A Solution for Faults at Two Locations in Three-Phase Power Systems

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This paper is an outgrowth of studies of double faults to ground in three-phase power systems made by the author in connection with work of the Joint Subcommittee on Development and Research, Edison Electric Institute and Bell Telephone System. The paper provides a systematic solution, based on the method of symmetrical components, by means of which currents and voltages can be determined at times of fault involving any combination of phases at one or two locations on three-phase power systems.

1. Introduction

A KNOWLEDGE of the magnitude and phase relation of power system voltages and currents for various types of faults in three-phase systems is of importance in the study of various problems, among which are relaying studies, the efficacy of current limiting devices and their reaction on the power network, and estimates of induction in paralleling communication circuits.

The method of symmetrical components as developed by Fortescue [1] and others is now extensively used in the solution for currents and voltages in three-phase power systems under fault (short-circuit) conditions. Formulas for special cases of faults, such as single and double line-to-ground faults, can be found in various text books on this subject. The solution for simultaneous faults at two locations has been treated by Miss Clark,[2] in a form particularly adaptable to the use of a calculating board.

The present development provides a complete and systematic solution for currents and voltages at times of fault on any number of phases at one or two locations in a three-phase system, in which generators may be assumed in phase and of the same internal voltage, and where load currents can be neglected. These are the usual assumptions made in computing fault currents, except for certain special problems, such as that of power system stability. The methods employed herein could be extended to cases where generators of different phase angles and voltages of more than two points of fault are involved. Formally such cases can be treated in a manner similar to that given in the paper. The number of impedances to which an n-terminal network

[1] Reference numbers refer to references appearing at the end of the article.

can be reduced is given by the expression $\frac{1}{2}n(n-1)$. For $n = 3$, the case treated in this paper, three impedances are required which necessitate six equations for the general solution of the six fault currents. For $n = 4$, which would be the case for three points of fault (or two points of fault and two generating voltages), six impedances would appear in the reduced network and this would necessitate twelve equations for the general solution of the fault currents. For larger values of n, the necessary number of equations increases rapidly, thus making the solution impractical. Such problems usually, as a practical matter, are more readily solved by the use of a-c. calculating boards.

While no departure from the general methods of symmetrical components is made in the present development, a systematic method of handling the equations is presented and means of determining the coefficients given so that numerical calculations can be directly carried out when the constants of the network are known.

2. General Solution

The equations developed in this paper are based on the sequence impedances looking into a three-phase network from two points of fault.

Consider the network shown in Fig. 1. This system can be reduced

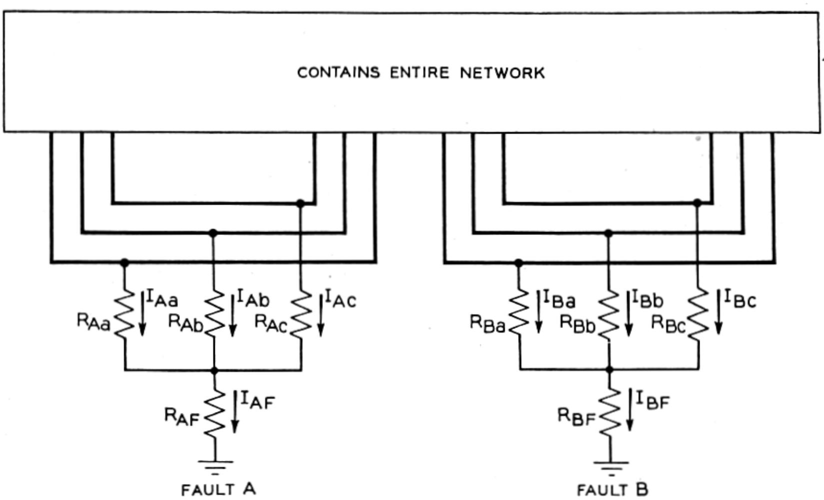

Fig. 1—General network diagram.

to an equivalent star for each of the positive, negative and zero sequence networks, with legs to the points of generation and to the faults at A and B. Figure 2 shows the reduced positive sequence network. Similar diagrams can be made for the negative and zero sequence systems except for the fact that in these cases there are no generated voltages, and the impedances and currents are the negative and zero sequence quantities.

The reduction of a network to an equivalent star is usually a tedious and sometimes a difficult process especially in large interconnected systems. Methods of accomplishing the reduction, such as delta-star transformations, simultaneous equations or direct measurements on calculating boards can be found in the literature.[3, 5]

Having reduced the three sequence networks to equivalent stars, the equations are developed as shown in the Appendix.

The following set of equations (1) is the general solution for fault currents during simultaneous three-phase faults to ground at two different locations in a power network. The set applies directly to the calculation of ground fault currents on a system having finite

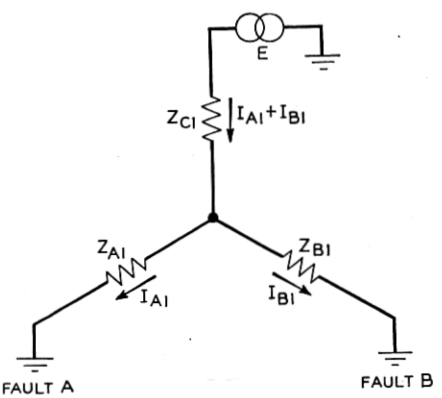

Fig. 2—Reduced positive sequence diagram.

neutral impedances or an isolated system in which the zero sequence capacitance is taken into account. For other types of faults, such as faults to ground in isolated systems in which zero sequence capacitance has been neglected, or for phase to phase faults, set (1) is not directly applicable since some of the constants become infinitely large. However, by certain transformations of set (1), more convenient sets (2) and (3) are obtained, directly applicable for solution of these latter cases.

