Effect of Space Charge and Transit Time on the Shot Noise in Diodes

By A. J. RACK

The theoretical analysis of the effect of space charge upon the "shot noise" in a planar diode shows that for practically all operating conditions, the tube noise is equivalent to the thermal resistance noise of the plate resistance at 0.644 times the cathode temperature. Noise in diodes of other than planar shapes is discussed and it is concluded that the same relation holds. It is shown that transit time produces the same high frequency modification for both the thermal and shot tube noise, and that the tube noise is decreased by transit time.

IN the study of noise in vacuum tubes, the effect of the space charge upon the shot noise has been a subject of considerable interest and practical importance. Several papers have been written in which it is shown that the shot noise is decreased by the space charge, and that the tube noise in a diode with space charge is equivalent to the thermal resistance noise of the plate impedance at a temperature slightly greater than half of that of the cathode.[1, 2, 3, 4] The most comprehensive analysis was made by Schottky and Spenke. These authors, employing a different method from the one here presented, have obtained the same general conclusions given in this paper, although they prefer to express the result in the form of a modified shot-noise equation, whereas for reasons developed below, the writer prefers the thermal form. The theoretical analysis and discussion presented here was undertaken to show in more detail the extent of the range of the operating condition for which the thermal resistance equivalent of tube noise is valid and to study the effect of transit time upon both the shot and thermal tube noise.

For convenience, the paper is divided into three parts. In the first section is given an exact mathematical treatment of the tube noise at low frequencies in a parallel plane diode for any degree of space charge. A discussion of the final tube noise equation obtained through this analysis, and the extension of these results for the planar diode to any other shape diode is given in Part II, where the presentation is such that the section may be read independently of the theoretical analysis in Part I. Through several approximations, Part III treats the effect of transit time upon tube noise in the planar diode.

Part I—General Low Frequency Analysis

In the development of the general equations for the direct current in vacuum tubes with space charge, account has been taken of the fact that the electrons are emitted from the cathode with Maxwellian velocity distribution. This fact has been verified experimentally by Germer,[5] and the resulting equations for the relation between current and voltage have been derived and investigated by Fry,[6] Langmuir,[7] and others. In the extension of this analysis to tube noise, it is only necessary to assume that the number of electrons emitted with any velocity does not remain constant, but fluctuates with time according to the well-known laws of probability. In the analysis on this basis, the frequencies involved will be considered to be sufficiently low so that any transit time effect is negligible.

Below is given a list of the definitions of various symbols to be used in the tube noise study of a parallel plane diode. The practical system of units is employed throughout.

$n(u_c)du_c$ = instantaneous rate of emission per unit area of the cathode of electrons with initial velocities between u_c and $u_c + du_c$ in the x-direction, regardless of the velocity components in the other directions,

= $n_0(u_c)du_c + \delta(u_c)du_c$,

$n_0(u_c)du_c$ = average rate of emission of electrons with x-directed velocities between u_c and $u_c + du_c$,

$\delta(u_c)du_c$ = instantaneous deviation from average rate of emission,

I = instantaneous anode current per unit area,

V = instantaneous potential with respect to cathode of a plane at a distance x from the cathode,

V' = instantaneous potential with respect to cathode of the potential minimum,

u = instantaneous velocity at x-plane of electrons which had an initial x-directed velocity of u_c at the cathode,

x' = instantaneous position of potential minimum,

e = charge on electron = -1.59×10^{-19} coulombs,

m = mass of electron = 9.01×10^{-28} grams,

h = ratio of dyne cms. to joules = 10^{-7},

ϵ = permittivity of a vacuum in practical units = 8.85×10^{-14} farads/cm.,

k = Boltzmann's gas constant = 1.372×10^{-23} watts/degree Kelvin,

N = average total number of electrons emitted per second per unit area from the cathode,

T = absolute temperature of the cathode.

In the following analysis, it is assumed that the electrodes of the planar diode are infinite in extent, and that the electron emission is random, so that the equipotential surfaces are parallel planes perpendicular to the x-axis.

