# The Measurement of the Performance Index of Quartz Plates

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## 15.00 Introduction

THE theory of the general behavior of crystals in oscillator circuits has been described by I. E. Fair[1]. In Fair's paper as well as in others[2], it has been pointed out that in the neighborhood of the operating frequency a crystal is equivalent to the circuit shown in Fig. 15.1A. The crystal possesses two resonant frequencies, a series resonant frequency determined by the effective inductance, $L_1$, and effective capacitance, $C_1$, and an anti-resonant frequency determined by these same elements plus the paralleling capacitance, $C_0$. This paralleling capacitance is the static capacitance between electrodes of the crystal and any capacitance connected thereto by the crystal holder and lead wires within the holder. The dotted resistor, $R_L$, shunting the equivalent crystal circuit represents the effective shunt loss of the holder. In the ideal case and in many practical instances this loss is negligible.

It is rather difficult to express the circuital merit of a crystal quantitatively in a single term such as has been found useful for inductances and capacitances. It is customary to express the circuital merit of these two elements in the form of the ratio of reactance to resistance. That is, for an inductance

$$Q = \frac{\omega L}{R} \tag{15.1}$$

and for a capacitance

$$Q = \frac{1}{\omega C R}. \tag{15.2}$$

For filter purposes, the $Q$ of a crystal involving only the inductance, $L_1$, and resistance, $R_1$, of Fig. 15.1A is adequate to express its usefulness in certain parts of a filter network, but for oscillator purposes it is insufficient. At frequencies other than the series resonant frequency the paralleling capacitor $C_0$ together with the associated shunt loss of the holder enters into the determination of a crystal's performance. The term $Q$ therefore is not completely indicative of the crystal performance. There has been devised, as pointed out in Fair's paper[1], a term called "figure of merit" for a crystal

[1] I. E. Fair, "Piezoelectric Crystals in Oscillator Circuits," this issue of the *B.S.T.J.*
[2] K. S. Van Dyke, "The Electrical Network of a Piezo-Electric Resonator", *Physical Review*, Vol. 25, pp. 895, 1925.

which involves all the elements in the effective crystal circuit, and this term is much more expressive of the quality of a crystal. The figure of merit is:

$$M = \frac{\omega L_1}{R_1} \frac{C_1}{C_0} = \frac{Q}{r} \tag{15.3}$$

where "$r$" is the ratio of the paralleling capacitance to the series branch capacitance. Figure of merit is useful for expressing the quality of a crystal

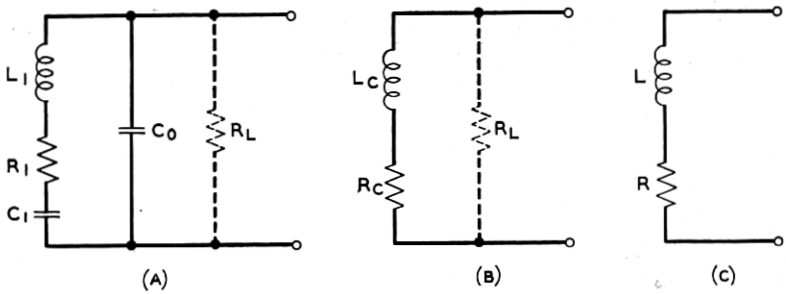


Fig. 15.1—Electrical equivalent circuits of a piezoelectric crystal—(A) At any frequency between the resonant frequency $\omega_1$, and anti-resonant frequency $\omega_2$; (B) and (C) at any specific frequency between $\omega_1$ and $\omega_2$.

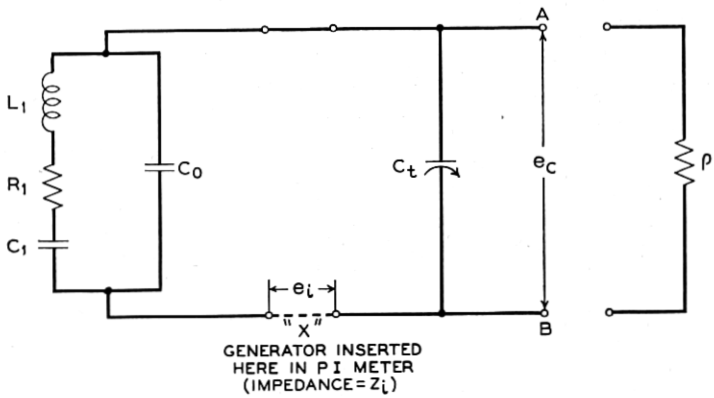


Fig. 15.2—Generalized oscillator circuit of the Pierce or Miller type.

in its holder or mount; however by definition it is independent of the value of $R_L$, and does not permit a ready evaluation of the performance of the crystal in an oscillator of the type that may be represented by Fig. 15.2. Any oscillator that operates the crystal in the positive region of the reactance vs. frequency characteristic exhibits capacitive reactance and negative resistance paralleled across the terminals to which the crystal is connected. The operation of the crystal when connected to an oscillator will be influenced by the magnitude of these two terms, and the combination must operate at such a frequency that the total reactance is zero and at such an amplitude that the total resistance is zero. The performance of the crystal will therefore not depend solely upon its figure of merit, but will involve the impedance of the remainder of the oscillator. Up to the present time circuit design engineers have not devised standards or units to express the quality of their oscillator circuits without the crystal, so there are no corresponding circuital units of quality with which to correlate figures of merit of crystals to ascertain the suitability of one for the other.

It was a practice for many years for manufacturers to test crystals in a model of the oscillator in which the crystal was to be used. This required manufacturers to keep on hand models of all oscillators for which they expected to make crystals. To avoid the mounting number of such test oscillators a special test set was developed which could be adjusted to simulate any oscillator. By correlating various oscillator circuits to a set of adjustments on the test set, the actual model of the oscillator can be dispensed with. This special test set usually referred to as the '"D" spec. test set', eliminated the "file" of oscillators, and substituted a file of adjustment readings that would be their equivalent. However, the '"D" spec. test set' is still inadequate to the development engineer since it defines "activity" in terms of oscillator grid current rather than in terms of the electrical equivalent circuit of the crystal. The activity as expressed by grid current is a purely arbitrary standard and serves only as a means of determining the relative activity as against other crystals of the same frequency operated under the same circuit conditions.

The need for a system of measurement using units that are fundamental and not empirical has led to the proposal of "Performance Index". An instrument to make such measurements is to be described in this paper.

Specifically the Performance Index is

$$PI = \frac{\omega L}{\omega C_t R} \tag{15.4}$$

where $C_t$ is the paralleling capacitance that is found in the oscillator circuit to which the crystal is attached, and $L$ and $R$ represent the effective inductance and resistance of the crystal as measured at the operating frequency indicated in Fig. 15.1C which is its equivalent *at that frequency*. If the loss in the holder is so low that the resistance, $R_L$, may be neglected, then $PI$ may be expressed in other relations that are more useful such as,

$$\left.\begin{aligned} PI &= \frac{M}{\omega C_0 \left(1 + \dfrac{C_t}{C_0}\right)^2} \\ \text{or} \quad PI &= P^2 R_1 \end{aligned}\right\} \tag{15.5}$$

where the symbols $R_1$ and $C_0$ are as shown in Figs. 15.1A and 15.2 and $P$ is expressed as

$$P = \frac{M}{1 + \frac{C_t}{C_0}} \tag{15.6}$$

With the effective capacitance, $C_t$, of the remainder of the oscillator added to the paralleling capacitance, $C_0$, in Fig. 15.2, the operating frequency will be that frequency at which the combination will exhibit a pure resistance at the terminals $AB$ (excluding the generator "$X$" which is involved in the measuring technique). This leads to the definition:

*The Performance Index is the anti-resonant resistance of the crystal and holder having in parallel with it the capacitance introduced by the remainder of the oscillator.*

The Performance Index is therefore a term to express performance not in terms of the grid current of some particular oscillator, but in fundamental circuital units—impedance. The Performance Index is a term that may be used to compare performance of crystals at different frequencies. Its value is independent of plate voltage, grid leak resistance, or of plate impedance. It provides a measuring stick that should replace the "activity" figures of grid current in so far as the crystal is concerned. It paves the way for the oscillator circuit designers to come forth with standards of measurement for the oscillator circuit without the crystal in the hope that the two may be quantitatively associated and lend themselves to theoretical calculation of full oscillator performance.

## 15.10 Theory of Measurement

The problems of measurement are most readily explained by reference to Fig. 15.2. The crystal provides elements $L_1$, $C_1$, $R_1$ and $C_0$. The circuit of the oscillator provides an effective capacitance, $C_t$, which is composed of grid and lead wire capacitances plus capacitance introduced from the plate circuit. The frequency at which this combination exhibits anti-resonance as measured at $AB$ is the oscillating frequency. The resistance when added to negative resistance, $\rho$, will be zero. Oscillations will start with $\rho$ numerically smaller than the anti-resonant resistance measured at $AB$, but the amplitude of oscillations will increase causing $\rho$ to increase until $\rho$ and $Z_{AB}$ are equal numerically. The primary problem is to measure the anti-resonant resistance at $AB$ at the anti-resonant frequency with $\rho$ disconnected.

Measurement of anti-resonant resistance directly is very difficult. The current flowing into an anti-resonant circuit is too small to measure with the usual meters. Other devices for measuring the current are likely to introduce paralleling capacitance that will vitiate the readings. The suggested method of measurement utilizes a suitable driving voltage at "X" (Fig. 15.2) and a means to indicate the voltage at "X" as well as at points $AB$. From these and other measured constants, the anti-resonant impedance can be computed.

This method of measurement has its own difficulties, but it is believed corrections can be made to allow for errors introduced. Fundamentally, the series resonant frequency and the anti-resonant frequency are the same only when the resistances in the inductive and capacitive branches are equal. When the resistance is practically all in the inductive branch, which is true in this case, the impedance between terminals $AB$, at the series resonant frequency will exhibit capacitive reactance, though the total impedance will scarcely be different from that at the anti-resonant frequency. In the Performance Index meter, although the voltage is introduced in series with the circuit, the frequency is adjusted to the point of maximum voltage across $AB$, which further minimizes this frequency difference. A second error is inherently introduced by the loss in the crystal holder. This means that the series resonant frequency is also altered by the presence of this loss. Errors of any seriousness will result from the assumption that the series resonant and anti-resonant frequencies are identical only when the resistance in the inductive branch and loss in the crystal holder approach the effective crystal reactance in magnitude. These errors will be discussed in greater detail in a succeeding section.

The development and operation of a satisfactory meter to measure *PI* (Performance Index) depends upon a number of factors such as:

1. A method to determine capacitance, $C_t$, of the circuit (Fig. 15.2).
2. A generator "X" to produce the driving voltage $e_i$ having variability in frequency and negligible internal impedance.
3. A current indicator that introduces a minimum of reactance and resistance.
4. A circuit or method to indicate *PI* directly, or with a minimum of calculations.
5. A number of other factors associated with the above which will be mentioned at the logical times.