Neutral Grounded System

I_{Aa}	I_{Ab}	I_{Ac}	I_{Ba}	I_{Bb}	I_{Bc}		
A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}	$3E$	(Aa)
A_{21}	A_{22}	A_{23}	A_{24}	A_{25}	A_{26}	$3a^2E$	(Ab)
A_{31}	A_{32}	A_{33}	A_{34}	A_{35}	A_{36}	$3aE$	(Ac)
A_{41}	A_{42}	A_{43}	A_{44}	A_{45}	A_{46}	$3E$	(Ba)
A_{51}	A_{52}	A_{53}	A_{54}	A_{55}	A_{56}	$3a^2E$	(Bb)
A_{61}	A_{62}	A_{63}	A_{64}	A_{65}	A_{66}	$3aE$	(Bc)

(1)

The six equations are written in matrix form with the currents and voltages outside the system matrix. For example the first row in (1) is interpreted as:

$$A_{11}I_{Aa} + A_{12}I_{Ab} + A_{13}I_{Ac} + A_{14}I_{Ba} + A_{15}I_{Bb} + A_{16}I_{Bc} = 3E$$

The values of the A's in (1) are given in Table I. It should be noted that of the 36 constants only 13 are distinct. Six of these are in the nature of self-impedances, two are transfer impedances between phases at A and two between phases at B. The remaining three are transfer impedances between the two faults at A and at B.

Considerable reductions in the constants are obtained when the positive and negative sequence impedances are assumed equal. These values are given in Table II.

Faults to ground on less than three phases at one or both locations are accounted for by assuming the corresponding fault resistances infinitely large. The currents to ground in the sound phases are zero. Striking out the columns containing these currents and the corresponding rows, indicated by the index at right in equation (1), a reduced set of equations is obtained from which the desired currents can be found. A few examples are given in subsequent sections.

In power networks with isolated neutral the zero sequence impedance Z_{C0} reduces essentially to the capacitance of the system. In this case equations (1) are still appropriate and will give a rigorous solution for the six currents. However, in many cases it is sufficiently accurate to neglect the capacitance of the system. This results in infinitely large values of all of the A's in Table I (Table II), since each depends on Z_{C0} which is infinitely large. For this condition it is desirable to transform the set of equations in (1) to a more convenient set with finite constants.

The transformation required is obtained by observing that, with Z_{C0} infinitely large, the sum of the zero sequence currents $I_{A0} + I_{B0}$ must be equal to zero. Making use of this relation the difference of the zero sequence voltages at A and B (equation (50) of appendix) reduces to:

$$V_{A0} - V_{B0} = (Z_{A0} + Z_{B0})I_{B0}$$

The last equation shows that subtraction of equations associated with phases at A from those at B removes the infinitely large element Z_{C0}. This can be done in nine ways (ignoring reversals of sign), but three of these result in the single equation:

$$I_{Aa} + I_{Ab} + I_{Ac} + I_{Ba} + I_{Bb} + I_{Bc} = 0$$

This equation with any five of the remaining six constitutes an independent set; for convenience in dealing with special cases the redundant set of seven equations is shown in the following array:

Isolated System—Capacitance Neglected

I_{Aa}	I_{Ab}	I_{Ac}	I_{Ba}	I_{Bb}	I_{Bc}			
B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	$3(1-a^2)E$	$(Aa-Bb)$	
B_{21}	B_{22}	B_{23}	B_{24}	B_{25}	B_{26}	$3(1-a)E$	$(Aa-Bc)$	
B_{31}	B_{32}	B_{33}	B_{34}	B_{35}	B_{36}	$3(a^2-1)E$	$(Ab-Ba)$	(2)
B_{41}	B_{42}	B_{43}	B_{44}	B_{45}	B_{46}	$3(a^2-a)E$	$(Ab-Bc)$	
B_{51}	B_{52}	B_{53}	B_{54}	B_{55}	B_{56}	$3(a-1)E$	$(Ac-Ba)$	
B_{61}	B_{62}	B_{63}	B_{64}	B_{65}	B_{66}	$3(a-a^2)E$	$(Ac-Bb)$	

$$I_{Aa} + I_{Ab} + I_{Ac} + I_{Ba} + I_{Bb} + I_{Bc} = 0 \tag{2a}$$

The index to the right indicates which of the equations in (1) have been used. The values of the B's are given in Table I and Table II.

In case of faults to ground on less than three phases, as in equations (1), columns and rows associated with sound phase currents are to be deleted; with respect to the rows, however, the index is double and all rows having the index of the sound phase or phases are deleted. If, for example, the sound phase is Aa, rows 1 and 2, each of which contains Aa in its index, as well as column 1, are deleted. This leaves only four equations, which together with (2*a*) give the necessary five equations for the five currents. For this reason all six equations are given in (2), since any phase might be involved in special cases.

