345	0.35416	0.35487	0.35558
.54	0.35629	0.35701	0.35772	0.35844	0.35915	0.35986	0.36058	0.36129	0.36201	0.36272

Superscripts—Corrington's values.

TABLE III—*Continued*

f/f_0	0	1	2	3	4	5	6	7	8	9
.55	0.36343	0.36415	0.36487	0.36559	0.36631	0.36702	0.36774	0.36846	0.36918	0.36989
.56	0.37061	0.37133	0.37205	0.37277	0.37350	0.37422	0.37494	0.37566	0.37638	0.37710
.57	0.37782	0.37855	0.37927	0.38000	0.38072	0.38145	0.38217	0.38290	0.38362	0.38435
.58	0.38507	0.38580	0.38653	0.38726	0.38799	0.38872	0.38945	0.39018	0.39091	0.39164
.59	0.39237	0.39310	0.39383	0.39457	0.39530	0.39603	0.39677	0.39750	0.39823	0.39897
.60	0.39970	0.40044	0.40117	0.40191	0.40265	0.40339	0.40413	0.40486	0.40560	0.40634
.61	0.40708	0.40782	0.40856	0.40930	0.41004	0.41079	0.41153	0.41227	0.41301	0.41375
.62	0.41450	0.41524	0.41599	0.41674	0.41748	0.41823	0.41898	0.41972	0.42047	0.42122
.63	0.42197	0.42272	0.42347	0.42422	0.42497	0.42572	0.42647	0.42723	0.42798	0.42873
.64	0.42948	0.43024	0.43100	0.43175	0.43251	0.43327	0.43402	0.43478	0.43554	0.43629
.65	0.43705	0.43781	0.43857	0.43934	0.44010	0.44086	0.44162	0.44238	0.44315	0.44391
.66	0.44467	0.44544	0.44620	0.44697	0.44774	0.44851	0.44927	0.45004	0.45081	0.45158
.67	0.45234	0.45312	0.45389	0.45466	0.45544	0.45621	0.45698	0.45776	0.45853	0.45930
.68	0.46008	0.46086	0.46164	0.46242	0.46319	0.46397	0.46475	0.46553	0.46631	0.46709
.69	0.46787	0.46866	0.46944	0.47023	0.47101	0.47180	0.47258	0.47337	0.47415	0.47494
.70	0.47573	0.47652	0.47731	0.47810	0.47889	0.47968	0.48047	0.48127	0.48206	0.48285
.71	0.48365	0.48444	0.48524	0.48604	0.48684	0.48763	0.48843	0.48923	0.49004	0.49084
.72	0.49164	0.49244	0.49325	0.49405	0.49485	0.49566	0.49647	0.49727	0.49808	0.49889
.73	0.49970	0.50051	0.50132	0.50213	0.50295	0.50376	0.50457	0.50539	0.50621	0.50702
.74	0.50784	0.50866	0.50948	0.51030	0.51112	0.51194	0.51276	0.51358	0.51441	0.51523
.75	0.51605	0.51688	0.51771	0.51854	0.51937	0.52019	0.52103	0.52186	0.52269	0.52352
.76	0.52436	0.52519	0.52603	0.52686	0.52770	0.52854	0.52938	0.53022	0.53106	0.53190
.77	0.53274[5]	0.53359	0.53444	0.53528	0.53613	0.53697	0.53783	0.53868	0.53953	0.54038
.78	0.54123	0.54208	0.54294	0.54380	0.54465	0.54551	0.54637	0.54723	0.54809	0.54895
.79	0.54981	0.55068	0.55154	0.55241	0.55327	0.55414	0.55501	0.55588	0.55676	0.55763
.80	0.55850	0.55938	0.56025	0.56113	0.56201	0.56288	0.56377	0.56465	0.56553	0.56642
.81	0.56730	0.56819	0.56908	0.56996	0.57085	0.57174	0.57264	0.57353	0.57443	0.57532
.82	0.57622	0.57712	0.57802	0.57892	0.57982	0.58072	0.58163	0.58254	0.58345	0.58436
.83	0.58526	0.58618	0.58709	0.58801	0.58892	0.58984	0.59076	0.59168	0.59260	0.59353
.84	0.59445	0.59538	0.59631	0.59723	0.59816	0.59909	0.60003	0.60097	0.60190	0.60284
.85	0.60378	0.60472	0.60567	0.60661	0.60756	0.60850	0.60945	0.61041	0.61136	0.61231
.86	0.61327	0.61423	0.61519	0.61615	0.61711	0.61808	0.61905	0.62002	0.62099	0.62196
.87	0.62293	0.62391	0.62489	0.62587	0.62685	0.62783	0.62882	0.62981	0.63080	0.63179
.88	0.63278	0.63378	0.63478	0.63578	0.63678	0.63779	0.63880	0.63981	0.64082	0.64183
.89	0.64284	0.64387	0.64489	0.64591	0.64693	0.64796	0.64899	0.65003	0.65106	0.65210
.90	0.65313	0.65418	0.65523	0.65628	0.65733	0.65837	0.65943	0.66050	0.66156	0.66262
.91	0.66368	0.66476	0.66583	0.66691	0.66798	0.66906	0.67015	0.67124	0.67233	0.67343
.92	0.67452	0.67562	0.67672	0.67783	0.67894	0.68006	0.68118	0.68230	0.68342	0.68456
.93	0.68569	0.68683	0.68797	0.68911	0.69026	0.69141	0.69257	0.69373	0.69490	0.69607
.94	0.69724	0.69843	0.69961	0.70080	0.70199	0.70319	0.70439	0.70560	0.70681	0.70804
.95	0.70926	0.71049	0.71172	0.71297	0.71421	0.71547	0.71673	0.71800	0.71927	0.72055
.96	0.72183	0.72313	0.72443	0.72574	0.72706	0.72838	0.72971	0.73106	0.73240	0.73377
.97	0.73513	0.73651	0.73790	0.73930	0.74070	0.74213	0.74356	0.74501	0.74646	0.74794
.98	0.74942	0.75092	0.75243	0.75397	0.75552	0.75709[8]	0.75867	0.76028	0.76192	0.76358
.99	0.76527	0.76698	0.76874[3]	0.77053	0.77236	0.77425	0.77620	(refer to table below)		