The potential distribution in such a planar diode operating with space charge is shown in Fig. 1. The origin of coordinates is taken at the cathode, and the potential minimum formed by space charge occurs at a distance $x = x'$ from the cathode. The subscript α will be used to denote the space between cathode and potential minimum while β applies similarly to the space between minimum and anode. Of all

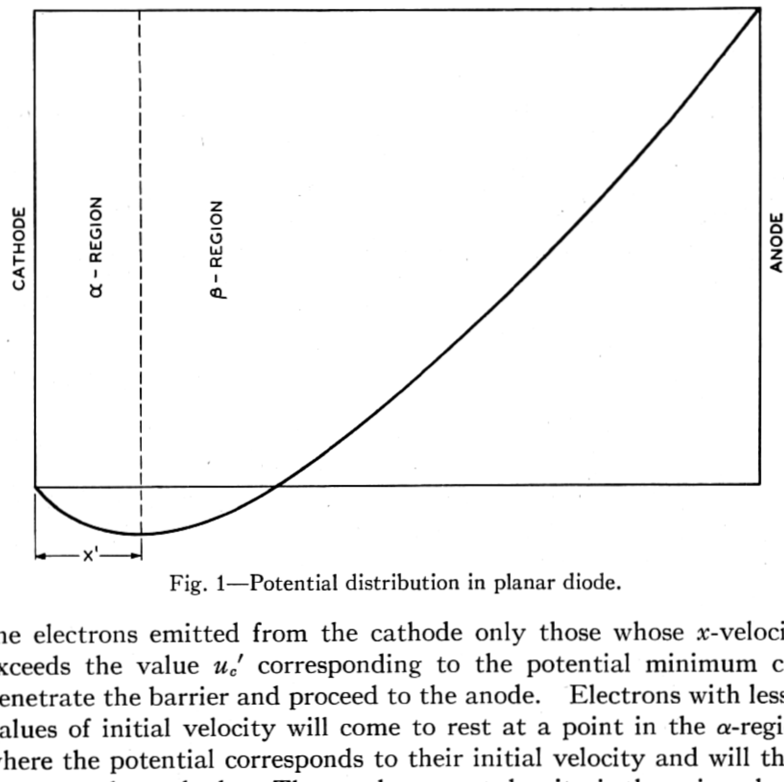

Fig. 1—Potential distribution in planar diode.

the electrons emitted from the cathode only those whose x-velocity exceeds the value u_c' corresponding to the potential minimum can penetrate the barrier and proceed to the anode. Electrons with lesser values of initial velocity will come to rest at a point in the α-region where the potential corresponds to their initial velocity and will then return to the cathode. The anode current density is thus given by

$$I = e \int_{u_c'}^{\infty} n(u_c) du_c, \tag{1}$$

while the relation between velocity u and potential V at a given value of x is

$$u^2 = u_c^2 - \frac{2e}{hm} V. \tag{2}$$

A third fundamental relation is Poisson's equation which becomes in the parallel plane case under consideration

$$\epsilon \frac{d^2 V}{dx^2} = -\rho. \tag{3}$$

In the α-region the total charge density is made up of three classes of electrons, namely

1. Those destined to pass the potential minimum and arrive at the anode.
2. Those moving away from the cathode but which will not travel as far as the minimum point.
3. Those returning to the cathode.

Corresponding to each class of electrons, there is an associated current, ρu, so that each of the three densities ρ_1, ρ_2 or ρ_3, may be expressed by a relation of the form,

$$\rho_n = \frac{I_n}{u}. \tag{4}$$

When it is remembered that the potential and velocity at a given value of x are uniquely related through (2), then it is easy to see that the total density for a given plane in the α-region is given by

$$\rho_\alpha = e \int_{u_c'}^{\infty} \frac{n(u_c)}{u} du_c + 2e \int_{v}^{u_c'} \frac{n(u_c)}{u} du_c, \tag{5}$$

where the first term represents the contribution of electrons in class 1 above, while the second term represents the contribution of electrons in classes 2 and 3. The contribution of class 3 is equal to that of class 2. The lower integration limit v of the second term of (5) represents the initial velocity of an electron which would just arrive at the value of x under consideration before coming to rest and starting back toward the cathode and the limit u_c' in both terms represents the initial velocity of an electron which comes to rest just at the potential minimum. Thus, from (2)

$$v = \sqrt{\frac{2e}{hm} V} \qquad \text{and} \qquad u_c' = \sqrt{\frac{2e}{hm} V'}. \tag{6}$$