Phase-to-phase faults are obtained from the general case (1) by allowing the resistances R_{AF} and R_{BF} to become infinite. In this case phase-to-phase quantities at the same location remain finite and the appropriate set of equations is obtained by subtracting equations having the corresponding phase indexes; thus $Aa - Ab$, $Aa - Ac$ and $Ab - Ac$ indicate subtractions at A. There are six possible ways of doing this, ignoring reversals of sign. The resulting set is given in (3). The four equations obtained by taking any two of the first three and any two of the last three equations in this set together with the two equations (3*a*) relating to the sum of the currents at each fault location, which from physical considerations equal zero, constitute an independent set. For convenience in dealing with special cases all eight equations are given below:

Phase-to-Phase Faults

I_{Aa}	I_{Ab}	I_{Ac}	I_{Ba}	I_{Bb}	I_{Bc}			
C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	$3(1-a^2)E$	$(Aa-Ab)$	
C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}	$3(1-a)E$	$(Aa-Ac)$	
C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}	$3(a^2-a)E$	$(Ab-Ac)$	(3)
C_{41}	C_{42}	C_{43}	C_{44}	C_{45}	C_{46}	$3(1-a^2)E$	$(Ba-Bb)$	
C_{51}	C_{52}	C_{53}	C_{54}	C_{55}	C_{56}	$3(1-a)E$	$(Ba-Bc)$	
C_{61}	C_{62}	C_{63}	C_{64}	C_{65}	C_{66}	$3(a^2-a)E$	$(Bb-Bc)$	

$$I_{Aa} + I_{Ab} + I_{Ac} = 0 \qquad (3a)$$
$$I_{Ba} + I_{Bb} + I_{Bc} = 0$$

The index to the right indicates which equations of (1) have been used. The values of the C's are given in Table I and Table II.

The total ground fault currents at the two fault locations are:

$$I_{AF} = I_{Aa} + I_{Ab} + I_{Ac} \tag{4}$$

$$I_{BF} = I_{Ba} + I_{Bb} + I_{Bc} \tag{5}$$

and the total residual current in the two faults is:

$$I_R = I_{AF} + I_{BF} \tag{6}$$

In an isolated system in which capacitance has been neglected this current is zero and equation (6) will be identical with (2a).

The distribution of these currents in the network can be found as follows. From equations (51) and (52) of the appendix the calculated fault currents are transformed into sequence currents. By working back into the original sequence networks the sequence currents in each branch of the system can be found and later combined from similar expressions as shown in (43) and (44) to obtain the actual branch currents.

A combination of the equations in (1) and (3) can be used for cases involving faults to ground at one location and faults between phases (not involving ground) at the other location.

For faults to ground at A and between phases at B, the three first equations in (1) and the three last in (3) together with the last in (3a) constitute the most convenient set of equations for this type of fault.

It should be noted that if all three phases are involved at B any two of the three last equations in (3) together with the last in (3a) can be used, while for less than three phases involved, the rules for striking out rows and columns automatically will result in the proper equations to be used.

Vice versa the three first equations in (3) together with the first in (3a) and the three last equations in (1) will give the solutions for phase-to-phase faults at A and ground faults at B.

This will be illustrated with an example in a later section.

The voltages to ground at the two locations of faults can be obtained directly from equations (53) and (54) of the Appendix, after the currents have been evaluated. At any other point in the system the voltages to ground are found by adding the voltage drops of the lines in question to these voltages, treating each sequence network separately, then adding the sequence voltages together according to equations (46) or (47).

TABLE I

CONSTANTS FOR EQUATIONS (1), (2) AND (3)