.9960	0.77620	.9984	0.78125	.9992	0.78315
.9965	0.77720	.9985	0.78148	.9993	0.78340
.9970	0.77822	.9986	0.78171	.9994	0.78366
.9975	0.77928	.9987	0.78195	.9995	0.78392
.9980	0.78036	.9988	0.78218	.9996	0.78419
.9981	0.78058	.9989	0.78242	.9997	0.78446
.9982	0.78080	.9990	0.78266	.9998	0.78475
.9983	0.78103	.9991	0.78290	.9999	0.78505
				1.0000	0.78540

TABLE IV—RADIANS PHASE (±.000015) FOR SEMI-INFINITE ATTENUATION SLOPE $k = 1$

$f > f_0$

f_0/f	0	1	2	3	4	5	6	7	8	9
.00	1.57080	1.57016	1.56952	1.56889	1.56825	1.56761	1.56698	1.56634	1.56570	1.56507
.01	1.56443	1.56379	1.56316	1.56252	1.56188	1.56125	1.56061	1.55997	1.55934	1.55870
.02	1.55806	1.55743	1.55679	1.55615	1.55552	1.55488	1.55424	1.55361	1.55297	1.55233
.03	1.55170	1.55106	1.55042	1.54979	1.54915	1.54851	1.54787	1.54724	1.54660	1.54596
.04	1.54533	1.54469	1.54405	1.54342	1.54278	1.54214	1.54150	1.54087	1.54023	1.53959
.05	1.53896	1.53832	1.53768	1.53704	1.53641	1.53577	1.53513	1.53450	1.53386	1.53322
.06	1.53258	1.53195	1.53131	1.53067	1.53003	1.52940	1.52876	1.52812	1.52748	1.52685
.07	1.52621	1.52557	1.52493	1.52430	1.52366	1.52302	1.52238	1.52174	1.52111	1.52047
.08	1.51983	1.51919	1.51855	1.51792	1.51728	1.51664	1.51600	1.51536	1.51472	1.51409
.09	1.51345	1.51281	1.51217	1.51153	1.51089	1.51026	1.50962	1.50898	1.50834	1.50770
.10	1.50706	1.50642	1.50578	1.50515	1.50451	1.50387	1.50323	1.50259	1.50195	1.50131
.11	1.50067	1.50003	1.49939	1.49875	1.49811	1.49748	1.49684	1.49620	1.49556	1.49492
.12	1.49428	1.49364	1.49300	1.49236	1.49172	1.49108	1.49044	1.48980	1.48916	1.48852
.13	1.48788	1.48724	1.48660	1.48596	1.48532	1.48468	1.48404	1.48339	1.48275	1.48211
.14	1.48147	1.48083	1.48019	1.47955	1.47891	1.47827	1.47763	1.47698	1.47634	1.47570
.15	1.47506	1.47442	1.47378	1.47314	1.47249	1.47185	1.47121	1.47057	1.46993	1.46929
.16	1.46864	1.46800	1.46736	1.46672	1.46607	1.46543	1.46479	1.46414	1.46350	1.46286
.17	1.46222	1.46157	1.46093	1.46029	1.45964	1.45900	1.45836	1.45772	1.45707	1.45643
.18	1.45579	1.45514	1.45450	1.45385	1.45321	1.45257	1.45192	1.45128	1.45063	1.44999
.19	1.44934	1.44870	1.44805	1.44741	1.44677	1.44612	1.44548	1.44483	1.44419	1.44354
.20	1.44290	1.44225	1.44161	1.44096	1.44031	1.43967	1.43902	1.43837	1.43773	1.43708
.21	1.43644	1.43579	1.43514	1.43450	1.43385	1.43320	1.43256	1.43191	1.43127	1.43062
.22	1.42997	1.42933	1.42868	1.42803	1.42738	1.42673	1.42608	1.42544	1.42479	1.42414
.23	1.42349	1.42284	1.42219	1.42155	1.42090	1.42025	1.41960	1.41895	1.41831	1.41766
.24	1.41701	1.41636	1.41571	1.41506	1.41441	1.41376	1.41311	1.41246	1.41181	1.41116
.25	1.41050	1.40985	1.40920	1.40855	1.40790	1.40725	1.40660	1.40595	1.40530	1.40465
.26	1.40400	1.40335	1.40270	1.40204	1.40139	1.40074	1.40008	1.39943	1.39878	1.39813
.27	1.39747	1.39682	1.39617	1.39551	1.39486	1.39421	1.39356	1.39290	1.39225	1.39160
.28	1.39094	1.39029	1.38963	1.38898	1.38832	1.38767	1.38701	1.38636	1.38570	1.38505
.29	1.38439	1.38374	1.38308	1.38242	1.38177	1.38111	1.38046	1.37980	1.37915	1.37849
.30	1.37784	1.37718	1.37652	1.37586	1.37520	1.37455	1.37389	1.37323	1.37257	1.37191
.31	1.37125	1.37060	1.36994	1.36928	1.36862	1.36796	1.36730	1.36665	1.36599	1.36533
.32	1.36467	1.36401	1.36335	1.36269	1.36203	1.36136	1.36070	1.36004	1.35938	1.35872
.33	1.35806	1.35740	1.35673	1.35607	1.35541	1.35475	1.35409	1.35343	1.35277	1.35210
.34	1.35144	1.35078	1.35011	1.34945	1.34878	1.34812	1.34745	1.34679	1.34613	1.34546
.35	1.34480	1.34413	1.34347	1.34280	1.34214	1.34147	1.34081	1.34014	1.33948	1.33881
.36	1.33815	1.33748	1.33681	1.33614	1.33548	1.33481	1.33414	1.33347	1.33280	1.33213
.37	1.33147	1.33080	1.33013	1.32946	1.32879	1.32812	1.32746	1.32679	1.32612	1.32545
.38	1.32478	1.32411	1.32344	1.32277	1.32209	1.32142	1.32075	1.32008	1.31941	1.31873
.39	1.31806	1.31739	1.31672	1.31604	1.31537	1.31470	1.31403	1.31336	1.31268	1.31201
.40	1.31134	1.31066	1.30999	1.30931	1.30864	1.30796	1.30729	1.30661	1.30594	1.30526
.41	1.30458	1.30391	1.30323	1.30255	1.30187	1.30120	1.30052	1.29984	1.29916	1.29849
.42	1.29781	1.29713	1.29645	1.29577	1.29509	1.29441	1.29373	1.29305	1.29237	1.29169
.43	1.29101	1.29033	1.28965	1.28897	1.28828	1.28760	1.28692	1.28624	1.28556	1.28487
.44	1.28419	1.28351	1.28282	1.28214	1.28145	1.28077	1.28008	1.27940	1.27872	1.27803
.45	1.27735	1.27666	1.27597	1.27529	1.27460	1.27391	1.27323	1.27254	1.27185	1.27116
.46	1.27048	1.26979	1.26910	1.26841	1.26772	1.26703	1.26634	1.26565	1.26496	1.26427
.47	1.26358	1.26289	1.26220	1.26150	1.26081	1.26012	1.25943	1.25874	1.25804	1.25735
.48	1.25666	1.25596	1.25527	1.25457	1.25388	1.25318	1.25249	1.25179	1.25110	1.25040
.49	1.24971	1.24901	1.24831	1.24762	1.24692	1.24622	1.24552	1.24482	1.24413	1.24343
.50	1.24273	1.24203	1.24133	1.24063	1.23992	1.23922	1.23852	1.23782	1.23712	1.23642
.51	1.23572	1.23502	1.23431	1.23361	1.23290	1.23220	1.23150	1.23079	1.23009	1.22938
.52	1.22868	1.22797	1.22726	1.22656	1.22585	1.22514	1.22444	1.22373	1.22302	1.22231
.53	1.22161	1.22090	1.22019	1.21948	1.21876	1.21805	1.21734	1.21663	1.21592	1.21521
.54	1.21450	1.21379	1.21307	1.21236	1.21165	1.21093	1.21022	1.20950	1.20879	1.20808