To construct a measuring circuit to determine the anti-resonant impedance by means of a series circuit so as to avoid any unnecessary measurements and computations involves the following basic principle. Excluding $\rho$, Fig. 15.2 is essentially equivalent to the circuit used in $Q$ meters. The ratio of voltage $e_c$ to the driving voltage $e_i$ is the voltage stepup or the $Q$ of that part of the circuit containing the resistance. In this case, the resistance is in the crystal which at the operating frequency has an effective $Q$ of

$$Q_1 = \frac{\omega L}{R} \tag{15.7}$$

where $L$ and $R$ represent the effective values of the crystal. Equation (15.4) will be found to embody the relation

$$PI = \frac{\omega L}{\omega C_t R} = Q_1 X_t \tag{15.8}$$

where $X_t$ is the reactance of $C_t$, the capacitance introduced by the circuit at the operating frequency. If $e_i$ is kept constant, $e_c$ will at all times be proportional to $Q_1$. By insertion of an attenuator network, whose attenuation varies with frequency in the same manner as does the reactance of $C_t$, between terminals $AB$ and the voltmeter, the meter indication will be proportional to the product of these quantities or *proportional to PI*. With

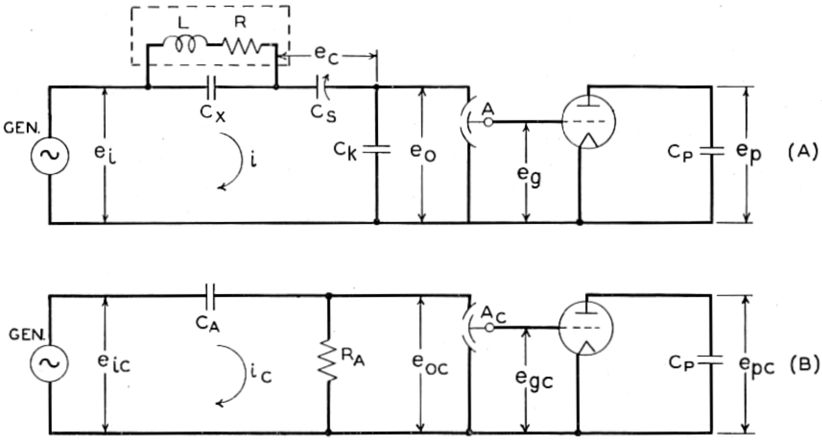


Fig. 15.3—Measuring circuits of the performance index meter.

suitable calibrations, therefore, it should be possible to get indications of $PI$ as readily as is now done for $Q$.

The circuit shown in Fig. 15.2 is now best redrawn as in Fig. 15.3A. The crystal embodying elements $L_1$, $C_1$, $R_1$, and $C_0$ of Fig. 15.2 is now represented in Fig. 15.3A by the dotted rectangle and as having effective inductance, $L$, and effective resistance, $R$, both of which are functions of frequency. Capacitance, $C_t$, is simulated by capacitors $C_x$ plus $C_s$ and $C_k$ in series where $C_x$ represents the capacitance of the crystal socket. Zero internal impedance of the generator is simulated by maintaining the driving voltage constant at all times and at all frequencies. To facilitate explanation, the measured voltage $e_i$ at the place shown is considered to be the driving voltage from a zero internal impedance generator.

Instead of using an ammeter to indicate current in the circuit, a voltmeter is utilized to measure voltage across an element under such conditions as not to introduce disturbing capacitance. Splitting the series capacitance into two parts, $C_s$ and $C_k$, the latter fixed and large compared to $C_s$, provides the impedance element across which the voltmeter is connected. The input capacitance of the voltmeter is incorporated in the magnitude of $C_k$. A capacitance attenuator, $A$, of known or calibrated values interposed on the input of the voltmeter enables the voltmeter to be used to indicate voltage ratios in terms of the attenuator calibration.

The measuring voltmeter and a shunting capacitance, $C_p$, are connected in the plate circuit of the amplifier tube, V-1. This circuit provides sufficient gain to furnish an output voltage of measurable magnitude and also provides an output voltage inversely proportional to frequency. The indication of the output voltage is proportional to $PI$.

The utilization of a vacuum tube in a circuit leading to a quantitative measuring instrument such as the voltmeter across $C_p$ involves determination of tube constants or calibration. The determination of these constants is best evaluated experimentally. A calibrating circuit for that purpose is shown in Fig. 15.3B. A capacitance, $C_A$, of high impedance in series with comparatively negligible resistance, $R_A$, is connected across the driving voltage terminals of $e_i$ with a voltmeter measuring $e_i$ giving a reading $e_{ic}$. The second subscript "$c$" indicates calibration conditions. By connecting the input circuit of V-1 across this resistance, the attenuation variation with frequency of the $R_A - C_A$ network cancels the attenuation variation with frequency in the plate circuit of V-1. The ratio of $e_{ic}$ to $e_{pc}$ will then be independent of frequency. In the "calibrate" circuit (Fig. 15.3B), the capacitor attenuator, $A_c$, interposed in the grid circuit is set at unity (minimum insertion loss) for a given deflection of the meter indicating $e_p$. In the operate circuit (Fig. 15.3A), the attenuator is readjusted so that voltage $e_0$ produces the reading of $e_p$ as obtained in the calibrate position. The quantitative action of the amplifier then may be expressed in terms of $C_p$, $R_A$, $C_A$ and a reading from the attenuator $A$, as will be shown later, and it is constant and independent of frequency. By placing this resulting constant in an equation, which will also be derived later, the value of $PI$ may be determined in terms of such constant, of the reading of attenuator $A$, and of a reading on the scale of $C_s$ that has been calibrated in terms of $C_t$.

To facilitate still further the operation of the $PI$ meter, the voltage $e_i$ is produced as shown in Fig. 15.4 by arranging for the oscillator to have its frequency controlled by the crystal through feedback from capacitor, $C_k$. Automatic volume control is provided such that the amplitude of $e_i$ is essentially constant at all times and at all frequencies. The circuit is constructed to oscillate at the desired frequency, and adjustment for insuring this operation is provided in the form of a phase shifting circuit with variable capacitor, $C_r$. After a crystal has been inserted in its proper place, oscillations will begin, but may be slightly above or below the resonant frequency of the crystal plus $C_t$. By adjustment of $C_r$ the frequency can be shifted the slight amount necessary for resonance. This is observed by placing switch $S$ in the $PI$ position and making the adjustment to give maximum deflection of $e_p$.

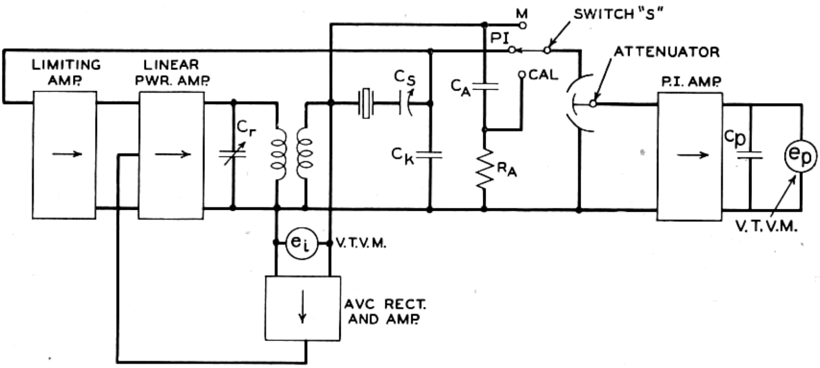


Fig. 15.4—Diagram of Performance Index meter.

### 15.20 Derivation of PI Circuit Equation

The following circuit relations derived from Figure 15.3 show first, that the ratio of $e_p$ to $e_i$ is a function of the Performance Index of the crystal, and second, that the calibration circuit permits an absolute evaluation of its magnitude.

At resonance, the effective circuit $Q$, designated as $Q_2$, is determined from

$$Q_2 = \frac{|e_c + e_0|}{e_i} = \frac{e_0}{e_i}\left(1 + \frac{C_k}{C_s}\right) \tag{15.9}$$

Since the circuit $Q$ includes the capacitance of $C_x$ as a part of the crystal, it is necessary to express $Q_2$ in terms of the crystal's properties (see Fig. 15.1). Since $Q_1\left(=\frac{X}{R}\right)$ of the crystal is independent of $C_x$, the relationship between $Q_1$ and $Q_2$ is readily obtained by equating the expressions for the anti-resonant impedance first, when $C_x$ is considered to be in shunt with the series capacitor, $C_t$, and second, when $C_x$ is considered as part of the crystal. This results in

$$Q_2 = Q_1 \frac{C_t - C_x}{C_t} \tag{15.10}$$

where

$$C_t = C_x + \frac{C_s C_k}{C_s + C_k} \tag{15.11}$$

enabling $Q_1$ to be expressed as

$$Q_1 = \frac{e_0}{e_i} \frac{C_t C_k}{(C_t - C_x)^2} \tag{15.12}$$

Expressing $e_0$ in terms of $e_p$, we have

$$|\mu e_g| = |i_p| \sqrt{r_p^2 + X_{C_p}^2} \tag{15.13}$$

$$|i_p| = \frac{e_p}{X_{C_p}} \tag{15.14}$$

hence

$$e_g = \frac{e_p \omega C_p}{G_m} \sqrt{1 + \left[\frac{X_{C_p}}{r_p}\right]^2} \tag{15.15}$$

and

$$\frac{e_0}{A} = e_g \tag{15.16}$$

With the above equations substituted in (15.12), we may express $Q_1$ as

$$Q_1 = \frac{e_p}{e_i} \omega \frac{C_p}{G_m} A \frac{C_k C_t}{(C_t - C_x)^2} \sqrt{1 + \left[\frac{X_{C_p}}{r_p}\right]^2} \tag{15.17}$$

Now

$$PI = \frac{Q_1}{\omega C_t} = Q_1 X_t \tag{15.18}$$

Therefore

$$PI = \frac{e_p}{e_i} A \frac{C_p}{G_m} \frac{C_k}{(C_t - C_x)^2} \sqrt{1 + \left[\frac{X_{C_p}}{r_p}\right]^2} \tag{15.19}$$

If

$$\sqrt{1 + \left[\frac{X_{C_p}}{r_p}\right]^2} \cong 1 \tag{15.20}$$

$$PI = \frac{e_p}{e_i} A \frac{C_p}{G_m} \frac{C_k}{(C_t - C_x)^2} \tag{15.21}$$

The simplified *PI* expression (15.21) assumes that the reactance of $C_p$ is small compared to the plate resistance and plate load resistance of V-1. The evaluation of *PI* from this expression has three obvious difficulties:

(1) $C_p/G_m$ is a quantity that is difficult to evaluate numerically, (2) the magnitude of $PI$ is measured in terms of the ratio of the two voltages, $e_i$ and $e_p$, and (3) the measurement is dependent upon the gain of a vacuum tube amplifier, V-1. These difficulties may be materially reduced in their consequence by an internal calibration circuit.

The internal calibration circuit (Fig. 15.3B) consists of a capacitor, $C_A$, and resistor, $R_A$, in series. If the reactance of $C_A$ is very much greater than $R_A$, and the plate resistance of V-1 is very much greater than the reactance of $C_p$, the calibration is essentially independent of frequency.