Equation (1)	Equation (2)	Equation (3)	Equalities
$A_{11} = S_a + 3R_{Aa} + 3R_{AF}$	$B_{11} = S_a - T_c + 3R_{Aa} + 3R_{AF}$	$C_{11} = S_a - S_c + 3R_{Aa}$	Equation (1)
$A_{12} = S_b + 3R_{AF}$	$B_{12} = S_b - T_a + 3R_{AF}$	$C_{12} = - S_a + S_b - 3R_{Ab}$	$A_{12} = A_{23} = A_{31} \quad A_{45} = A_{56} = A_{64}$
$A_{13} = S_c + 3R_{AF}$	$B_{13} = S_c - T_b + 3R_{AF}$	$C_{13} = S_c - S_b$	$A_{13} = A_{21} = A_{32} \quad A_{46} = A_{54} = A_{65}$
$A_{14} = T_a$	$B_{14} = - U_c + T_a - 3R_{BF}$	$C_{14} = T_a - T_c$	$A_{14} = A_{25} = A_{36} = A_{41} = A_{52} = A_{63}$
$A_{15} = T_b$	$B_{15} = - U_a + T_b - 3R_{Bb} - 3R_{BF}$	$C_{15} = T_b - T_a$	$A_{15} = A_{26} = A_{34} = A_{42} = A_{53} = A_{61}$
$A_{16} = T_c$	$B_{16} = - U_b + T_c - 3R_{BF}$	$C_{16} = T_c - T_b$	$A_{16} = A_{24} = A_{35} = A_{43} = A_{51} = A_{62}$
$A_{22} = S_a + 3R_{Ab} + 3R_{AF}$	$B_{21} = S_a - T_b + 3R_{Aa} + 3R_{AF}$	$C_{21} = S_a - S_b + 3R_{Aa}$	Equation (2)
$A_{33} = S_a + 3R_{Ac} + 3R_{AF}$	$B_{22} = S_b - T_c + 3R_{AF}$	$C_{23} = - S_a + S_c - 3R_{Ac}$	$B_{12} = B_{43} = B_{51} \quad B_{22} = B_{33} = B_{61}$
$A_{44} = U_a + 3R_{Ba} + 3R_{BF}$	$B_{23} = S_c - T_a + 3R_{AF}$	$C_{32} = S_a - S_c + 3R_{Ab}$	$B_{13} = B_{41} = B_{52} \quad B_{23} = B_{31} = B_{62}$
$A_{45} = U_b + 3R_{BF}$	$B_{24} = - U_b + T_a - 3R_{BF}$	$C_{33} = - S_a + S_b - 3R_{Ac}$	$B_{14} = B_{45} = B_{56} \quad B_{24} = B_{35} = B_{66}$
$A_{46} = U_c + 3R_{BF}$	$B_{25} = - U_c + T_b - 3R_{BF}$	$C_{44} = U_a - U_c + 3R_{Ba}$	$B_{16} = B_{44} = B_{55} \quad B_{25} = B_{36} = B_{64}$
$A_{55} = U_a + 3R_{Bb} + 3R_{BF}$	$B_{26} = - U_a + T_c - 3R_{Bc} - 3R_{BF}$	$C_{45} = - U_a + U_b - 3R_{Bb}$	Equation (3)
$A_{66} = U_a + 3R_{Bc} + 3R_{BF}$	$B_{32} = S_a - T_b + 3R_{Ab} + 3R_{AF}$	$C_{46} = U_c - U_b$	$C_{13} = - C_{22} = C_{31}$
	$B_{34} = - U_a + T_c - 3R_{Ba} - 3R_{BF}$	$C_{54} = U_a - U_b + 3R_{Ba}$	$C_{14} = - C_{26} = C_{35} = C_{41} = - C_{53} = C_{62}$
	$B_{42} = S_a - T_c + 3R_{Ab} + 3R_{AF}$	$C_{56} = - U_a + U_c - 3R_{Bc}$	$C_{15} = - C_{24} = C_{36} = C_{42} = - C_{51} = C_{63}$
	$B_{46} = - U_a + T_b - 3R_{Bc} - 3R_{BF}$	$C_{65} = U_a - U_c + 3R_{Bb}$	$C_{16} = - C_{25} = C_{34} = C_{43} = - C_{52} = C_{61}$
	$B_{53} = S_a - T_c + 3R_{Ac} + 3R_{AF}$	$C_{66} = - U_a + U_b - 3R_{Bc}$	$C_{46} = - C_{55} = C_{64}$
	$B_{54} = - U_a + T_b - 3R_{Ba} - 3R_{BF}$		
	$B_{63} = S_a - T_b + 3R_{Ac} + 3R_{AF}$		
	$B_{65} = - U_a + T_c - 3R_{Bb} - 3R_{BF}$		

where:

$S_a = (Z_{A1} + Z_{C1}) + (Z_{A2} + Z_{C2}) + (Z_{A0} + Z_{C0}) \quad T_a = Z_{C1} + Z_{C2} + Z_{C0} \quad U_a = (Z_{B1} + Z_{C1}) + (Z_{B2} + Z_{C2}) + (Z_{B0} + Z_{C0})$

$S_b = a(Z_{A1} + Z_{C1}) + a^2(Z_{A2} + Z_{C2}) + (Z_{A0} + Z_{C0}) \quad T_b = aZ_{C1} + a^2Z_{C2} + Z_{C0} \quad U_b = a(Z_{B1} + Z_{C1}) + a^2(Z_{B2} + Z_{C2}) + (Z_{B0} + Z_{C0})$

$S_c = a^2(Z_{A1} + Z_{C1}) + a(Z_{A2} + Z_{C2}) + (Z_{A0} + Z_{C0}) \quad T_c = a^2Z_{C1} + aZ_{C2} + Z_{C0} \quad U_c = a^2(Z_{B1} + Z_{C1}) + a(Z_{B2} + Z_{C2}) + (Z_{B0} + Z_{C0})$