f_0/f	0	1	2	3	4	5	6	7	8	9
.55	1.20736	1.20664	1.20593	1.20521	1.20449	1.20377	1.20306	1.20234	1.20162	1.20090
.56	1.20019	1.19946	1.19874	1.19802	1.19730	1.19658	1.19586	1.19514	1.19442	1.19369
.57	1.19297	1.19225	1.19152	1.19080	1.19007	1.18935	1.18862	1.18790	1.18717	1.18645
.58	1.18572	1.18499	1.18426	1.18353	1.18280	1.18208	1.18135	1.18062	1.17989	1.17916
.59	1.17843	1.17770	1.17696	1.17623	1.17550	1.17476	1.17403	1.17330	1.17256	1.17183
.60	1.17110	1.17036	1.16962	1.16888	1.16815	1.16741	1.16667	1.16593	1.16519	1.16446
.61	1.16372	1.16298	1.16224	1.16149	1.16075	1.16001	1.15927	1.15853	1.15778	1.15704
.62	1.15630	1.15555	1.15481	1.15406	1.15331	1.15257	1.15182	1.15107	1.15032	1.14958
.63	1.14883	1.14808	1.14733	1.14658	1.14582	1.14507	1.14432	1.14357	1.14282	1.14207
.64	1.14131	1.14056	1.13980	1.13904	1.13829	1.13753	1.13677	1.13602	1.13526	1.13450
.65	1.13375	1.13298	1.13222	1.13146	1.13070	1.12994	1.12917	1.12841	1.12765	1.12689
.66	1.12613	1.12536	1.12459	1.12382	1.12306	1.12229	1.12152	1.12075	1.11999	1.11922
.67	1.11845	1.11768	1.11690	1.11613	1.11536	1.11459	1.11381	1.11304	1.11227	1.11149
.68	1.11072	1.10994	1.10916	1.10838	1.10760	1.10682	1.10604	1.10526	1.10448	1.10371
.69	1.10293	1.10214	1.10135	1.10057	1.09978	1.09900	1.09821	1.09743	1.09664	1.09586
.70	1.09507	1.09428	1.09349	1.09270	1.09191	1.09112	1.09032	1.08953	1.08874	1.08794
.71	1.08715	1.08635	1.08556	1.08476	1.08396	1.08316	1.08236	1.08156	1.08076	1.07996
.72	1.07916	1.07836	1.07755	1.07675	1.07594	1.07514	1.07433	1.07352	1.07271	1.07191
.73	1.07110	1.07029	1.06947	1.06866	1.06785	1.06704	1.06622	1.06541	1.06459	1.06378
.74	1.06296	1.06214	1.06132	1.06050	1.05968	1.05886	1.05804	1.05721	1.05639	1.05557
.75	1.05474	1.05391	1.05309	1.05226	1.05143	1.05060	1.04977	1.04894	1.04811	1.04727
.76	1.04644	1.04560	1.04477	1.04393	1.04309	1.04226	1.04142	1.04058	1.03973	1.03889
.77	1.03805	1.03721	1.03636	1.03551	1.03467	1.03382	1.03297	1.03212	1.03127	1.03042
.78	1.02957	1.02871	1.02786	1.02700	1.02615	1.02529	1.02443	1.02357	1.02271	1.02185
.79	1.02099	1.02012	1.01925	1.01839	1.01752	1.01666	1.01578	1.01491	1.01404	1.01317
.80	1.01230	1.01142	1.01054	1.00967	1.00879	1.00791	1.00703	1.00615	1.00526	1.00438
.81	1.00350	1.00261	1.00172	1.00083	.99994	.99905	.99816	.99726	.99637	.99547
.82	.99458	.99368	.99278	.99188	.99097	.99007	.98916	.98826	.98735	.98644
.83	.98553	.98462	.98370	.98279	.98187	.98096	.98004	.97912	.97819	.97727
.84	.97635	.97542	.97449	.97356	.97263	.97170	.97077	.96983	.96889	.96796
.85	.96702	.96607	.96513	.96418	.96324	.96229	.96134	.96039	.95944	.95848
.86	.95753	.95657	.95561	.95464	.95368	.95272	.95175	.95078	.94981	.94884
.87	.94787	.94689	.94591	.94493	.94395	.94297	.94198	.94099	.93999	.93900
.88	.93801	.93701	.93601	.93501	.93401	.93301	.93200	.93099	.92998	.92896
.89	.92795	.92693	.92591	.92488	.92386	.92284	.92180	.92077	.91973	.91870
.90	.91766	.91662	.91557	.91452	.91347	.91242	.91136	.91030	.90924	.90818
.91	.90712	.90604	.90496	.90389	.90281	.90174	.90064	.89955	.89846	.89737
.92	.89628	.89518	.89407	.89296	.89185	.89074	.88962	.88850	.88737	.88624
.93	.88511	.88397	.88283	.88168	.88054	.87938	.87823	.87706	.87590	.87473
.94	.87355	.87237	.87119	.87000	.86881	.86761	.86641	.86519	.86398	.86276
.95	.86154	.86031	.85907	.85783	.85658	.85533	.85407	.85280	.85153	.85025
.96	.84896	.84766	.84637	.84505	.84374	.84241	.84108	.83974	.83839	.83703
.97	.83567	.83428	.83290	.83150	.83009	.82867	.82724	.82579	.82434	.82286
.98	.82138	.81988	.81836	.81683	.81528	.81371	.81212	.81051	.80888	.80722
.99	.80553	.80381	.80206	.80027	.79844	.79655	.79460	(refer to table below)		