The internal calibration circuit (Figure 15.3B) enables the evaluation of $C_p/G_m$ to be carried out. The additional subscript, $c$, indicates "calibrate" conditions.

$$i_c = \frac{e_{ic}}{\sqrt{R_A^2 + X_{C_A}^2}} \tag{15.22}$$

$$e_{0c} = i_c R_A = \frac{e_{ic}\,\omega C_A R_A}{\sqrt{1 + \left[\dfrac{R_A}{X_{C_A}}\right]^2}} \tag{15.23}$$

$$\frac{e_{0c}}{A_c} = e_{g_c} \tag{15.24}$$

Equation (15.15) remains the same for both "operate" and "calibrate" conditions with the exception of the second subscript reserved for the "calibrate" operation. Therefore, by solving for $C_p/G_m$ we find

$$\left|\frac{C_p}{G_m}\right| = \frac{e_{gc}}{e_{p_c}\,\omega\sqrt{1 + \left[\dfrac{X_{C_p}}{r_p}\right]^2}} \tag{15.25}$$

Equation (15.25) may be rewritten as (15.26), if (15.23) and (15.24) are substituted in (15.25)

$$\left|\frac{C_p}{G_m}\right| = \frac{e_{ic} R_A C_A}{e_{pc} A_c}\left[\frac{1}{\sqrt{1 + \left[\dfrac{X_{C_p}}{r_p}\right]^2}}\right]\left[\frac{1}{\sqrt{1 + \left[\dfrac{R_A}{X_{C_A}}\right]^2}}\right] \tag{15.26}$$

If (15.26) is substituted in (15.19), it is found that $PI$ may be expressed as follows:

$$PI = \frac{e_p}{e_i}\frac{e_{ic}}{e_{pc}}\frac{A}{A_c} R_A C_A \left[\frac{C_k}{(C_t - C_x)^2}\right]\frac{1}{\sqrt{1 + \left[\dfrac{R_A}{X_{C_A}}\right]^2}} \tag{15.27}$$

The above equation involves only the original approximation that maximum current indicates resonance. If $R_A$ and $C_A$ are selected such that $R_A << X_{C_A}$ and if $A_c$ equals unity, then

$$PI = \left[\frac{e_p}{e_i}\frac{e_{ic}}{e_{pc}}\right] A \frac{C_k}{(C_t - C_x)^2} R_A C_A \qquad (15.28)$$

From this expression it can be seen that the $PI$ measurement is independent of calibration of both the $e_p$ and $e_i$ vacuum tube voltmeters, provided that the same voltmeter scale factors are used for the "operate" and "calibrate" conditions. The absolute calibration then depends on the magnitude of $A$, $R_A$, $C_A$, $C_k$, $C_t$ and $C_x$. The "multiply-by" factor that is to appear on the $C_s$ dial is determined by the magnitude of $\frac{R_A C_A C_k}{(C_t - C_x)^2}$.

Accurate evaluation of this quantity by capacitance and resistance measurements is a little difficult since the denominator represents the square of the difference of two small capacitances. When $C_t$ is large, the evaluation of this factor is helped considerably. This "multiply-by" factor may be experimentally determined by a voltage measuring means which permits an evaluation of this factor to a higher degree of accuracy. Substituting (15.11) in (15.28) we have

$$PI = \left[\frac{e_p}{e_{pc}}\frac{e_{ic}}{e_i}\right] A \frac{R_A C_A}{C_k}\left(1 + \frac{C_k}{C_s}\right)^2 \qquad (15.29)$$

Now by shorting the crystal socket terminals (Fig. 15.3A) and applying a voltage $e_1$ at the $e_i$ generator terminals of external origin (the crystal oscillator circuit itself may be used if self-excitation is provided), the current $i_1$ through the capacitors $C_k$ and $C_s$ is given as

$$i_1 = \frac{e_1 \omega C_s C_k}{C_s + C_k} = \frac{e_1 \omega C_k}{\left(1 + \frac{C_k}{C_s}\right)} \qquad (15.30)$$

Now the voltage, $e_2$, across the series capacitor, $C_k$, is

$$e_2 = \frac{i_1}{\omega C_k} \qquad (15.31)$$

The ratio of $e_1/e_2$ may be expressed as given in (15.32) when (15.31) is substituted in (15.30)

$$\frac{e_1}{e_2} = \left(1 + \frac{C_k}{C_s}\right) \qquad (15.32)$$

If (15.32) is substituted in (15.29), we find

$$PI = \left[\frac{e_p}{e_{pc}}\frac{e_{ic}}{e_i}\right] A \frac{R_A C_A}{C_k}\left[\frac{e_1}{e_2}\right]^2 \qquad (15.33)$$

The quantity $\frac{e_1}{e_2}$ is readily determined by the attenuator, $A$, when the switch, $S$, (Fig. 15.4) is operated between "$M$" and "$PI$" for the above described conditions. The absolute calibration then depends upon $A$, $R_A$, $C_A$ and $C_k$. All four of these quantities may be determined within a few per cent.

## 15.30 Oscillator Correlation

The equivalent crystal circuit has been discussed in so far as the measurement of $PI$ is concerned; however, for correlation with an oscillator, the behavior of the crystal in that oscillator must be duplicated. Correlation of the $PI$ meter with an oscillator is a function of both amplitude and frequency. It is obviously necessary from the derivation of (15.28) that the frequency of operation be duplicated, but the necessity for amplitude correlation can only be explained from the practical consideration that the equivalent circuit components of Fig. 15.1 are parameters that may be functions of amplitude. Crystals having nonlinear characteristics of the type that necessitate amplitude correlation may in part be attributed to either the method of mounting the crystal or couplings to other modes of vibration whose coupling coefficients are functions of amplitude.

In most oscillators the voltage across the terminals of a crystal is a function of many parameters such as plate voltage, vacuum tubes, etc. With an average set of conditions, however, reasonable correlation is obtained with the $PI$ meter for a single adjustment of the generator voltage, $e_i$, for all crystals. The magnitude of $e_i$ must, of course, be chosen to produce a voltage across the crystal equal to the average value obtained in the oscillator circuit for which the crystal is intended.

Frequency correlation with an external oscillator is a function of the effective capacitance, $C_t$, in shunt with the crystal. In order to duplicate the oscillator frequency with the $PI$ meter, the capacitance, $C_s$, (Fig. 15.4) must be adjusted until the frequency of oscillation in the $PI$ meter is the same as that in the oscillator for a crystal having average activity. In Fig. 15.4, the capacitance, $C_s$, is variable, and its dial is calibrated in terms of both the total effective capacitance across the crystal, $C_t$, and the resulting multiplying factor $\frac{R_A C_A C_k}{(C_t - C_x)^2}$. The magnitude of $C_t$ may be measured by means of a capacitance bridge connected across the crystal socket terminals with the generator impedance shorted.

The determination of the dynamic or effective capacitance, $C_t$, across the crystal for an oscillator may similarly be obtained by adjusting the magnitude of $C_s$ in the $PI$ meter until the frequencies of oscillation in the $PI$

meter and in the oscillator under test are identical for the same amplitude of oscillation. By this means, the *PI* meter directly indicates the effective oscillator capacitance, $C_t$. The amplitude of oscillation must be duplicated in as much as $C_t$ is not independent of the amplitude in most oscillators.

### 15.40 Description of Oscillator Generating "$e_i$"

The generator plays no part in the theory of *PI* measurement as it could be replaced by a signal generator or any other suitable source of radio frequency energy. It is convenient, however, to utilize the voltage appearing across $C_k$ as an input to an amplifier whose output represents the generator. This in effect constitutes a feedback oscillator whose frequency is controlled by the crystal under test. Initial consideration of the over-all characteristics of the *PI* meter oscillator leads to the following requirements. The oscillator must,

1. Be capable of oscillating all crystals usable in other oscillator circuits.
2. Be capable of operating the crystal over a wide range of shunting capacitances in order to duplicate all the frequencies of oscillators now in the field.
3. Be capable of permitting high degrees of *AVC* control in order to maintain the generator voltage constant while the frequency is adjusted for reasonance.

If the generator voltage, $e_i$, is constant, resonance of the crystal circuit is essentially indicated by maximum crystal current, and oscillation is maintained at that resonant frequency. The adjustment to obtain maximum current is such that the phase shift throughout the oscillator loop is $2\pi n$ where $n = 0, 1, 2, 3,$ etc. As previously described the phase shift and resulting frequency of oscillation are varied by a tuned circuit. The generator voltage, $e_i$, is held constant by an automatic amplitude control similar to the automatic volume control which is often applied to amplifiers. The manual control of the magnitude of the generator, $e_i$, is provided by an adjustment of the bias voltage of the automatic amplitude control circuit. In this way the maximum or start gain is independent of the setting of the amplitude control.

Automatic amplitude control (commonly referred to as automatic volume control, *AVC*) of an oscillator may be applied by the separation of the limiter from the linear amplifier. This means that in order to apply a high degree of *AVC* to the *PI* oscillator (Fig. 15.4), the input voltage of the limiter must be above the threshold of limiting by an amount exceeding the variation in the $\beta$ path caused by the *AVC* control. This enables the limiter to absorb the changes in the gain of the linear amplifier such that $\mu\beta = 1$ at all times.

The time constant of the limiter is fast compared to that of the *AVC* circuit, a condition which permits damping of transients set up by changes in gain occurring from *AVC* action. The input of the linear amplifier is held constant by the limiter. Gain changes in the linear amplifier produced by the variation of $C_r$ (Fig. 15.4) are absorbed by *AVC*, while the variation of activity in the crystal is absorbed by the limiting amplifier.

### 15.50 Evaluating Performance Index

From (15.28) it can be seen that the attenuator, $A$, the effective variable capacitance, $C_t$, together with the vacuum tube voltmeters, $e_i$ and $e_p$, offer a number of possible variations in the method of evaluating the constants used to determine the *PI* of quartz crystals. There are, however, two principal methods—the first provides direct reading, while the second is more accurate but requires an indirect evaluation.

The first method utilizes a means of calibration of the meter scales directly in terms of *PI*. The attenuator is adjusted such that its indicator reading times the multiplying factor associated with the dial attached to $C_s$ is some multiple of 10. If in the calibrate position, $A_c$ is set at unity, and $e_{pc}$ and $e_{ic}$ are adjusted by varying the capacitive load to some reference deflection, then the expression for *PI* becomes

$$PI = \frac{e_p}{e_i} K_1 \tag{15.34}$$

where

$$K_1 = \left[\frac{e_{ic}}{e_{pc}} A \frac{C_k R_A C_A}{(C_t - C_x)^2}\right]$$

The Performance Index then is indicated by the two readings of $e_p$ and $e_i$. The absolute magnitudes of $e_i$ and $e_p$ need not be known since it is possible to use as a reference, the arbitrary calibrating deflections of $e_{ic}$ and $e_{pc}$. The magnitude of $e_p$ indicates the significant figures while $e_i$ is a multiplying factor.