TABLE II

CONSTANTS FOR EQUATIONS (1), (2) AND (3) WHEN POSITIVE AND NEGATIVE SEQUENCE IMPEDANCES ARE EQUAL

Equation (1)	Equation (2)	Equation (3)	Equalities
$A_{11} = S_a + 3R_{Aa} + 3R_{AF}$	$B_{11} = S_a - T_b + 3R_{Aa} + 3R_{AF}$	$C_{11} = S_a - S_b + 3R_{Aa}$	Equation (1)
$A_{12} = S_b + 3R_{AF}$	$B_{12} = S_b - T_a + 3R_{AF}$	$C_{12} = -S_a + S_b - 3R_{Ab}$	$A_{12} = A_{13} = A_{21} = A_{23} = A_{31} = A_{32}$
$A_{14} = T_a$	$B_{13} = S_b - T_b + 3R_{AF}$	$C_{13} = 0$	$A_{45} = A_{46} = A_{54} = A_{56} = A_{64} = A_{65}$
$A_{15} = T_b$	$B_{14} = -U_b + T_a - 3R_{AF}$	$C_{14} = T_a - T_b$	$A_{14} = A_{25} = A_{36} = A_{41} = A_{52} = A_{63}$
$A_{22} = S_a + 3R_{Ab} + 3R_{AF}$	$B_{15} = -U_a + T_b - 3R_{Bb} - 3R_{BF}$	$C_{23} = -S_a + S_b - 3R_{Ac}$	$A_{15} = A_{16} = A_{24} = A_{26} = A_{34} = A_{35}$
$A_{33} = S_a + 3R_{Ac} + 3R_{AF}$	$B_{16} = -U_b + T_b - 3R_{AF}$	$C_{44} = U_a - U_b + 3R_{Ba}$	$= A_{42} = A_{43} = A_{51} = A_{53} = A_{61}$
$A_{44} = U_a + 3R_{Ba} + 3R_{BF}$	$B_{26} = -U_a + T_b - 3R_{Bc} - 3R_{BF}$	$C_{45} = -U_a + U_b - 3R_{Bb}$	$= A_{62}$
$A_{45} = U_b + 3R_{BF}$	$B_{32} = S_a - T_b + 3R_{Ab} + 3R_{AF}$	$C_{56} = -U_a + U_b - 3R_{Bc}$	
$A_{55} = U_a + 3R_{Bb} + 3R_{BF}$	$B_{34} = -U_a + T_b - 3R_{Ba} - 3R_{BF}$		Equation (2)
$A_{66} = U_a + 3R_{Bc} + 3R_{BF}$	$B_{53} = S_a - T_b + 3R_{Ac} + 3R_{AF}$		$B_{11} = B_{21}$ $B_{15} = B_{65}$
			$B_{12} = B_{23} = B_{31} = B_{43} = B_{51} = B_{62}$
			$B_{13} = B_{22} = B_{33} = B_{41} = B_{52} = B_{61}$
			$B_{14} = B_{24} = B_{35} = B_{45} = B_{56} = B_{66}$
			$B_{16} = B_{25} = B_{36} = B_{44} = B_{55} = B_{64}$
			$B_{32} = B_{42}$ $B_{34} = B_{54}$
			$B_{26} = B_{46}$ $B_{53} = B_{63}$
			Equation (3)
			$C_{13} = C_{16} = C_{22} = C_{25} = C_{31} = C_{34}$
			$= C_{43} = C_{46} = C_{52} = C_{55} = C_{61}$
			$= C_{64} = 0$
			$C_{14} = -C_{15} = C_{24} = -C_{26} = C_{35}$
			$= -C_{36} = C_{41} = -C_{42} = C_{51}$
			$= -C_{53} = C_{62} = -C_{63}$
			$C_{11} = C_{21}$ $C_{12} = -C_{32}$ $C_{23} = C_{33}$
			$C_{44} = C_{54}$ $C_{45} = -C_{65}$ $C_{56} = C_{66}$

where:

$$
\begin{aligned}
S_a &= 2(Z_{A1} + Z_{C1}) + (Z_{A0} + Z_{C0}) \\
S_b &= -(Z_{A1} + Z_{C1}) + (Z_{A0} + Z_{C0}) \\
T_a &= 2Z_{C1} + Z_{C0} \\
T_b &= -Z_{C1} + Z_{C0} \\
U_a &= 2(Z_{B1} + Z_{C1}) + (Z_{A0} + Z_{C0}) \\
U_b &= -(Z_{B1} + Z_{C1}) + (Z_{A0} + Z_{C0})
\end{aligned}
$$

3. Special Cases

The application of the three sets of equations (1), (2) and (3), will be illustrated with a few examples. For simple cases, such as a single or double line-to-ground fault at one location, the equations reduce to formulas frequently found in the literature on this subject.

From set (1) equations for faults to ground at one or two locations can be obtained directly when the zero sequence impedance is finite. Set (2), obtained from (1), is the most convenient set for solutions of faults to ground in isolated systems in which capacitance has been neglected. The phase-to-phase fault currents are best obtained from set (3).

3.1 Single Line-to-Ground Fault at A

Consider a fault to ground on phase "*b*" at *A*. The solution can be obtained from (1) by letting:

$$R_{Aa} = R_{Ac} = R_{Ba} = R_{Bb} = R_{Bc} = \infty \tag{7}$$

This results in:

$$I_{Aa} = I_{Ac} = I_{Ba} = I_{Bb} = I_{Bc} = 0 \tag{8}$$

Striking out all columns in (1) containing the currents in (8) and the corresponding rows indexed by *Aa*, *Ac*, *Ba*, *Bb* and *Bc* only one equation is left:

$$A_{22} I_{Ab} = 3a^2 E \tag{9}$$

The numerical value of A_{22} can be calculated directly from Table I, or on substituting the symbolic value of A_{22} in equation (9) the result will be:

$$I_{Ab} = \frac{3a^2 E}{Z_1 + Z_2 + Z_0 + 3R_F} \tag{10}$$

where

$$\begin{aligned} Z_1 &= Z_{A1} + Z_{C1} \\ Z_2 &= Z_{A2} + Z_{C2} \\ Z_0 &= Z_{A0} + Z_{C0} \\ R_F &= R_{Ab} + R_{AF} \end{aligned} \tag{11}$$

Equation (10) is the well-known formula for a single line-to-ground fault at one location in a three-phase system.