.9960	0.79460	.9984	0.78954	.9992	0.78765
.9965	0.79360	.9985	0.78931	.9993	0.78739
.9970	0.79257	.9986	0.78908	.9994	0.78714
.9975	0.79152	.9987	0.78885	.9995	0.78688
.9980	0.79044	.9988	0.78862	.9996	0.78661
.9981	0.79022	.9989	0.78838	.9997	0.78633
.9982	0.78999	.9990	0.78814	.9998	0.78605
.9983	0.78977	.9991	0.78789	.9999	0.78575
				1.0000	0.78540

of phase versus frequency and drawing a smooth curve weighting the points in accordance with the errors known by experience to occur for various types

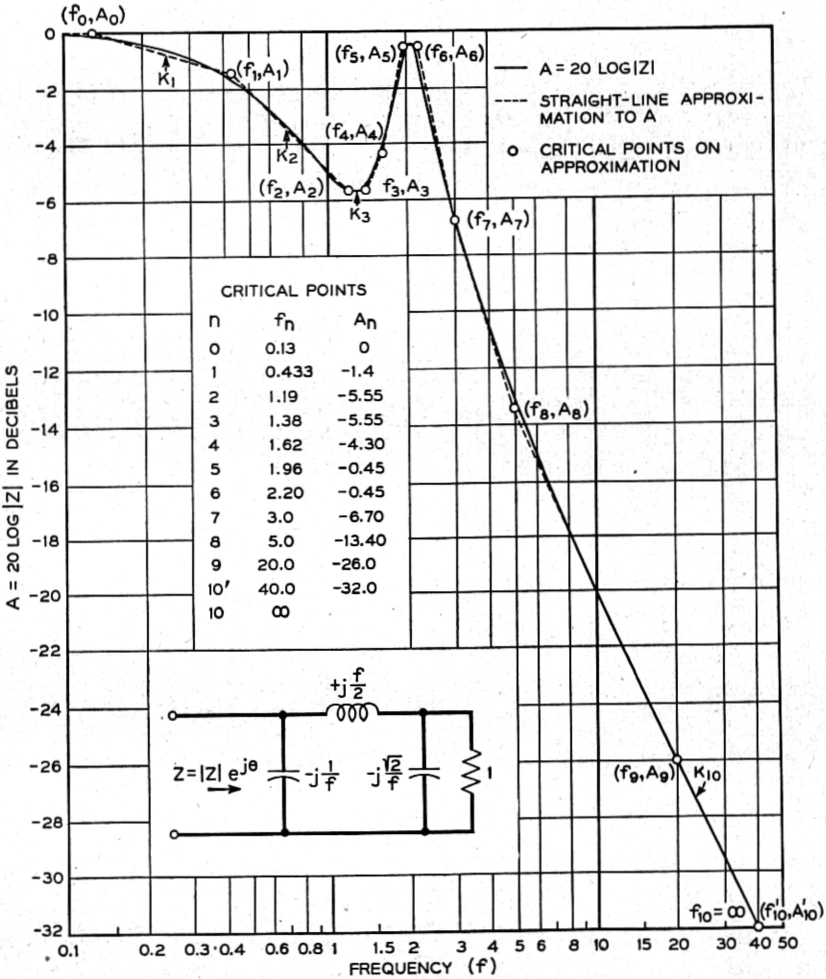

Fig. 5—20 log | Z |.

of departures of the straight line approximation from the exact characteristic.

Although the degree and *db* relationship is applicable to attenuation and phase computations, nepers and radians are proper theoretical units which can be used in other problems[9]. For instance, Tables III and IV give the

[9] Bode, "Network Analysis and Feedback Amplifier Design," Chapter XV, page 340.

reactance in ohms associated with a semi-infinite unit slope of resistance where a unit slope of resistance is one in which a one-ohm change in resistance

TABLE V

TABULATION OF CRITICAL POINTS AND DETERMINATION OF SLOPES OF STRAIGHT LINES APPROXIMATING CHARACTERISTIC OF FIG. 5

n	f_n	A_n	$A_n - A_{n-1}$	$20 \log \frac{f_n}{f_{n-1}}$	k_n
0	.13	0	—	—	—
1	.433	−1.40	−1.40	10.45	−.134
2	1.19	−5.55	−4.15	8.78	−.473
3	1.38	−5.55	.00	1.287	0
4	1.62	−4.30	+1.25	1.393	+.897
5	1.96	−.45	+3.85	1.655	+2.326
6	2.20	−.45	.00	1.003	0
7	3.00	−6.70	−6.25	2.694	−2.320
8	5.00	−13.40	−6.70	4.437	−1.510
9	20.0	−26.00	−12.60	12.04	−1.046
10′	40.0	−32.0	−6.0	6.02	
10	∞				−1.0

Note that $f'_{10} = 40.0$ is chosen to get k_{10} over a finite section of the semi-infinite slope extending to $f = \infty$.