The second method of evaluating Performance Index eliminates any calibration errors in the two vacuum tube voltmeters, $e_i$ and $e_p$. This method utilizes the attenuator to adjust $\left[\frac{e_p e_{ic}}{e_i e_{pc}}\right] = 1$. In the "calibrate" operation, $e_{ic}$ is set to equal $e_i$ and then $e_{pc}$ is adjusted for full scale or a convenient deflection. In the "operate" position, the attenuator is varied until $e_p$ equals $e_{pc}$. In this manner, the two readings of the attenuator are used to determine the ratio of $\frac{e_p}{e_{pc}}$ and the measurement is independent of the voltmeter calibration. The factor $R_A C_A$ is a constant; therefore, the P.I. equation (15.28) simplifies to

$$P.I. = (\Delta A) K_2 \tag{15.35}$$

where

$\Delta A$ = change in attenuator insertion loss between the "operate" and "calibrate" conditions in terms of output voltage ratio $\frac{e_p}{e_{pc}}$, given as $\frac{A}{A_c}$.

$K_2 = \frac{C_k R_A C_A}{(C_t - C_x)^2}$ where $C_t$ is the effective capacitance in series with the crystal, $C_k$ is the fixed series capacitance and $C_x$ is the crystal socket capacitance. (See Fig. 15.3)

## 15.60 P.I. Meter Applications

The application of this instrument can be extended to determine other properties of both crystal and oscillator. With the aid of a frequency measuring means and a capacitance bridge, the P.I. meter may be used to determine all the circuit constants designated in the electrical equivalent circuit of Fig. 15.1. If the loss in the holder is negligible then the equations are considerably simplified; however, in those instances where holder loss must be considered, the approximation that $X_t \ll R_L$ which may be allowed for most cases enables an evaluation of $M$ and $Q_c$ that is readily computed.

The dial controlling $C_s$, that is calibrated in terms of the total capacitance, $C_t$, makes possible the calculation of the magnitude of the input impedance to the crystal circuit, $R$, as well as $Q_1$.

$$\left.\begin{aligned} R &= \frac{X_t^2}{P.I.} \\ Q_1 &= \frac{P.I.}{X_t} \end{aligned}\right\} \tag{15.36}$$

The magnitude of $Q_1$ may also be measured directly from equation (15.12) where $Q_1$ was given as

$$Q_1 = \frac{e_0}{e_i} \frac{C_k C_t}{(C_t - C_x)^2} \tag{15.12}$$

As may be seen from Fig. 15.4, $e_0/e_i$ can be evaluated in terms of the attenuator calibration by enabling switch, $S$, to select the "*PI*" and "*M*" positions respectively and adjusting the attenuator such that the same output meter indication is obtained in the two cases.

The quantity $Q_1$ makes possible the calculation of $Q_c$ where $Q_c$ is defined as

$$Q_c = \frac{\omega L_c}{R_c} \cong \frac{X_t}{R_1\left(1 + \frac{C_0}{C_t}\right)^2} \tag{15.37*}$$

It can be shown that $Q_c$ in terms of $Q_1$ is given by the following equation if $R_L \gg X_t$.

$$Q_c = \frac{Q_1}{1 - Q_1 \frac{X_t}{R_L}} = \frac{Q_1}{1 - \frac{P.I.}{R_L}} \tag{15.38}$$

The Figure of Merit, $M$, defined by (15.3) at the series resonant frequency of the crystal, $\omega_1$, becomes

$$M \cong \frac{1}{\omega_1 C_0 R_1} = \frac{X_{c_0}}{R_1} \tag{15.39}$$

The measurement of $M$ can be determined from $Q_c$ provided $C_t$ and $C_0$ are known. $M$ may be determined from the following expression

$$M = \frac{Q_1}{1 - \frac{P.I.}{R_L}} \frac{C_t}{C_0}\left(1 + \frac{C_0}{C_t}\right)^2 \tag{15.40}$$

If $C_t$ is selected such that $C_t \gg C_0$, then for most cases $R_L$ is large compared to $Q_1X_t$. This means that $Q_1 \cong Q_c$ and $R \cong R_c$. With this approximation, (15.40) becomes

$$M = Q_1 \frac{C_t}{C_0}\left(1 + \frac{C_0}{C_t}\right)^2 \tag{15.41}$$

The relationship between $R_c$ and $R_1$ as a function of frequency may be expressed directly from the input impedance expression of the equivalent circuit. This is given as

$$R_c = \frac{R_1}{\left[\frac{\omega_2^2 - \omega^2}{\omega_2^2 - \omega_1^2}\right]^2 + \frac{1}{M^2}} \tag{15.42}$$

where $\omega$ is the unity power factor frequency in the P.I. meter, neglecting $R_L$. If

$$\frac{1}{M} \ll \left[\frac{\omega_2^2 - \omega^2}{\omega_2^2 - \omega_1^2}\right]$$

* The relationship $R_c \cong R_1\left(1 + \frac{C_0}{C_t}\right)^2$ was derived in Fair's paper*1.

which is a plausible assumption when $C_t \geq C_0$, and we assume that $\frac{\omega_1 + \omega_2}{2} = \omega_e$ then,

$$R_c = \frac{R_1}{\left[\frac{\omega_2 - \omega}{\omega_2 - \omega_1}\right]^2} \tag{15.43}$$

The relationship between $R_1$ and $R_c$ does not involve $R_L$ and may be expressed in terms of $C_0$ and $C_t$ instead of frequency. Neglecting $R_L$ to determine the relationship between capacitance and frequency in the P.I. meter as derived in Section 15.92, we find

$$\frac{\omega_2 - \omega}{\omega_2 - \omega_1} = \frac{1}{1 + \frac{C_0}{C_t}} \frac{\left[1 + \sqrt{1 + \frac{4}{P_1^2}}\right]}{2} \tag{15.44}$$

where,

$$P_1 = \frac{M}{\left(1 + \frac{C_0}{C_t}\right)} \tag{15.45}$$

If $P_1 \gg 2$ then the expression between $R_c$ and $R_1$ may be written

$$R_c = R_1 \left(1 + \frac{C_0}{C_t}\right)^2 \tag{15.46}$$

The restriction that $C_t \geq C_0$ may be removed if the error between (15.42) and (15.46) is taken into account. This error may be expressed as

$$\text{Per Cent Error in } R_c = 100 \left[\frac{1}{P_1^2}\right] \left[\frac{2}{1 + \sqrt{1 + 4/P_1^2}}\right] \tag{15.47}$$

The resonant resistance, $R_1$, together with (15.46) provides a means of checking the P.I. meter. The magnitude of $R_1$ may be determined by the substitution method and from this, the value of $R_c$ calculated. Fig. 15.5 represents the agreement between the P.I. meter and those expected from resonant frequency measurements.

The remaining crystal constants $L_1$ and $C_1$ (Fig. 15.1) may be evaluated from the measurement of $\omega_1$, $\omega_2$ and $C_0$. The resonant frequency, $\omega_1$, is defined as

$$\omega_1^2 = \frac{1}{L_1 C_1} \tag{15.48}$$

The anti-resonant frequency, $\omega_2$, is defined as

$$\omega_2^2 = \frac{1}{L_1}\left(\frac{1}{C_1} + \frac{1}{C_0}\right) \tag{15.49}$$

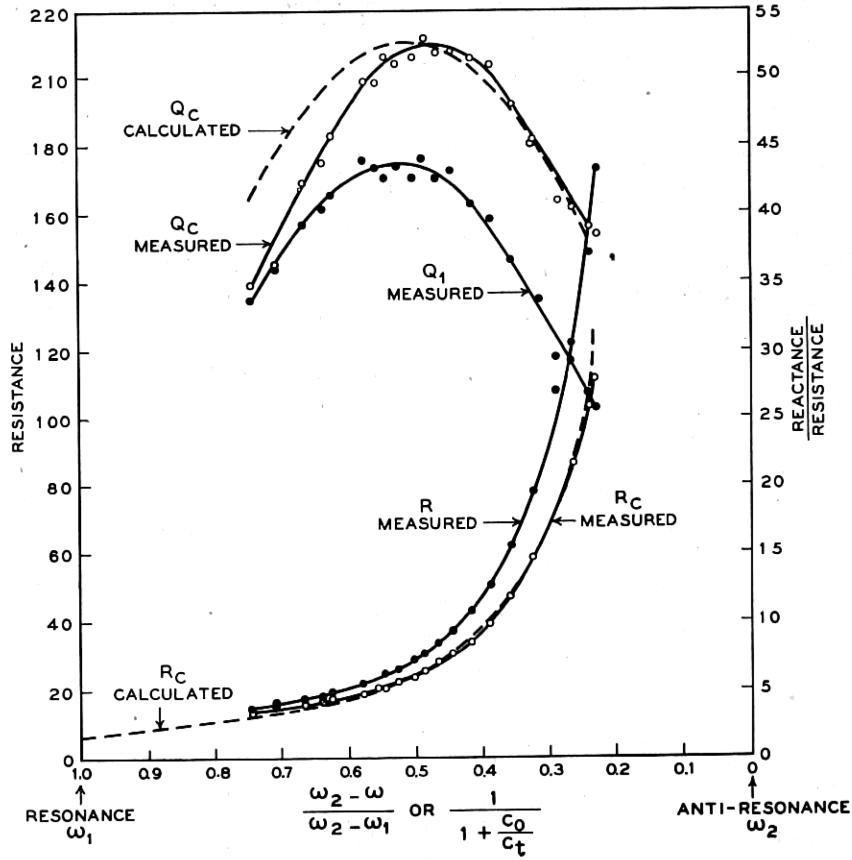


Fig. 15.5—Typical characteristic of a quartz crystal measured by the Performance Index meter.

Solving these equations simultaneously, it is found that

$$\left.\begin{aligned} L_1 &= \frac{1}{C_0(\omega_2^2 - \omega_1^2)} \cong \frac{1}{2\omega_e C_0(\omega_2 - \omega_1)} \\ C_1 &= \frac{C_0(\omega_2^2 - \omega_1^2)}{\omega_1^2} \cong \frac{2C_0(\omega_2 - \omega_1)}{\omega_1} \end{aligned}\right\} \qquad (15.50)$$

## 15.70 Experimental Data

The performance of the P.I. meter may best be illustrated by experimental data. The following data indicate the correlation which may be obtained between the P.I. meter and various types of oscillator circuits. Experimental considerations are extended to

(1) Frequency and amplitude correlation with a "Pierce" and "Tuned-Plate" oscillator

(2) The measurement of the effective capacitance, $C_t$, of an oscillator as a function of tuning, and

(3) The variation of P.I. as a function of voltage across the crystal.

The results presented are not to be considered as generalized data, but are intended only to show a set of measurements obtained for a specific set of operating conditions for each type of circuit.