3.2 Double Line-to-Ground Fault at A

Consider a double line-to-ground fault on phases "*a*" and "*b*" at *A*. Then:

$$R_{Ac} = R_{Ba} = R_{Bb} = R_{Bc} = \infty \tag{12}$$

and

$$I_{Ac} = I_{Ba} = I_{Bb} = I_{Bc} = 0 \tag{13}$$

Striking out the columns of (1) containing the currents in (13) and the corresponding rows (Ac, Ba, Bb and Bc) the following two equations remain:

$$\begin{aligned} A_{11}I_{Aa} + A_{12}I_{Ab} &= 3E \\ A_{21}I_{Aa} + A_{22}I_{Ab} &= 3a^2E \end{aligned} \tag{14}$$

from which on substituting the numerical values for the A's from Table I the two currents I_{Aa} and I_{Ab} can be found. The total fault current to ground at A is:

$$I_{AF} = I_{Aa} + I_{Ab} \tag{15}$$

In the special case where R_{Aa}, R_{Ab} and R_{AF} are zero the expression for I_{AF} can be reduced to the following expression after a direct substitution for the A's in (14) is made:

$$I_{AF} = \frac{-3aZ_2E}{Z_0Z_1 + Z_0Z_2 + Z_1Z_2} \tag{16}$$

where:

$$\begin{aligned} Z_1 &= Z_{A1} + Z_{C1} \\ Z_2 &= Z_{A2} + Z_{C2} \\ Z_0 &= Z_{A0} + Z_{C0} \end{aligned} \tag{17}$$

3.3 Simultaneous Double Line-to-Ground Fault at A and Double Line-to-Ground Fault at B

Consider a fault-to-ground on phases "a" and "b" at A and phases "a" and "c" at B. Then:

$$R_{Ac} = R_{Bb} = \infty \tag{18}$$

Hence:

$$I_{Ac} = I_{Bb} = 0 \tag{19}$$

Striking out the two columns containing I_{Ac} and I_{Bb} and the two corresponding rows (Ac and Bb), the four following equations remain:

$$\begin{aligned} A_{11}I_{Aa} + A_{12}I_{Ab} + A_{14}I_{Ba} + A_{16}I_{Bc} &= 3E \\ A_{21}I_{Aa} + A_{22}I_{Ab} + A_{24}I_{Ba} + A_{26}I_{Bc} &= 3a^2E \\ A_{41}I_{Aa} + A_{42}I_{Ab} + A_{44}I_{Ba} + A_{46}I_{Bc} &= 3E \\ A_{61}I_{Aa} + A_{62}I_{Ab} + A_{64}I_{Ba} + A_{66}I_{Bc} &= 3aE \end{aligned} \tag{20}$$

A symbolic solution in terms of the sequence impedances for these currents becomes quite involved and it is advisable to substitute numerical values of the constants before solving for the four currents. The total fault currents at A and B, respectively, are (from (4) and (5) in connection with (19)):

$$I_{AF} = I_{Aa} + I_{Ab} \tag{21}$$

$$I_{BF} = I_{Ba} + I_{Bc} \tag{22}$$

In a similar manner faults to ground for any other combination of faulted phases can be found.

3.4 Single Line-to-Ground Faults at A and B in an Isolated System

Consider a fault-to-ground on phases "a" at A and "b" at B in an isolated system in which capacity can be neglected. Then:

$$R_{Ab} = R_{Ac} = R_{Ba} = R_{Bc} = \infty \tag{23}$$

and

$$I_{Ab} = I_{Ac} = I_{Ba} = I_{Bc} = 0 \tag{24}$$

Striking out the columns of (2) containing the currents in (24) and the corresponding rows $Aa - Bc$, $Ab - Ba$, $Ab - Bc$, $Ac - Ba$ and $Ac - Bb$ (all rows containing Ab, Ac, Ba and Bc), leaves only one equation in (2), which together with (2a) gives:

$$\begin{aligned} B_{11}I_{Aa} + B_{15}I_{Bb} &= 3(1 - a^2)E \\ I_{Aa} + I_{Bb} &= 0 \end{aligned} \tag{25}$$

Solving for these currents the result is:

$$I_{Aa} = -I_{Bb} = \frac{3(1 - a^2)E}{B_{11} - B_{15}} \tag{26}$$

Inserting the values of the B's from Table I this reduces to:

$$I_{Aa} = -I_{Bb} = \frac{3(1 - a^2)E}{Z_{1i} + Z_{2i} + Z_{0i} + 3(R_A + R_B)} \tag{27}$$

$$\begin{aligned} Z_{1i} &= Z_{A1} + Z_{B1} + 3Z_{C1} \\ Z_{2i} &= Z_{A2} + Z_{B2} + 3Z_{C2} \\ Z_{0i} &= Z_{A0} + Z_{B0} \\ R_A &= R_{Aa} + R_{AF} \\ R_B &= R_{Bb} + R_{BF} \end{aligned} \tag{28}$$

The subscript i (isolated) is used to distinguish these impedances for the isolated system from those used in (11), (17) and (38).

3.5 *Phase-to-Phase Fault at A*

Consider a fault between phases "a" and "b" at A. Then let:

$$I_{Ac} = I_{Ba} = I_{Bb} = I_{Bc} = 0 \qquad (29)$$

Striking out the columns in (3) containing the currents in (29) and the corresponding rows (all rows containing Ac, Ba, Bb and Bc) only one equation is left:

$$C_{11}I_{Aa} + C_{12}I_{Ab} = 3(1 - a^2)E \qquad (30)$$

It is further known from (3a) that:

$$I_{Ab} = -I_{Aa} \qquad (31)$$

Substituting (31) and the constants C_{11} and C_{12} from Table I in (30) the result is:

$$I_{Aa} = -I_{Ab} = \frac{(1 - a^2)E}{Z_1 + Z_2 + R_{Aa} + R_{Ab}} \qquad (32)$$

$$\left.\begin{aligned} Z_1 &= Z_{A1} + Z_{C1} \\ Z_2 &= Z_{A2} + Z_{C2} \end{aligned}\right\} \qquad (33)$$

which is a well-known expression for a phase-to-phase fault.