TABLE VI

SUMMATION OF PHASE ASSOCIATED WITH 20 LOG $|Z|$ OF FIG. 5 AT $f = 1.5$

n	f_n from Table V	$\frac{f_n}{f}$	$\frac{f}{f_n}$	θ_n Degrees	$\theta_{n-1} - \theta_n$ Degrees	k_n from Table V	$k_n(\theta_{n-1} - \theta_n)$ Degrees
0	.13	.087		86.824			
1	.433	.289		79.357	7.467	−.134	−1.00
2	1.19	.793		58.349	21.008	−.473	−9.94
3	1.38	.920		51.353	6.996	0	
4	1.62		.926	39.029	12.324	+.897	+11.05
5	1.96		.765	30.283	8.746	+2.326	+20.34
6	2.20		.682	26.450	3.833	0	
7	3.00		.5	18.797	7.653	−2.320	−17.75
8	5.00		.3	11.056	7.741	−1.510	−11.69
9	20.0		.075	2.737	8.319	−1.046	−8.70
10	∞		0	.000	2.737	−1.00	−2.74
						$\Sigma\, k_n\, (\theta_{n-1} - \theta_n) =$	−20.43
						$\theta\ (f = 1.5) =$	−20.4°

Note that for f_0 to f_3 the ratio of f to f_n must be taken f_n/f to be less than unity and θ_n is therefore read from Table II, whereas for f_4 to f_{10} the ratio must be taken f/f_n and θ_n is therefore read from Table I.

occurs between frequencies which are in the ratio $e = 2.7183$. The same technique described above for the determination of the phase associated

TABLE VII
CRITICAL POINTS FOR FIVE LINE APPROXIMATION TO CHARACTERISTIC OF FIG. 5

n	f_n	A_n
0	.25	0
1	1.40	−5.8
2	2.10	0
3	3.00	−7.0
4	10.0	−20.0
5′	40.0	−32.0
5	∞	

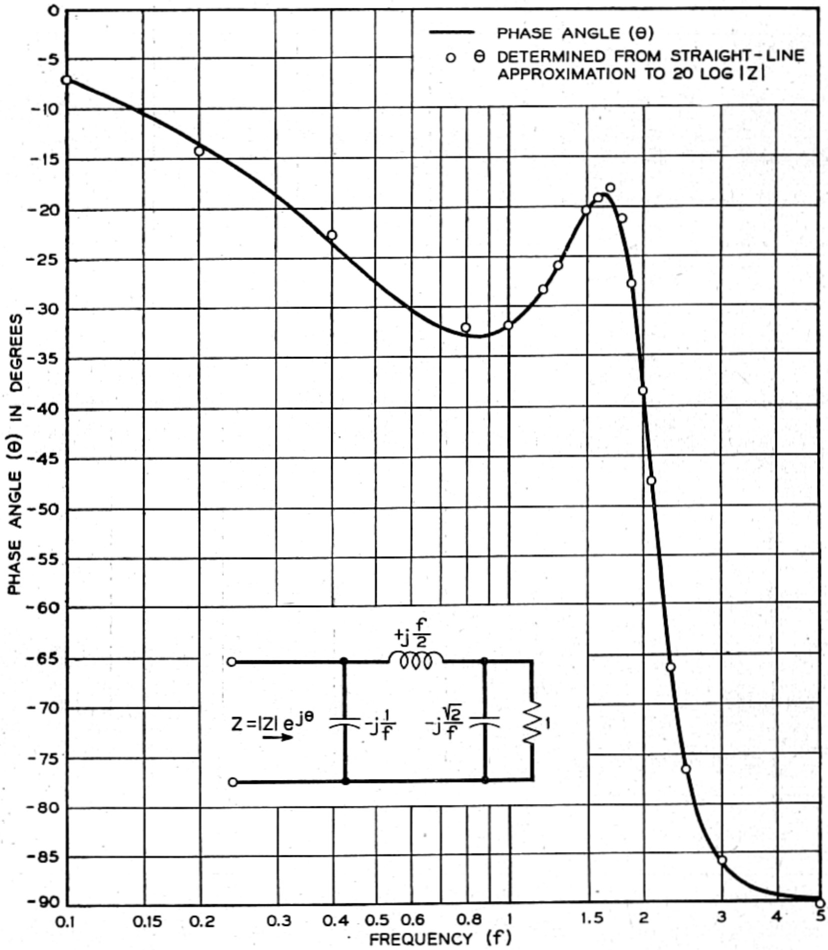

Fig. 6—Phase associated with 20 log | Z | of Fig. 5.

with a given attenuation characteristic may therefore be used to determine the reactance associated with a given resistance characteristic. The only

TABLE VIII
TABULATION OF CRITICAL POINTS AND DETERMINATION OF SLOPES OF STRAIGHT LINES APPROXIMATING RESISTANCE CHARACTERISTIC OF FIG. 7

n	f_n	R_n	$R_n - R_{n-1}$	$2.303 \log \frac{f_n}{f_{n-1}}$	k_n
0	.078	1.000	0		
1	.185	.912	−.088	.864	−.102
2	.290	.805	−.107	.450	−.238
3	.900	.400	−.405	1.133	−.357
4	1.20	.400	0	—	
5	1.50	.547	+.147	.2231	+.659
6	1.67	.840	+.293	.1074	+2.728
7	1.84	1.280	+.440	.0969	+4.54
8	1.92	1.280	0		
9	2.20	.335	−.945	.1361	−6.94
10	2.45	.094	−.241	.1076	−2.24
11	2.85	.015	−.079	.1512	−.52
12	5.00	.000	−.015	.562	−.027