It has been pointed out that the frequency of oscillation is a function of $R_1$, $\omega_1$, $\omega_2$, $C_0$ and $C_t$. Since $R_1$, $\omega_1$, $\omega_2$ and $C_0$ are explicit parameters of the crystal, the capacitor, $C_t$, becomes the only frequency determining element in the P.I. meter. From analytical methods to be described in

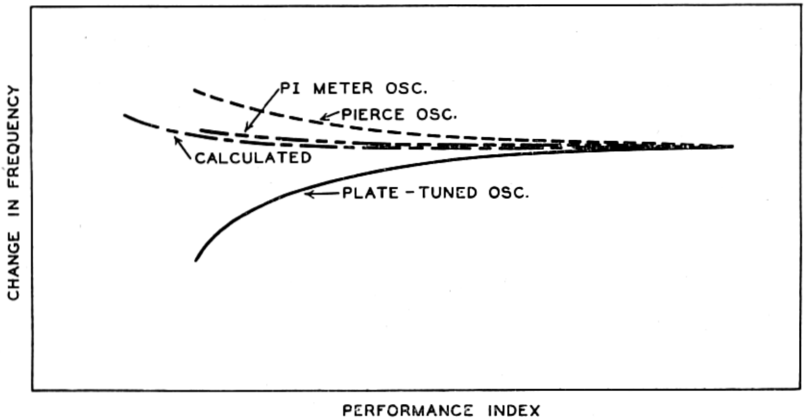


Fig. 15.6—Frequency variations in oscillators as a function of Performance Index.

Sections 15.80 and 15.92 the frequency difference between the P.I. meter and the generalized oscillator (neglecting $R_L$) is given as

$$\Delta\omega \cong \frac{(\omega_2 - \omega_1)}{M^2}\left(1 + \frac{C_0}{C_t}\right)\frac{2}{\left[1 + \sqrt{1 + \frac{4}{P_1^2}}\right]} - \frac{(\omega_2 - \omega_1)}{M^2}\left(1 + \frac{C_t}{C_0}\right) \tag{15.51}$$

Since the magnitude of frequency change in the above equation is small compared to the variations caused by changes in operating conditions, the P.I. meter may be used as a frequency correlation medium. Fig. 15.6 is an example of the correlation between the "Pierce" and "Tuned-Plate" oscillator and the P.I. meter. The P.I. meter falls between these two oscillators in frequency for any crystal activity. It must be recognized that Fig. 15.6 is not conclusive to the extent of generalization; however, it is indicative of possible correlation with these two popular oscillator circuits.

Figures 15.7 (A) and (B) show the amplitude correlation between the Performance Index meter and the grid current of the "Tuned-Plate" and "Pierce" oscillators, respectively. The P.I. vs. grid current characteristic was arbitrarily taken at three frequencies—4.5 mcs, 5.66 mcs and 7.81 mcs. The change in grid current of the oscillator shown in Fig. 15.7 (A) with frequency is caused by the varying *L-C* ratio in the plate circuit. (See curves *A*, *B* and *C* for constant crystal activity.) The curves *A* and *D* represent the effect of changing bands by switching coils, varying the *L-C* ratio 2 to 1 in the plate circuit for the same crystal frequency. Fig. 15.7

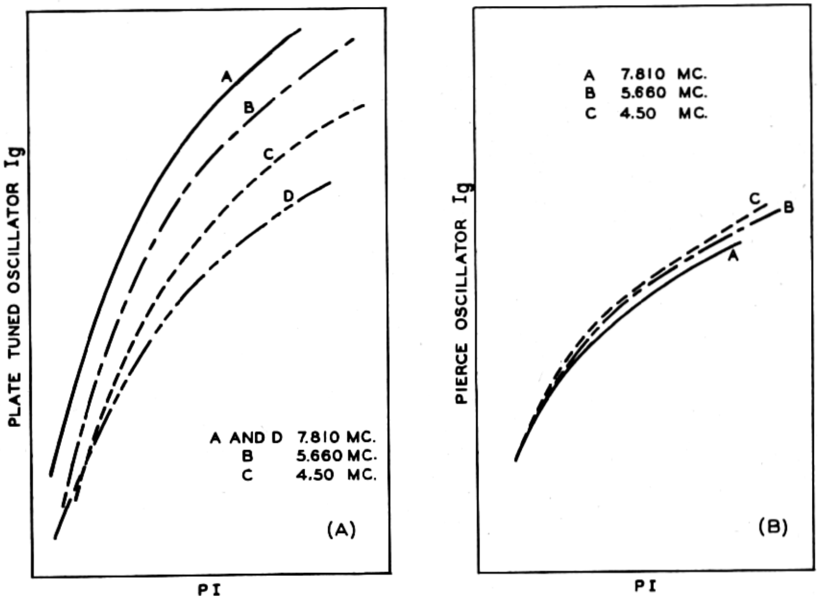


Fig. 15.7—Typical oscillator characteristics.

(B) represents the correlation between P.I. and grid current of the "Pierce" type oscillator. The results of this correlation indicate that the grid current is essentially independent of the operating frequency.

The measurement of P.I. is independent of the level of crystal vibration, provided that the electrical equivalent circuit parameters of Fig. 15.1 become constants; however, in actual practice these are not constants, particularly $R_1$. Variations of this type, as previously discussed, make it necessary to duplicate the amplitude of oscillation of the P.I. meter with the oscillator. Fig. 15.8 represents the variation of P.I. as a function of voltage across the crystal terminals for five crystals arbitrarily selected. It is readily observed that P.I. may be a random function of amplitude.

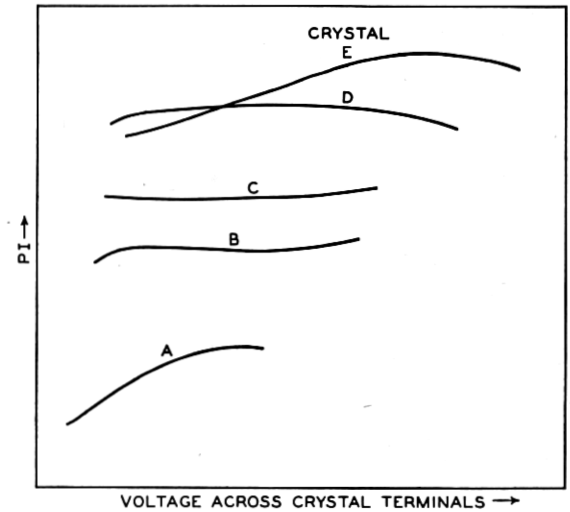


Fig. 15.8—Observed variation of Performance Index as a function of voltage across the crystal terminals.

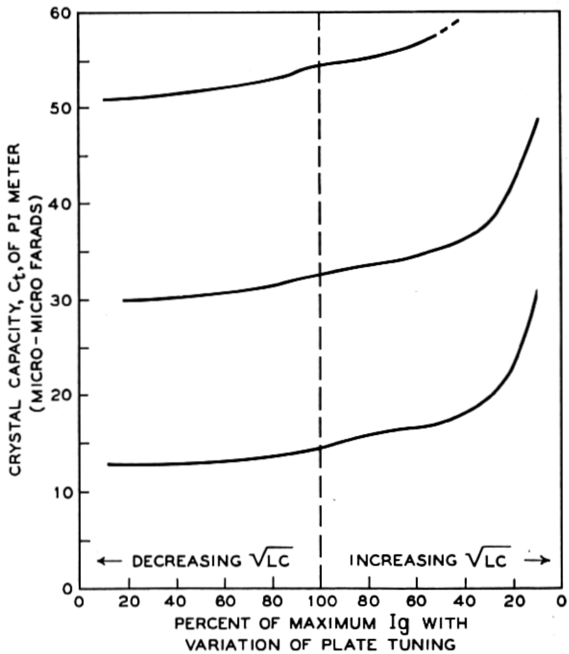


Fig. 15.9—Observed variation of effective crystal capacitance in a Miller oscillator.

Equation (15.44) indicates that the capacitance, $C_t$, determines the operating frequency between $\omega_1$ and $\omega_2$ of any given crystal. The capacitance, however, may include reflected reactances from associated circuits or possibly from circuits unintentionally coupled to the oscillator. Normally, crystals are adjusted to frequency for a specified value of $C_t$. This makes it of interest to measure the magnitude of $C_t$ over the range of the manual oscillator tuning adjustments, as well as over the frequency range of the oscillator. Fig. 15.9 shows the circuit capacitance, $C_t$, plotted as a function of tuning of the plate circuit of a Tuned-Plate oscillator. Tuning of the plate circuit is expressed in terms of percentage of maximum grid current.

## 15.80 Circuit Analysis Involving the Accuracy of P.I. Measurements

The method of P.I. measurement just described involved a number of unverified approximations. These approximations under the majority of conditions will be proven to be justified and the resulting expressions for percentage error will be obtained.

It is necessary to apply a method of analysis that is most readily adaptable to the crystal circuit. The analysis described in this section involves the use of Conformal representation as a means of determining (1) the behavior of the equivalent crystal circuit, (2) the error resulting in P.I. from assuming operation at the resonant frequency rather than the frequency for minimum impedance, and (3) the comparison of frequency of oscillation in the P.I. meter with other oscillator circuits.

Generally, the variations of reactance, $X$, and resistance, $R$, of the equivalent circuit (Fig. 15.1) are plotted as a function of frequency, and the analysis of the impedance, $R + jX$, between the resonant frequency, $\omega_1$, and anti-resonant frequency, $\omega_2$, are handled in precisely the same way as any linear passive element. This was essentially the procedure used to derive the equations in section 15.60. The analysis required to evaluate the errors leads to rather an elaborate study; however, in Section 15.81, it will be shown that it is very helpful analytically if the impedance of the crystal is plotted in the form of a circle diagram, that is, with the ordinate representing reactance, $X$, and the abscissa representing resistance, $R$.

### 15.81 *Conformal Representation*

Conformal Representation or Mapping is a convenient tool which for this application enables the physical operating condition to be expressed quantitatively from its graphical counterpart. Physical interpretation also makes it possible to draw many other conclusions that by other methods prove clumsy and laborious.

The basis for this analysis depends upon the ability to utilize the following equation to represent any impedance whose frequency of operation is controlled by a crystal. This equation is known as a linear fractional transformation

$$W = \frac{\alpha + \beta Z}{\gamma + \delta Z} \tag{15.52}$$

The terms $\alpha$, $\beta$, $\gamma$, and $\delta$ represent complex constants and $Z$ represents a complex variable later to be chosen to represent a linear function of frequency. Since $W$ and $Z$ represent the dependent and independent variable, they may also be considered as representing two separate planes. The abscissa and ordinate of these two planes represent their real and imaginary components respectively. The planes are linked by (15.52), that is, this equation will transform a specific point from one plane to the other.