3.6 *Three-Phase Fault at A*

For this case:

$$I_{Ba} = I_{Bb} = I_{Bc} = 0 \qquad (34)$$

Striking out the columns in (3) containing the currents in (34) and the corresponding rows, three equations remain, any two of which together with the equation from (3a) relating to the currents at A give:

$$\begin{aligned} C_{11}I_{Aa} + C_{12}I_{Ab} + C_{13}I_{Ac} &= 3(1 - a^2)E \\ C_{21}I_{Aa} + C_{22}I_{Ab} + C_{23}I_{Ac} &= 3(1 - a)E \\ I_{Aa} + I_{Ab} + I_{Ac} &= 0 \end{aligned} \qquad (35)$$

from which the currents can be found.

In the special case where the fault resistances are all zero, the three currents are equal in magnitude and related as follows:

$$I_{Aa} = aI_{Ab} = a^2I_{Ac} \qquad (36)$$

The rank of the system determinant in (35) is therefore 1. Using any of the first two equations in (35) in connection with (36) and the constants in Table I, the result is:

$$I_{Aa} = aI_{Ab} = a^2I_{Ac} = \frac{E}{Z_1} \qquad (37)$$

$$Z_1 = Z_{A1} + Z_{C1} \qquad (38)$$

3.7 Phase-to-Phase Fault at A and Phase-to-Ground Fault at B

Consider a fault between phases "a" and "b" at A and a fault to ground on phase "c" at B. Then:

$$R_{Ac} = R_{Ba} = R_{Bb} = \infty \tag{39}$$

and

$$I_{Ac} = I_{Ba} = I_{Bb} = 0 \tag{40}$$

As explained in a preceding section the three first equations in (3) together with the first in (3a) and the three last equations in (1) may be used for this case.

Striking out the columns I_{Ac}, I_{Ba} and I_{Bb} and the corresponding rows $Aa - Ac$, and $Ab - Ac$ in the three first equations in (3) leaves only the first equation. Similarly by striking out the columns I_{Ba}, I_{Bb}, and the corresponding rows Ba and Bb in the three last equations in (1) leaves only the last equation. Hence:

$$\begin{aligned} C_{11}I_{Aa} + C_{12}I_{Ab} + C_{16}I_{Bc} &= 3(1 - a^2)E \\ A_{61}I_{Aa} + A_{62}I_{Ab} + A_{66}I_{Bc} &= 3aE \end{aligned} \tag{41}$$

and finally from (3a):

$$I_{Aa} + I_{Ab} = 0 \tag{42}$$

from which the three currents can be found. The A's and C's are given in Table I and Table II.

4. Conclusion

While the probability of all phases being faulted at both locations simultaneously is very remote, the three sets of equations (1), (2) and (3) have been given in such a form that they conveniently will provide a solution for any combination of phases faulted from a single line-to-ground fault at one location to the most involved fault condition.

In Section (3) of this paper, in which special cases have been treated, only simple types of fault conditions have been shown in order to illustrate the method to be used and to prove that the general equations reduce to well-known formulas.

The constants given in Table I consist of the nine quantities S_a, S_b, S_c, T_a, T_b, T_c, U_a, U_b and U_c arranged as shown for each set of equations. Table II gives somewhat simpler values for the constants in cases where the positive and negative sequence impedances are assumed equal.

The voltages to ground at the two fault locations are given by (46) and (47) in the Appendix.

It is hoped that this development will provide a more unified presentation of fault current calculations in power networks.

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APPENDIX

The standard notation for phase and sequence quantities is usually indicated by a subscript. Thus I_a, I_b, etc., means the current at the point of fault of phase "a" and "b," respectively. I_1, a^2I_1 and aI_1 are the positive sequence currents in phase "a," "b" and "c," respectively. In this treatment, however, complication arises from the fact that two points of faults are involved and it will be necessary to distinguish between the quantities at these two locations. This is most conveniently done by a double subscript, the first referring to the point of fault and the second to the phase or sequence in question. Thus I_{Aa}, I_{Ba}, etc., are the currents in the fault at A and B of phase "a" and I_{A1}, I_{B1} the positive sequence current at the two points of fault, respectively. Making use of this notation the fundamental equations for the sequence currents at fault A are:

$$\begin{aligned} I_{A0} + I_{A1} + I_{A2} &= I_{Aa} \\ I_{A0} + a^2I_{A1} + aI_{A2} &= I_{Ab} \\ I_{A0} + aI_{A1} + a^2I_{A2} &= I_{Ac} \end{aligned} \tag{43}$$

And at fault B:

$$\begin{aligned} I_{B0} + I_{B1} + I_{B2} &= I_{Ba} \\ I_{B0} + a^2I_{B1} + aI_{B2} &= I_{Bb} \\ I_{B0} + aI_{B1} + a^2I_{B2} &= I_{Bc} \end{aligned} \tag{44}$$

where the coefficient "a" is the sequence operator, having the value:

$$\begin{aligned} a &= -\frac{1}{2} + j\frac{\sqrt{3}}{2} \\ a^2 &= -\frac{1}{2} - j\frac{\sqrt{3}}{2} \end{aligned} \tag{45}$$