TABLE IX
SUMMATION OF REACTANCE ASSOCIATED WITH RESISTANCE OF FIG. 7 AT $f = 1.0$

n	f_n (From Table VIII)	$\frac{f_n}{f}$	$\frac{f}{f_n}$	X_n Ohms	$X_{n-1} - X_n$ Ohms	k_n (From Table VIII)	$k_n (X_{n-1} - X_n)$ Ohms
0	.078	.078		1.52111			
1	.185	.185		1.45257	.06854	−.102	−.0070
2	.290	.290		1.38439	.06818	−.238	−.0162
3	.90	.900		.91766	.46673	−.357	−.1666
4	1.20		.833	.58801	.32965		
5	1.50		.667	.45004	.13797	+.659	+.0909
6	1.67		.599	.39897	.05107	+2.728	+.1393
7	1.84		.543	.35844	.04053	+4.54	+.1840
8	1.92		.521	.34282	.01562		
9	2.20		.455	.29688	.04594	−6.94	−.3188
10	2.45		.408	.26486	.03202	−2.24	−.0717
11	2.85		.351	.22666	.03820	−.52	−.0199
12	5.00		.200	.12790	.09876	−.027	−.0027

$\Sigma k_n (X_{n-1} - X_n) = -.1887$
$X (f = 1.0) = -.189$ Ohm

difference is that the slopes of the straight lines approximating the resistance plotted on a log frequency scale are determined by the expression below:

$$k_n = \frac{R_n - R_{n-1}}{\log_e \frac{f_n}{f_{n-1}}} = \frac{R_n - R_{n-1}}{2.303 \log \frac{f_n}{f_{n-1}}}$$

where:

R_n is the resistance at f_n on the straight line approximation to R.

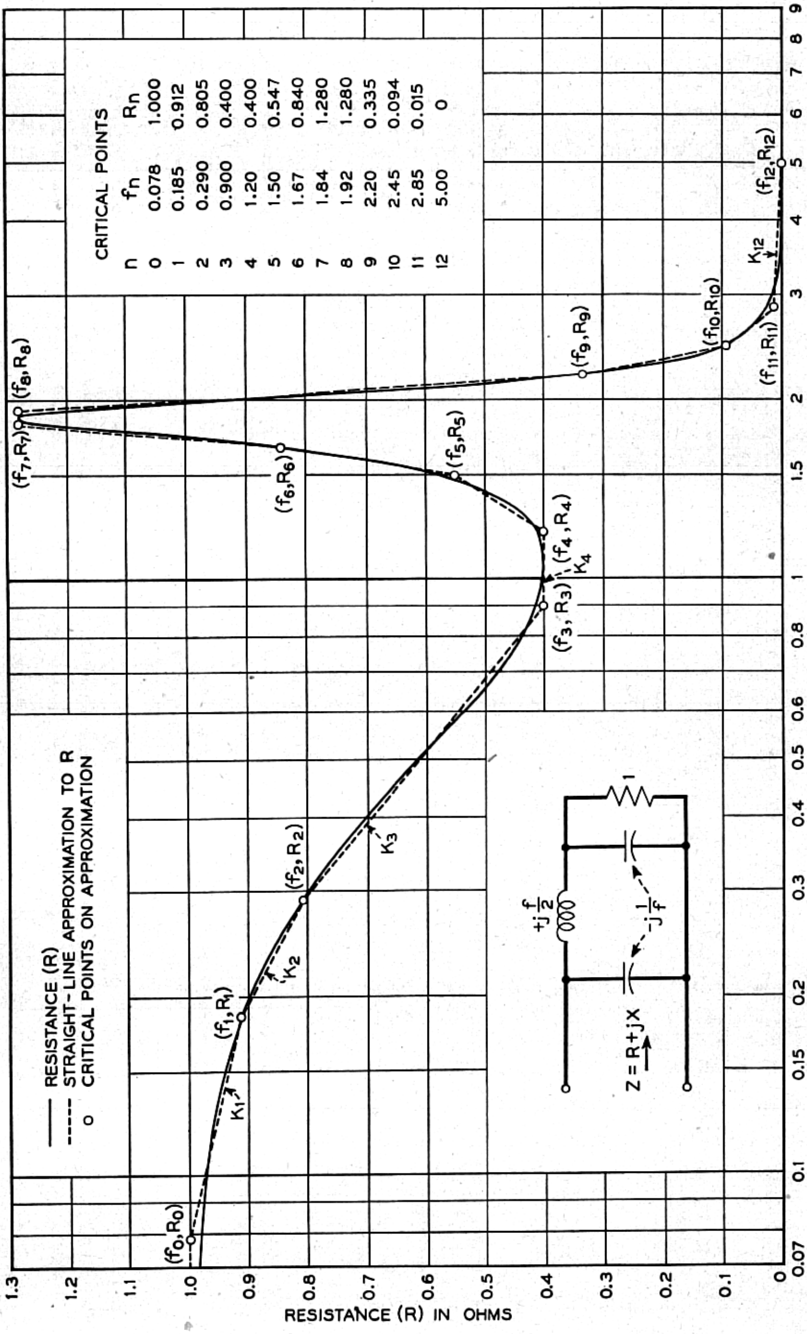

Fig. 7—Resistance characteristic.