The constants $\alpha, \beta, \gamma$ and $\delta$ for the equivalent crystal circuit are determined by writing the expression for $Z_c$ in the form of (15.52). For example (neglecting $R_L$), $Z_c$ from Fig. 15.1 may be written as

$$Z_c = \frac{1}{j\omega C_0} \frac{\left[R_1 + j\left(\omega L_1 - \frac{1}{\omega C_1}\right)\right]}{\left[R_1 + j\left(\omega L_1 - \frac{1}{\omega}\left(\frac{1}{C_1} + \frac{1}{C_0}\right)\right)\right]} \tag{15.53}$$

By substituting (15.48) and (15.49) in (15.53), this impedance may be written as

$$Z_c = \frac{1}{j\omega C_0} \frac{[\omega R_1 + jL_1(\omega - \omega_1)(\omega + \omega_1)]}{[\omega R_1 + jL_1(\omega - \omega_2)(\omega + \omega_2)]} \tag{15.54}$$

Since the operating frequency, $\omega$, represents some frequency between $\omega_1$ and $\omega_2$, and $\omega \gg \omega_2 - \omega_1$, we can make the following approximations in this operating range. The symbol $\omega_e$ is defined as the average operating radian frequency.

$$\left.\begin{aligned} \omega_e &= \frac{\omega_1 + \omega_2}{2} \\ \omega_e &\cong \omega \end{aligned}\right\} \tag{15.55}$$

If (15.55) is substituted in (15.54), factor $\omega_e$ out, add and subtract $\omega_2$ from the imaginary component in both numerator and denominator, we may write $Z_c$ as

$$Z_c = \frac{\left[\frac{1}{\omega_e C_0} + j\frac{1}{\omega_e C_0}\frac{2L_1}{R_1}(\omega_2 - \omega_1)\right] + \left[\frac{1}{\omega_e C_0}\right] j\frac{2L_1}{R_1}(\omega - \omega_2)}{[j] + [j]j\frac{2L_1}{R_1}(\omega - \omega_2)} \tag{15.56}$$

The constants $\alpha$, $\beta$, $\gamma$ and $\delta$ may be written immedately from (15.56); however, for purposes of simplification let

$$\left.\begin{aligned} \tau &= \frac{1}{\omega_e C_0} \\ Z &= j\frac{2L_1}{R_1}(\omega - \omega_2) \\ M &= \frac{2L_1}{R_1}(\omega_2 - \omega_1) \quad \text{(See equation 15.50)} \end{aligned}\right\} \tag{15.57}$$

then,

$$Z_c = \frac{[\tau + jM\tau] + [\tau]Z}{j + jZ} \tag{15.58}$$

Now if $Z_c$ may be represented by W in (15.52) the remaining constants must be

$$\left.\begin{aligned} \alpha &= \tau + j\tau M \qquad & \gamma &= j \\ \beta &= \tau & \delta &= j \end{aligned}\right\} \tag{15.59}$$

Now $Z$ represents the frequency variable and graphically represents a line coincident with the $Y$-axis in the $Z$-plane, and $W$, the corresponding impedance variable, represents a circle in the $W$-plane. The coordinates of the center of the circle, $W_0$, in the $W$-plane is given by

$$W_0 = \frac{\beta}{\delta}\frac{\left[\frac{\alpha}{\beta} - \frac{\bar{\gamma}}{\bar{\delta}}\right]}{\left[\frac{\gamma}{\delta} + \frac{\bar{\gamma}}{\bar{\delta}}\right]} \tag{15.60}^*$$

when $\bar{\gamma}$ and $\bar{\delta}$ represent the conjugate functions of $\gamma$ and $\delta$ respectively. The radius of the circle is given by

$$\sigma = \left|\frac{\beta}{\delta}\right| \frac{\left|\frac{\alpha}{\beta} - \frac{\gamma}{\delta}\right|}{\left|\frac{\gamma}{\delta} + \frac{\bar{\gamma}}{\bar{\delta}}\right|} \tag{15.61}^*$$

* E. C. Titchmarch, "Theory of Functions," Oxford 1932, pp. 191–192. Note: These equations are not derived in Titchmarch; however, by taking the limit as the diameter of the circle in the Z-plane approaches infinity, (15.60) and (15.61) result.

Now substituting the values given in (15.59) in (15.60) and (15.61) to determine the coordinates of the center of the circle and the radius respectively, we find

$$\left.\begin{aligned} W_0 &= \frac{\tau M}{2} - j\tau \\ \sigma &= \frac{M\tau}{2} \end{aligned}\right\} \tag{15.62}$$

As might be expected, the radius of the circle is equal to the real component of the coordinates of the center of the circle. This indicates that Fig. 15.10

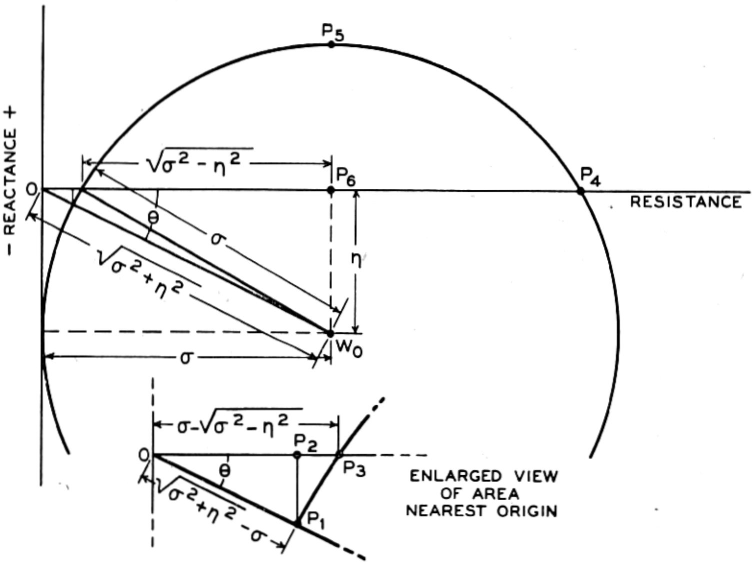


Fig. 15.10—Circle diagram for crystal circuit combinations.

graphically represents the circle diagram where $\eta \cong \tau$. The impedance, $Z_c$, is represented for a given frequency, as a vector from the origin to the corresponding point on the perimeter of the circle. From Fig. 15.10 the following crystal properties may be deduced.

1. The anti-resonant impedance designated as $O - P_4$ is simply $\sigma + \sqrt{\sigma^2 - \eta^2}$, and when evaluated equals $R_1(M^2 - 1) \cong R_1M^2$.
2. The resonant resistance, $O - P_3$ given by $\sigma - \sqrt{\sigma^2 - \eta^2}$ is equal to $R_1$.
3. The maximum positive reactance between $\omega_1$ and $\omega_2$ is represented by the distance from $P_5$ to $P_6$. It is given as $\sigma - \eta$ which equals $\left(\frac{R_1M^2}{2} - X_{c_0}\right)$.

4. The condition which must exist when the crystal reactance is zero between $\omega_1$ and $\omega_2$ occurs when $\sigma = \eta$ or $M = 2$.

It follows that the error of measuring, say the series resistance $R_1$, by varying the frequency for maximum transmission and assuming true resonance when the crystal is between two low non-inductive resistors is associated with the difference between the length of the two vectors $O - P_3$ and $O - P_1$. The per cent error caused by the crystal capacitor, $C_0$, by this method of measurement of $R_1$ is given as

$$\begin{aligned} \text{Per cent error of } R_1 &= 100\left[1 - \frac{O - P_1}{O - P_3}\right] \\ &= 100\left[1 - \frac{\sqrt{\sigma^2 + \eta^2} - \sigma}{\sigma - \sqrt{\sigma^2 - \eta^2}}\right] \end{aligned} \tag{15.63}$$

If $\sigma$ and $\eta$ are substituted in (15.63), we find

$$\text{Per cent error of } R_1 = 100\left[1 - \frac{1 + \sqrt{1 - \frac{4}{M^2}}}{1 + \sqrt{1 + \frac{4}{M^2}}}\right] \tag{15.64}$$

This difference in amplitude was caused by the difference in frequency between resonance and minimum impedance. This frequency difference may be determined by transforming the points $P_1$ and $P_3$ into the $Z$ plane by (15.52) and subtracting them arithmetically.

In order to express the coordinates of any point in the $W$-plane by its real and imaginary components let

$$W = R_0 + jX_0 \tag{15.65}$$

Now the coordinates of $P_3$ may be expressed as,

$$R_0 = \sigma\left[1 - \sqrt{1 - \left(\frac{\eta}{\sigma}\right)^2}\right]; \qquad X_0 = 0 \tag{15.66}$$

The coordinates of $P_1$ are similarly given by

$$\left.\begin{aligned} R_0 &= \sigma\left[1 - \frac{1}{\sqrt{1 + \left(\frac{\eta}{\sigma}\right)^2}}\right] \\ X_0 &= -\eta\left[1 - \frac{1}{\sqrt{1 + \left(\frac{\eta}{\sigma}\right)^2}}\right] \end{aligned}\right\} \tag{15.67}$$

The coordinates of these two points represent values of $W$; now (15.52) may be solved for $Z$.

$$Z = \frac{\alpha - \gamma W}{\delta W - \beta} \tag{15.68}$$

Substituting in values for $\alpha, \beta, \gamma$ and $\delta$ given by (15.59), we find

$$Z = j\frac{2L_1}{R_1}(\omega - \omega_2) = j\frac{M(\omega - \omega_2)}{(\omega_2 - \omega_1)} = -j2\frac{\sigma(\eta + X_0)}{(\eta + X_0)^2 + R_0^2} \tag{15.69}$$

The real component of $Z$ must be zero since the function of $Z$ is coincident with the $Y$-axis. By substituting values of $R_0$ and $X_0$ from (15.66) and (15.67), we find

$$\omega - \omega_2 = -\frac{(\omega_2 - \omega_1)}{M}\frac{\eta/\sigma}{[1 - \sqrt{1 - (\eta/\sigma)^2}]} \tag{15.70}$$

also

$$\omega - \omega_2 = -\frac{(\omega_2 - \omega_1)}{M}\frac{\eta/\sigma}{\left[\sqrt{1 + \left(\frac{\eta}{\sigma}\right)^2} - 1\right]} \tag{15.71}$$

Subtracting (15.71) from (15.70) to get $\Delta\omega$ we have

$$\Delta\omega = \frac{\eta}{\sigma}\frac{(\omega_2 - \omega_1)}{M}A \tag{15.72}$$

where

$$A = \frac{\sqrt{1 - (\eta/\sigma)^2} - \sqrt{1 + (\eta/\sigma)^2}}{(\eta/\sigma)^2} \tag{15.73}$$

Now the $\lim_{\eta/\sigma \to 0} A = 1$. When $\frac{\eta}{\sigma} = \frac{2}{M}$ then $\Delta\omega = \frac{\omega_2 - \omega_1}{M}A$

### 15.82 *Circuit Analysis Involving Crystals*

The same procedure could be followed for the impedance, $Z_i$, in Fig. 15.3; however, the impedance expression conforming to (15.52) may be written directly if a more general expression is derived for impedances added in parallel or series.

Paralleling the impedance, $W$, with an impedance, $T$, (Fig. 15.11) modifies the constants in (15.52) but not its form providing $T$ is essentially constant between $\omega_1$ and $\omega_2$. For parallel impedances the impedance equation becomes,

$$\frac{WT}{W + T} = \frac{\alpha + \beta Z}{\left(\gamma + \frac{\alpha}{T}\right) + \left(\delta + \frac{\beta}{T}\right)Z} = \frac{\alpha' + \beta' Z}{\gamma' + \delta' Z} \tag{15.74}$$

The numerator remains the same; however, the denominator has an additional term added to both $\gamma$ and $\delta$.