The voltages to ground at the two fault locations are given by:

$$\begin{aligned} V_{Aa} &= V_{A0} + V_{A1} + V_{A2} \\ &= (R_{Aa} + R_{AF})I_{Aa} + R_{AF}I_{Ab} + R_{AF}I_{Ac} \\ V_{Ab} &= V_{A0} + a^2V_{A1} + aV_{A2} \\ &= R_{AF}I_{Aa} + (R_{Ab} + R_{AF})I_{Ab} + R_{AF}I_{Ac} \\ V_{Ac} &= V_{A0} + aV_{A1} + a^2V_{A2} \\ &= R_{AF}I_{Aa} + R_{AF}I_{Ab} + (R_{Ac} + R_{AF})I_{Ac} \end{aligned} \tag{46}$$

$$
\begin{aligned}
V_{Ba} &= V_{B0} + V_{B1} + V_{B2} \\
&\qquad = (R_{Ba} + R_{BF})I_{Ba} + R_{BF}I_{Bb} + R_{BF}I_{Bc} \\
V_{Bb} &= V_{B0} + a^2V_{B1} + aV_{Ba} \\
&\qquad = R_{BF}I_{Ba} + (R_{Bb} + R_{BF})I_{Bb} + R_{BF}I_{Bc} \\
V_{Bc} &= V_{B0} + aV_{B1} + a^2V_{B2} \\
&\qquad = R_{BF}I_{Ba} + R_{BF}I_{Bb} + (R_{Bc} + R_{BF})I_{Bc}
\end{aligned}
\tag{47}
$$

Consider the positive sequence diagram in Fig. 2. Evidently:

$$
\begin{aligned}
V_{A1} &= E - (Z_{A1} + Z_{C1})I_{A1} - Z_{C1}I_{B1} \\
V_{B1} &= E - Z_{C1}I_{A1} - (Z_{B1} + Z_{C1})I_{B1}.
\end{aligned}
\tag{48}
$$

where V_{A1} and V_{B1} are the positive sequence voltages to ground at the two fault locations. Similar expressions can be obtained for V_{A2}, V_{B2}, V_{A0} and V_{B0}, except for the fact the E is zero in these cases and the impedances and currents are the negative and zero sequence quantities. They are:

$$
\begin{aligned}
V_{A2} &= -(Z_{A2} + Z_{C2})I_{A2} - Z_{C2}I_{B2} \\
V_{B2} &= -Z_{C2}I_{A2} - (Z_{B2} + Z_{C2})I_{B2}
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
V_{A0} &= -(Z_{A0} + Z_{C0})I_{A0} - Z_{C0}I_{B0} \\
V_{B0} &= -Z_{C0}I_{A0} - (Z_{B0} + Z_{C0})I_{B0}
\end{aligned}
\tag{50}
$$

Solving (43) and (44) for the sequence currents the result is:

$$
\begin{aligned}
I_{A0} &= \tfrac{1}{3}(I_{Aa} + I_{Ab} + I_{Ac}) \\
I_{A1} &= \tfrac{1}{3}(I_{Aa} + aI_{Ab} + a^2I_{Ac}) \\
I_{A2} &= \tfrac{1}{3}(I_{Aa} + a^2I_{Ab} + aI_{Ac})
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
I_{B0} &= \tfrac{1}{3}(I_{Ba} + I_{Bb} + I_{Bc}) \\
I_{B1} &= \tfrac{1}{3}(I_{Ba} + aI_{Bb} + a^2I_{Bc}) \\
I_{B2} &= \tfrac{1}{3}(I_{Ba} + a^2I_{Bb} + aI_{Bc})
\end{aligned}
\tag{52}
$$

Substituting the expressions for the sequence currents in (48), (49) and (50) the result is:

$$
\begin{aligned}
V_{A1} &= E - \tfrac{1}{3}(Z_{A1} + Z_{C1})(I_{Aa} + aI_{Ab} + a^2I_{Ac}) \\
&\qquad - \tfrac{1}{3}Z_{C1}(I_{Ba} + aI_{Bb} + a^2I_{Bc}) \\
V_{A2} &= -\tfrac{1}{3}(Z_{A2} + Z_{C2})(I_{Aa} + a^2I_{Ab} + aI_{Ac}) \\
&\qquad - \tfrac{1}{3}Z_{C2}(I_{Ba} + a^2I_{Bb} + aI_{Bc}) \\
V_{A0} &= -\tfrac{1}{3}(Z_{A0} + Z_{C0})(I_{Aa} + I_{Ab} + I_{Ac}) \\
&\qquad - \tfrac{1}{3}Z_{C0}(I_{Ba} + I_{Bb} + I_{Bc})
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
V_{B1} &= E - \tfrac{1}{3}Z_{C1}(I_{Aa} + aI_{Ab} + a^2I_{Ac}) \\
&\qquad - \tfrac{1}{3}(Z_{B1} + Z_{C1})(I_{Ba} + aI_{Bb} + a^2I_{Bc}) \\
V_{B2} &= -\tfrac{1}{3}Z_{C2}(I_{Aa} + a^2I_{Ab} + aI_{Ac}) \\
&\qquad - \tfrac{1}{3}(Z_{B2} + Z_{C2})(I_{Ba} + a^2I_{Bb} + aI_{Bc}) \\
V_{B0} &= -\tfrac{1}{3}Z_{C0}(I_{Aa} + I_{Ab} + I_{Ac}) \\
&\qquad - \tfrac{1}{3}(Z_{B0} + Z_{C0})(I_{Ba} + I_{Bb} + I_{Bc})
\end{aligned}
\qquad (54)
$$

Substituting (53) and (54) in (46) and (47) the six original equations in (1) are obtained.

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