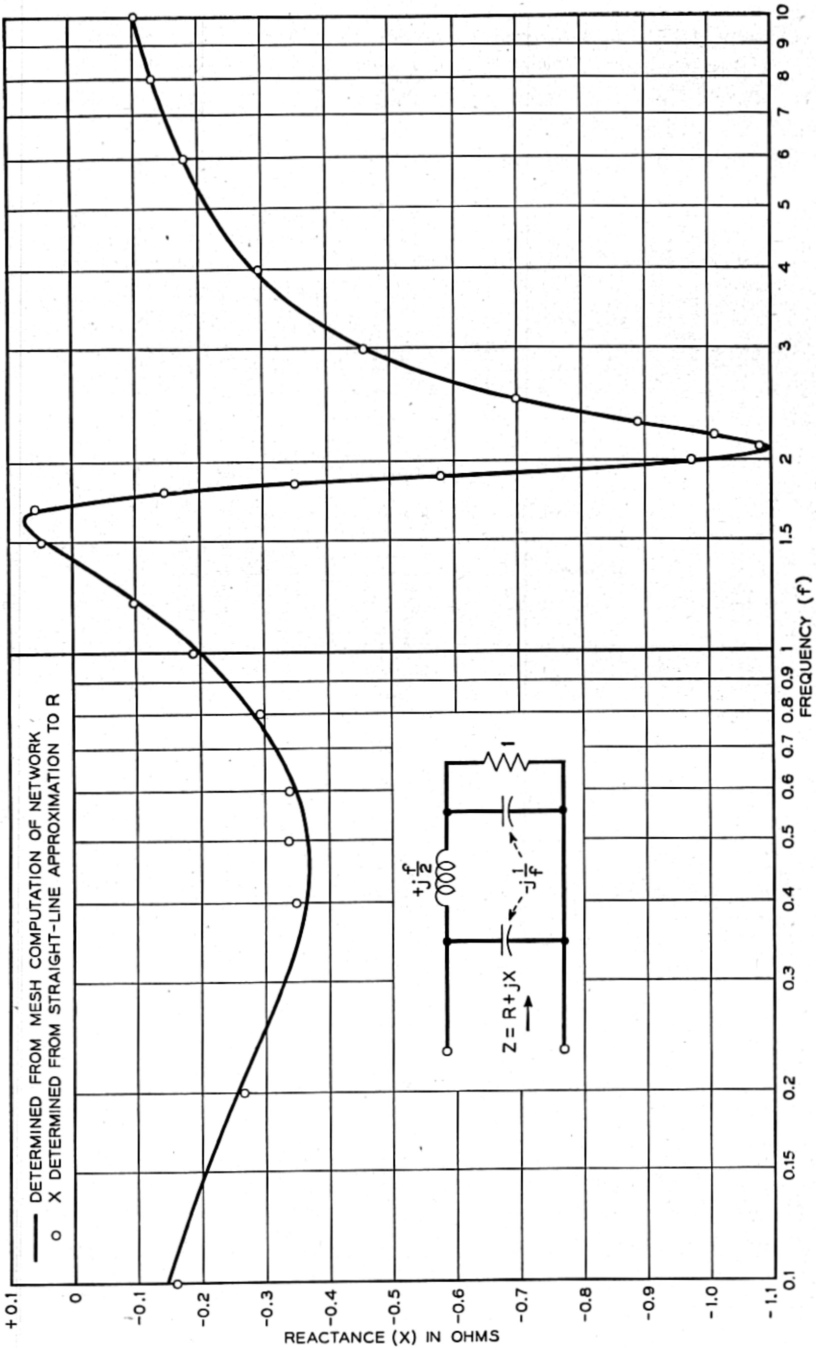

Fig. 8—Reactance associated with resistance characteristic of Fig. 7.

As an example of the determination of the reactance associated with a given resistance characteristic, consider the resistance characteristic of Fig. 7 and the straight line approximation shown in dotted form. The slopes of the straight lines are determined as illustrated in Table VIII.

Having determined the slopes of the various straight lines of the approximation, the reactance can be summed at any desired frequency. As an illustration the reactance is summed at $f = 1.0$, in Table IX.

The mesh computed reactance of the network of Fig. 7 is plotted in Fig. 8 and the reactance summed for $f = 1.0$ is seen to be within .01 ohm of the true reactance. The reactance was summed at a considerable number of frequencies and the results plotted as individual points in Fig. 8. The degree of approximation to the true reactance should be similar to the

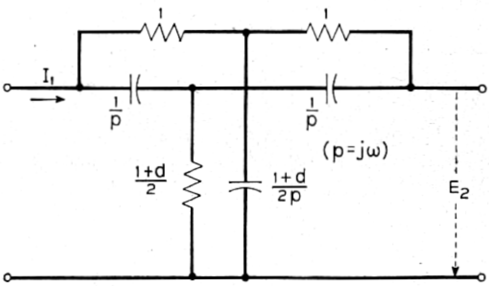

Fig. 9—Parallel T network.

degree of approximation to the original resistance and this is borne out by the example where the straight line approximation to the resistance characteristic is within $\pm$ .03 ohm and the maximum departure of the reactance determined from the straight line approximation is $\pm$ .025 ohm.

As was pointed out in the attenuation example a much simpler straight line approximation to the resistance characteristic would have resulted in a reactance determination without too much greater error than the determination of the illustration.

A word of caution is necessary in connection with the use of the straight line approximation method discussed above. The true phase or reactance is reliably obtained only in those cases where the problem in question is a minimum phase one. In order to illustrate the failure of the method in those problems in which non-minimum phase conditions exist consider the parallel T network of Fig. 9. The transfer impedance Z_{012} defined by the