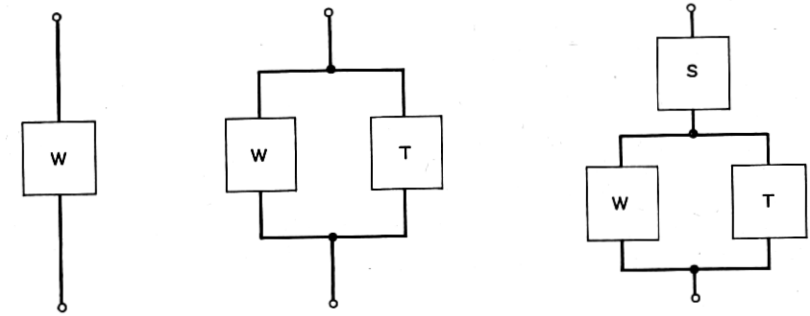


Fig. 15.11—Various crystal circuit combinations.

In the same way, a series element, $S$, may be added as shown in Fig. 15.11. The impedance of this combination is similarly modified and the input impedance may be expressed as

$$S + \frac{WT}{W+T} = \frac{\left[\alpha + S\left(\gamma + \frac{\alpha}{T}\right)\right] + \left[\beta + S\left(\delta + \frac{\beta}{T}\right)\right]Z}{\left(\gamma + \frac{\alpha}{T}\right) + \left(\delta + \frac{\beta}{T}\right)Z} \tag{15.75}$$

$$= \frac{\alpha'' + \beta'' Z}{\gamma'' + \delta'' Z}$$

The addition of a series element leaves the denominator unchanged but adds a term to the numerator. The form of (15.75) is exactly the same as (15.52) except that the magnitudes of the constants, $\alpha$, $\beta$, $\gamma$ and $\delta$ have been modified.

As an example, apply (15.74) to obtain an expression for the impedance $Z_{AB}$ (Fig. 15.2). This expression for impedance may be written directly if the following values are substituted in (15.74).

$$\left.\begin{aligned} \alpha &= \tau + jM\tau \\ \beta &= \tau \\ \gamma &= \delta = j \\ T &= \frac{1}{j\omega C_t} \end{aligned}\right\} \tag{15.76}$$

$$Z_{AB} = \frac{WT}{W+T} = \frac{[\tau + jM\tau] + [\tau]Z}{\left[-M\frac{C_t}{C_0} + j\left(1 + \frac{C_t}{C_0}\right)\right] + \left[j\left(1 + \frac{C_t}{C_0}\right)\right]Z} \tag{15.77}$$

From this expression we can again see the same form as (15.52) except that the coefficients of $\alpha, \beta, \gamma$ and $\delta$ are modified by the capacitance, $C_t$, in parallel with the equivalent crystal circuit. It has been previously explained that the frequency of oscillation in the generalized oscillator is determined by the anti-resonant frequency of the impedance, $Z_{AB}$, and that this impedance represents the exact definition of P.I. P.I. therefore is represented in magnitude by $O - P_4$ in Fig. 15.10.

Equation (15.75) enables us to write the expression for the input impedance, $Z_i$, for the P.I. meter directly and furthermore, it enables us to compute the error produced by the adjustment of $C_r$ to minimum impedance rather than unity power factor as assumed in the derivation of (15.28). This error obviously will be a function of $M$, $C_0$, $C_t$ and $R_L$. Writing the impedance expression for $Z_i$ directly, requires a little modification in that the crystal is shunted by two elements, $C_x$ (the crystal socket capacitance) and $R_L$ (the holder loss) as well as having the combination in series with the capacitor, $C_3$, where $C_3 = \dfrac{C_s C_k}{C_s + C_k}$. Shunting the crystal with $C_x$ modifies (15.77) only in that $C_t = C_x$; from this we can use (15.75) directly in a form readily adaptable to the determination of our original coefficients, $\alpha''$, $\beta''$, $\gamma''$ and $\delta''$. Adding a series capacitor, $C_3$, and a shunt resistor, $R_L$, modifies (15.77) in a manner specified by (15.75). Here,

$$\left.\begin{aligned} \alpha &= \tau + jM\tau & \delta &= j\left(1 + \frac{C_x}{C_0}\right) \\ \beta &= \tau & T &= R_L \\ \gamma &= -M\frac{C_x}{C_0} + j\left(1 + \frac{C_x}{C_0}\right) & S &= \frac{1}{j\omega C_3} \end{aligned}\right\} \qquad (15.78)$$

Substitute these constants given by (15.78) in (15.75) then $\alpha''$, $\beta''$, $\gamma''$ and $\delta''$ become,

$$\left.\begin{aligned} \alpha'' &= \alpha + S\left(\gamma + \frac{\alpha}{T}\right) = \left[\tau + X_{c_3}\left(1 + \frac{C_x}{C_0}\right) + \frac{M\tau X_{c_3}}{R_L}\right] \\ &\qquad + j\left[M\tau + \frac{MC_x X_{c_3}}{C_0} - \frac{\tau}{R_L}X_{c_3}\right] \\ \beta'' &= \beta + S\left(\delta + \frac{\beta}{T}\right) = \left[\tau + X_{c_3}\left(1 + \frac{C_x}{C_0}\right)\right] - j\left[\frac{X_{c_3}\tau}{R_L}\right] \\ \gamma'' &= \left(\gamma + \frac{\alpha}{T}\right) = \left[-M\frac{C_x}{C_0} + \frac{\tau}{R_L}\right] + j\left[1 + \frac{C_x}{C_0} + \frac{M\tau}{R_L}\right] \\ \delta'' &= \left(\delta + \frac{\beta}{T}\right) = \left[\frac{\tau}{R_L} + j\left(1 + \frac{C_x}{C_0}\right)\right] \end{aligned}\right\} \qquad (15.79)$$

Substitute these values in (15.60) and (15.61) to obtain the coordinates of the center of the circle, $W_0$, and the radius, $\sigma$.

$$W_0 = \tau \frac{\left[\left(M \frac{C_0}{C_0 + C_x} + 2 \frac{\tau}{R_L}\right) - 2j\left(1 + \frac{C_0 + C_x}{C_3} + \frac{\tau X_{c_3}}{R_L^2} + \frac{M\tau}{R_L} \frac{C_0}{C_3}\right)\right]}{2\left[1 + \left(\frac{\tau}{R_L}\right)^2 + \frac{MC_0}{C_0 + C_x} \frac{\tau}{R_L}\right]} \tag{15.80}$$

$$\sigma = \tau \frac{M \frac{C_0}{C_0 + C_x}}{\left[1 + \left(\frac{\tau}{R_L}\right)^2 + \frac{MC_0}{C_0 + C_x} \frac{\tau}{R_L}\right]} \tag{15.81}$$

From the above expressions, $\sigma$ is not exactly equal to the real component of $W_0$. If however, $2 \frac{\tau}{R_L} \ll \frac{MC_0}{C_0 + C_x}$, then the above equations reduce to

$$W_0 = \tau \frac{\left[M \frac{C_0}{C_0 + C_x} - 2j\left(1 + \frac{C_0 + C_x}{C_3} + \frac{\tau MC_0}{R_L C_3}\right)\right]}{\left[2\left(1 + \frac{MC_0}{C_0 + C_x} \frac{\tau}{R_L}\right)\right]} \tag{15.82}$$

and

$$\sigma = \frac{\tau\left[\frac{MC_0}{C_0 + C_x}\right]}{2\left[1 + \frac{MC_0}{C_0 + C_x} \frac{\tau}{R_L}\right]} \tag{15.83}$$

Now the radius, $\sigma$, equals the real component of $W_0$. The per cent error of P.I. resulting from tuning to minimum impedance rather than unity power-factor in terms of Fig. 15.10 is given as,

$$\text{Per Cent Error of P.I.} = 100\left[1 - \frac{O - P_3}{O - P_1}\right] = 100\left[1 - \frac{\left[1 - \sqrt{1 - \left(\frac{\eta}{\sigma}\right)^2}\right]}{\left[\sqrt{1 + \left(\frac{\eta}{\sigma}\right)^2} - 1\right]}\right] \tag{15.84}$$

Now since $\eta$ represents the imaginary component of (15.82) then

$$\frac{2\sigma}{\eta} = P_2 = \frac{M}{\left(1 + \frac{C_x}{C_0}\right)\left(1 + \frac{C_0 + C_x}{C_3} + M \frac{X_{c_t}}{R_L}\right)} \tag{15.85}$$

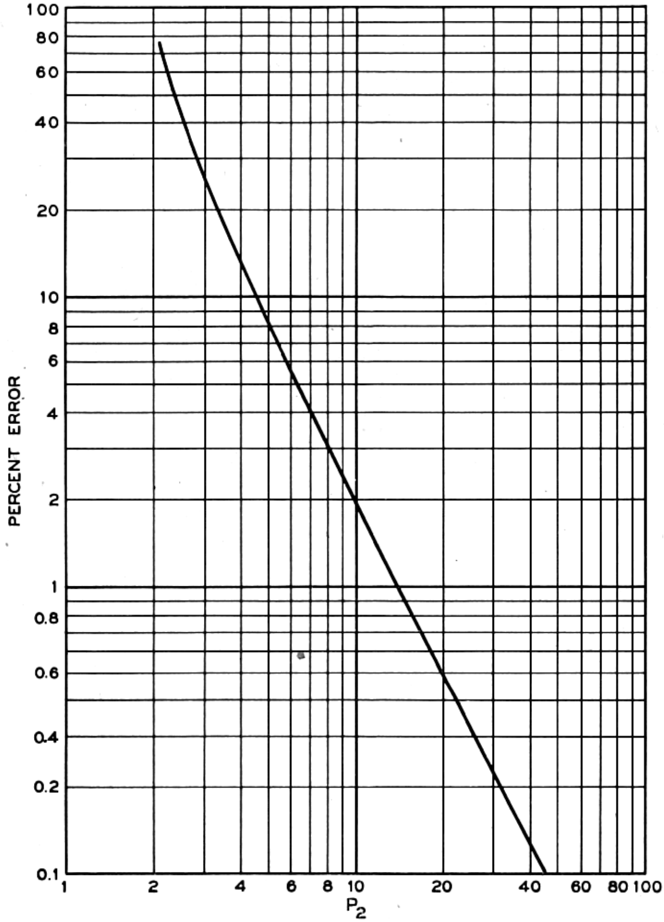


Fig. 15.12—Inherent error of the P.I. meter due to tuning for an indication of minimum impedance rather than unity power factor.

Rewriting (15.84) we have

$$\text{Per cent Error of P.I.} = 100\left[1 - \frac{\left[1 - \sqrt{1 - \frac{4}{P_2^2}}\right]}{\left[\sqrt{1 + \frac{4}{P_2^2}} - 1\right]}\right] \tag{15.86}$$

If $C_0 \gg C_x$ then

$$P_2 = \frac{M}{\left(1 + \frac{C_0}{C_t} + M\,\frac{X_{c_t}}{R_L}\right)} \tag{15.87}$$

The error given by (15.86) is plotted in Fig. 15.12 as a function of $P_2$.