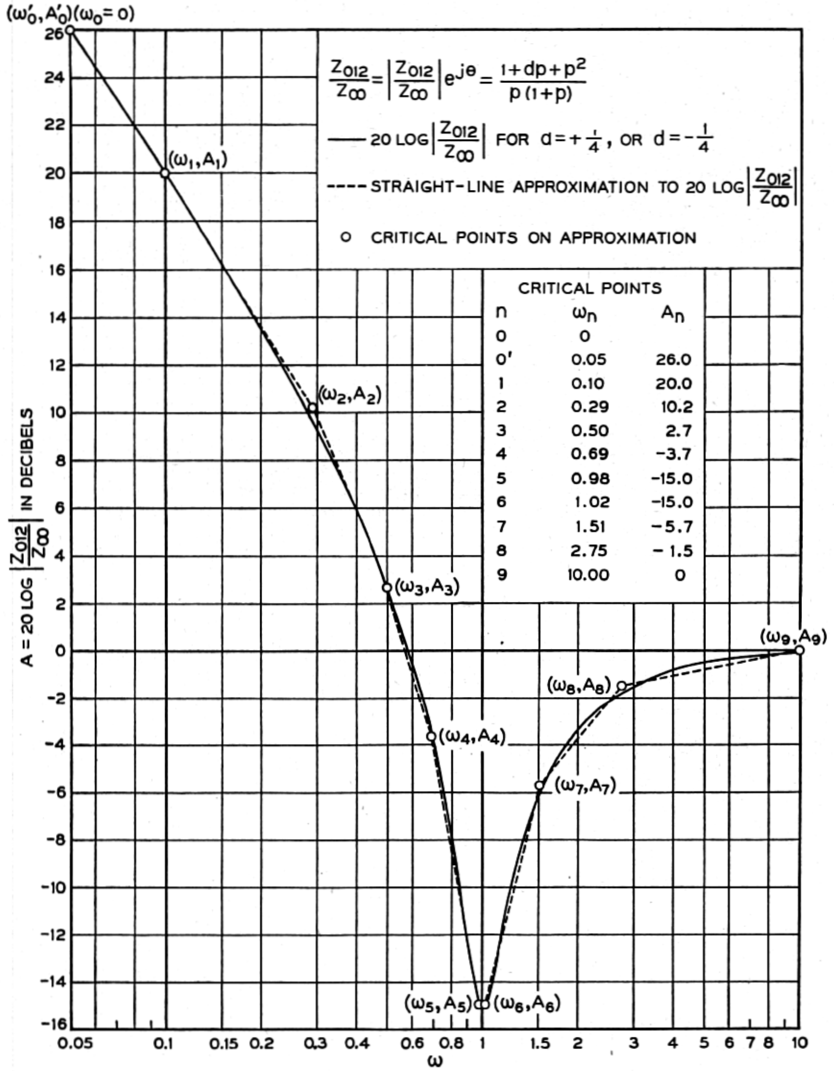

Fig. 10—20 log $\left|\frac{Z_{012}}{Z_\infty}\right|$ for network of Fig. 9.

ratio of the open circuit voltage E_2 to the open circuit driving current I_1 is given by:

$$Z_{012} = \frac{1}{2}\frac{1+d}{2+d}\frac{1+dp+p^2}{p(1+p)}.$$

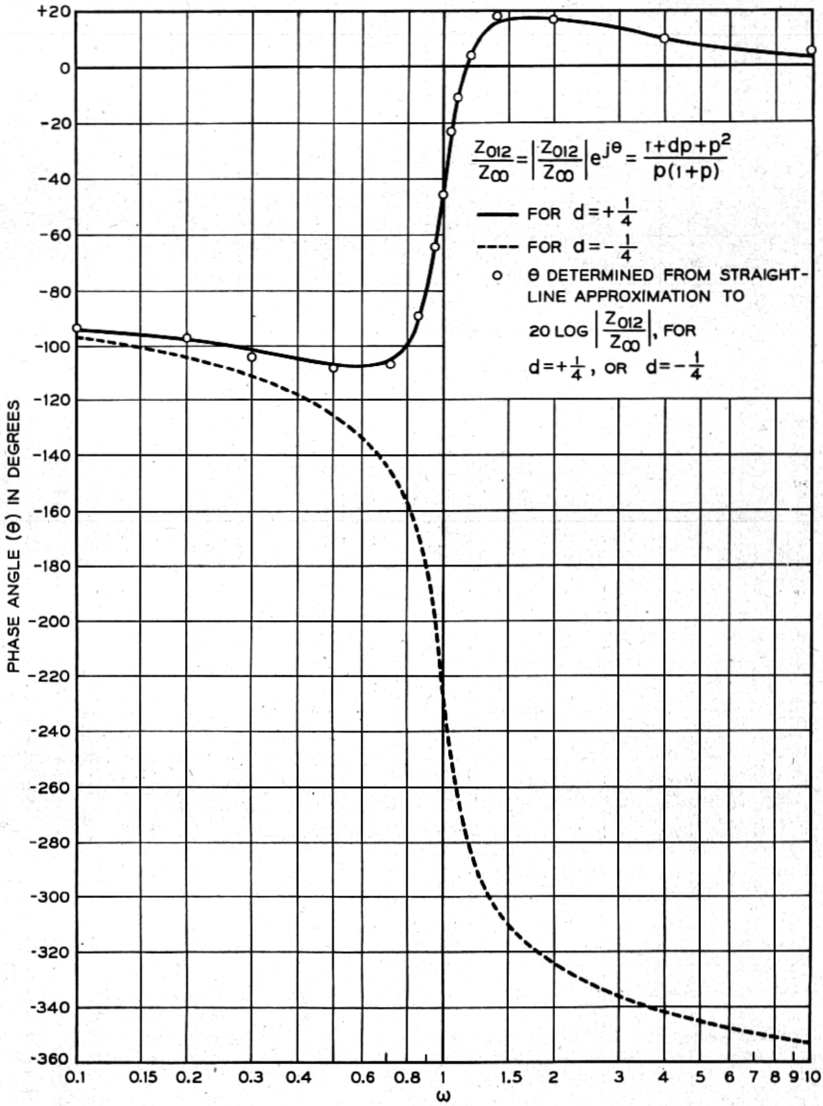

Fig. 11—Phase angle of $\frac{Z_{012}}{Z_{\infty}}$ for network of Fig. 9.

If we take the ratio of Z_{012} to its value for $\omega = \infty$ then:

$$\frac{Z_{012}}{Z_{\infty}} = \left|\frac{Z_{012}}{Z_{\infty}}\right| e^{j\theta} = \frac{1 + dp + p^2}{p(1 + p)}.$$

$20 \log \left| \frac{Z_{012}}{Z_\infty} \right|$ is plotted in Fig. 10 for $d = +1/4$ and it is apparent than $20 \log \left| \frac{Z_{012}}{Z_\infty} \right|$ for $d = -1/4$ is identical. This identity does not hold for θ, however. This is shown in Fig. 11 where θ for $d = +1/4$ and θ for $d = -1/4$ are plotted.

The real characteristic of Fig. 10 was then approximated by a series of straight lines determined by the critical points listed and the phase associated with this straight line approximation summed. The phase so determined is plotted as individual points in Fig. 11. It is seen that this summation determined the phase of the function in question for $d = +1/4$ but completely failed to do so for $d = -1/4$. The function for $d = -1/4$ is an example of a non-minimum phase function for which the above technique fails to determine the phase of the function from its attenuation characteristic.[10]

There are certain instances where the above technique can be usefully applied in connection with non-minimum phase systems in spite of the failure of the method to predict the total phase.[11] However, the necessity of checking for non-minimum phase conditions and, if such exist, determining whether the above method of computing phase is at all applicable, is illustrated by the non-minimum phase example above.

[10] This is the anticipated result since the function is identified as a non-minimum phase function by the fact that it has two zeros falling in the right half p plane.

[11] Bode, "Network Analysis and Feedback Amplifier Design," Chap. XIV, page 309.