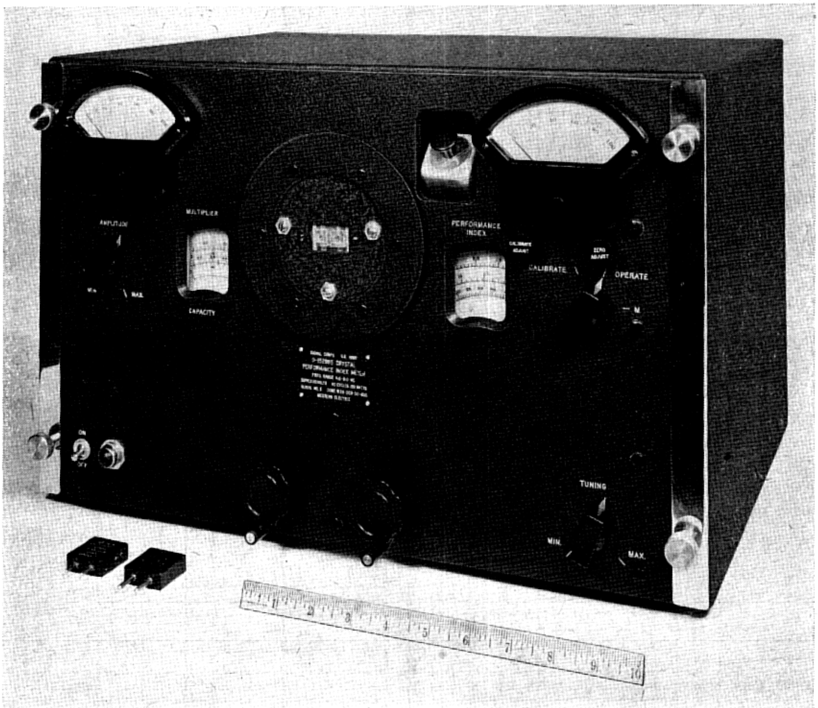


Fig. 15.13—Exterior view of crystal Performance Index meter.

### 15.83 *Frequency Errors*

As suggested in Section 15.81, conformal representation simplifies the mathematics required for the determination of frequency errors. In Section 15.81, the difference between the resonant frequency and the minimum impedance frequency was computed for the equivalent crystal circuit. This same procedure could be used to compute the frequency difference between the antiresonant frequency of the generalized oscillator circuit (Fig. 15.2) and the minimum impedance frequency of the P.I. meter for the same value of $C_t$. Comparison of the frequency of these two oscillators is plotted in Fig. 15.6 together with the measured values obtained from a "Pierce" and a "Tuned Plate" oscillator. This frequency comparison involves setting up two circle diagrams, one for $Z_{AB}$ (Fig. 15.2) and one for the impedance, $Z_i$ (Fig. 15.3) similar to Fig. 15.10. The impedance equations for both $Z_{AB}$ and $Z_i$ would be arranged such that they have the same function of $Z$ in order to have a common $Z$ plane. In this way, transformation of operating points such as the "anti-resonant frequency" operating point ($P_4$) for the $Z_{AB}$ impedance circle, could be subtracted from the "minimum impedance frequency" operating point ($P_1$) of the P.I. meter impedance circle in the common $Z$-plane. As in section 15.81, the frequency difference represents the arithmetic difference between $P_4$ and $P_1$ in the $Z$-plane in terms of $\frac{2L_1}{R_1}(\omega - \omega_2)$. As an example, look at the calculated curve in Fig. 15.6. This curve was computed for the case when $R_L$ is negligible. The derivation of (15.51) given in section 15.92 precisely follows the procedure just described.

It is of interest to note that (15.27) may also be derived by Conformal means. It is more laborious than the usual circuit equations of section 15.2; however, it does provide a check of the methods used.

### 15.84 *Errors of Other Approximations*

Further consideration of P.I. meter errors leads to the assumptions made in the derivation of (15.27). The derivation of (15.27) assumed that the resistor, $R_A$, was non-reactive. While actually it can be made essentially noninductive, we have neglected the effect of the input capacitance of the attenuator that is shunted across its terminals. The error from neglecting this capacitance in (15.28) is given by the following expression

$$\text{Per Cent Error} = 100\left[1 - \frac{\sqrt{1 + Q_A^2}}{\omega R_A(C_u Q_A^2 + C_A)}\right] \tag{15.88}$$

Where $Q_A$ equals the reactance of the shunt capacitance of the attenuator, $C_u$, divided by the magnitude of the calibration resistor, $R_A$.

It is interesting to note in the derivation of (15.28) that $R_A$ was assumed to be very much less than $X_{CA}$ which introduces an error of,

$$\text{Per Cent Error} = 100\left[1 - \sqrt{1 + \left[\frac{R_A}{X_{C_A}}\right]^2}\right] \tag{15.89}$$

## 15.90 *Derivation of Circuit Equations*

### 15.91 *Derivation of Equation* (15.42)

Other equations used in this paper may best be developed from Fig. 15.3. By analysis of the input impedance, $Z_i$, the basis for the development of (15.42) is as follows:

From Fig. 15.2

$$Z_i = \frac{1}{j\omega C_t} + \frac{1}{j\omega C_0} \frac{\left[R_1 + j\left(\omega L_1 - \frac{1}{\omega C_1}\right)\right]}{\left[R_1 + j\left[\omega L_1 - \frac{1}{\omega}\left(\frac{1}{C_0} + \frac{1}{C_1}\right)\right]\right]} \tag{15.90}$$

By substituting (15.48) and (15.49) in (15.90), we find

$$Z_i = \frac{1}{j\omega}\left[\frac{1}{C_t} + \frac{1}{C_0}\left[\frac{\omega R_1 + jL_1(\omega^2 - \omega_1^2)}{\omega R_1 + jL_1(\omega^2 - \omega_2^2)}\right]\right] \tag{15.91}$$

This may be expressed in the form,

$$Z_i = \frac{\frac{\omega R_1}{H} + jL_1\left[\frac{(\omega^2 - \omega_2^2)}{C_t} + \frac{(\omega^2 - \omega_1^2)}{C_0}\right]}{-\omega L_1(\omega^2 - \omega_2^2) + j\omega^2 R_1} \tag{15.92}$$

where

$$H = \frac{C_0 C_t}{C_0 + C_t}$$

Now adding and subtracting $\omega_2^2$ to the $\frac{\omega_2^2 - \omega_1^2}{C_0}$ term we have

$$Z_1 = \frac{\frac{\omega R_1}{H} + jL_1\left[\frac{(\omega^2 - \omega_2^2)}{C_t} + \frac{(\omega^2 - \omega_2^2)}{C_0} + \frac{(\omega_2^2 - \omega_1^2)}{C_0}\right]}{-\omega L_1(\omega^2 - \omega_2^2) + j\omega^2 R_1} \tag{15.93}$$

Rationalizing (15.93) and equating it to $R_e$ and substituting in $L_1 = \frac{1}{C_0(\omega_2^2 - \omega_1^2)}$ (obtained from (15.50)), we find

$$R_e = \frac{R_1}{\left[\frac{\omega^2 - \omega_2^2}{\omega_2^2 - \omega_1^2}\right]^2 + \left[\frac{\omega}{\omega_1}\frac{1}{M}\right]^2} \tag{15.94}$$

15.92 *Derivation of Equations* (15.51) *and* (15.44)

Equation (15.51) makes possible the theoretical computation of the frequency difference between the generalized oscillator and the minimum impedance frequency adjustment of the *PI* meter. The derivation assumes that $R_L$ is negligible and that the total capacitance across the crystal terminals is lumped in series with the crystal.

The impedance, $Z_{AB}$, in the generalized oscillator, Fig. 15.2, was given by (15.77). This equation may be expressed as follows:

$$Z_{AB}\,\omega_e(C_0 + C_t) = \frac{[1 + jM] + Z}{\left[-\frac{MC_t}{C_0 + C_t} + j\right] + jZ} \tag{15.95}$$

From this expression, as previously explained, (see Section 15.81), the following values for $\sigma$ and $\eta$ may be determined.

$$\left.\begin{aligned} \sigma &= \frac{M}{2}\,\frac{C_0}{C_0 + C_t} \\ \eta &= 1 \end{aligned}\right\} \tag{15.96}$$

The anti-resonant impedance of $Z_{AB}$ is represented by $O - P_4$ in Fig. 15.10. The left hand term of (15.95) for the $P_4$ operating point becomes

$$Z_{AB}\,\omega_e(C_0 + C_t) = \sigma + \sqrt{\sigma^2 - 1} \tag{15.97}$$

Substituting this value in (15.95) and solving for $Z$, we find

$$Z = -jM\left[K_3\,\frac{C_0}{C_0 + C_t} - 1\right] \tag{15.98}$$

where

$$K_3 = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{\sigma^2}}$$

If $K_3$ is expanded and all except the first two terms are neglected, $Z$ may be expressed as

$$Z = -j\left[\frac{M}{1 + \dfrac{C_0}{C_t}} + \frac{\left(1 + \dfrac{C_t}{C_0}\right)}{M}\right] \tag{15.99}$$

The next step is to obtain a similar expression to (15.99) only for the minimum frequency impedance of $Z_i$ in Fig. 15.3 with $\rho$ disconnected. For this application $S = \dfrac{1}{j\omega C_t}$ and $T = \infty$. Substituting these values, as well as those in (15.59), in (15.75), we have

$$Z_i = \frac{\left[\tau + jM\tau + \dfrac{1}{\omega C_t}\right] + \left[\tau + \dfrac{1}{\omega C_t}\right] Z}{j + jZ} \tag{15.100}$$

From this equation, values for $\sigma$ and $\eta$ may be determined as described in Section 15.81.

$$\left.\begin{aligned} \sigma &= \frac{M\tau}{2} \\ \eta &= \frac{C_0 + C_t}{C_t}\,\tau \end{aligned}\right\} \tag{15.101}$$

By the same procedure just described for evaluating $Z$ from the impedance expression $Z_{AB}$, the value of $Z$ corresponding to the minimum impedance operating point, $P_1$, (Fig. 15.10) must be determined. For this operating condition $Z_1$ may be expressed by (15.65). The coefficients of this operating point are given by (15.67) with the above values of $\sigma$ and $\eta$ (Equation 15.101). Utilizing (15.68) to solve for $Z$, we get

$$Z_i' = -j \frac{M}{\left(1 + \frac{C_0}{C_t}\right)} \frac{\left[1 + \sqrt{1 + \frac{4}{P_1^2}}\right]}{2} \quad \text{(minimum impedance)} \qquad (15.102)$$

now

$$Z_i - Z_i' = \frac{2L_1}{R_1} (\Delta\omega)$$

where $\Delta\omega$ = the difference in radian frequency between the frequency of oscillation in the generalized oscillator (anti-resonant frequency of the impedance, $Z_{AB}$) and the frequency of oscillation in the *PI* meter (minimum impedance frequency of the impedance, $Z_i'$)

$$\Delta\omega = \left[\frac{\omega_2 - \omega_1}{M^2}\left(1 + \frac{C_0}{C_t}\right)\frac{2}{\left[1 + \sqrt{1 + \frac{4}{P_1^2}}\right]} - \frac{\omega_2 - \omega_1}{M^2}\left(1 + \frac{C_t}{C_0}\right)\right] \qquad (15.51)$$

It is of interest to note that (15.102) becomes (15.44) when the value of $Z$ (15.69) is introduced.