In the β-region there is only one class of electrons, so that the density is more simply expressed. Thus,

$$\rho_\beta = e \int_{u_c'}^{\infty} \frac{n(u_c)}{u} du_c. \tag{7}$$

The value of ρ in (5) and (7) may each be expressed in terms of d^2V/dx^2 by the use of (3), and the integration of these two Poisson's relations for the common boundary condition that the electric force is zero at the potential minimum has the following result:

$$\frac{(dV_\alpha)^2}{(dx)} = \frac{2hm}{\epsilon}\int_{u_c'}^{\infty}(u - u')n(u_c)du_c + \frac{4hm}{\epsilon}\int_0^{u_c'} un(u_c)du_c, \tag{8}$$

$$\frac{(dV_\beta)^2}{(dx)} = \frac{2hm}{\epsilon}\int_{u_c'}^{\infty}(u - u')n(u_c)du_c, \tag{9}$$

where u' is the electronic velocity at the potential minimum, i.e., $(u')^2 = u_c^2 - (2e/hm)V'$.

At this point the analysis departs for the first time from the classic analyses of Fry [6] and Langmuir,[7] through the introduction of the concept that the instantaneous rate of emission may be expressed as the sum of an average rate of emission plus an instantaneous deviation. That is,

$$n(u_c) = n_0(u_c) + \delta(u_c), \tag{10}$$

transforms (8) and (9) into the following equations:

$$\frac{(dV_\alpha)^2}{(dx)} = \frac{2hm}{\epsilon}\int_{u_c'}^{\infty}(u - u')n_0(u_c)du_c + \frac{4hm}{\epsilon}\int_0^{u_c'} un_0(u_c)du_c + \alpha(\delta), \tag{11}$$

where

$$\alpha(\delta) = \frac{2hm}{\epsilon}\int_{u_c'}^{\infty}(u - u')\delta(u_c)du_c + \frac{4hm}{\epsilon}\int_0^{u_c'} u\delta(u_c)du_c$$

and

$$\frac{(dV_\beta)^2}{(dx)} = \frac{2hm}{\epsilon}\int_{u_c'}^{\infty}(u - u')n_0(u_c)du_c + \beta(\delta), \tag{12}$$

where

$$\beta(\delta) = \frac{2hm}{\epsilon}\int_{u_c'}^{\infty}(u - u')\delta(u_c)du_c.$$

Since the average rate of emission may be expressed by the Maxwellian relation,

$$n_0(u_c) = 2\alpha N u_c \epsilon^{-\alpha u_c^2},$$

where

$$\alpha = \frac{hm}{2kT},$$

the indicated integrations in (11) and (12) have as a result,

$$\frac{(kT)^2}{(e)}\frac{(d\eta_\alpha)^2}{(dx)} = \frac{Nhm}{\epsilon}\sqrt{\frac{\pi}{\alpha}}\,\epsilon^{-\eta'} \times \left[\epsilon^{\eta} - 1 + \epsilon^{\eta}P(\sqrt{\eta}) - 2\sqrt{\frac{\eta}{\pi}}\right] + \alpha(\delta) \tag{13}$$

and

$$\frac{(kT)^2}{(e)}\frac{(d\eta_\beta)^2}{(dx)} = \frac{Nhm}{\epsilon}\sqrt{\frac{\pi}{\alpha}}\,\epsilon^{-\eta'} \times \left[\epsilon^{\eta} - 1 - \epsilon^{\eta}P(\sqrt{\eta}) + 2\sqrt{\frac{\eta}{\pi}}\right] + \beta(\delta), \quad (14)$$

where

$$\eta = \frac{e}{kT}(V' - V), \qquad \eta' = \frac{eV'}{kT},$$

$$P(x) = \frac{2}{\sqrt{\pi}}\int_0^x \epsilon^{-x^2}dx.$$

The fact that both $\alpha(\delta)$ and $\beta(\delta)$ are very small greatly simplifies the solution for the distance coordinate x in (13) and (14). The process is to invert the two equations, respectively, extract the square root, and then expand the right-hand side in powers of $\alpha(\delta)$ and $\beta(\delta)$, respectively. This results in expressions for $dx/d\eta$ which can be integrated term by term. However, the small values of $\alpha(\delta)$ and $\beta(\delta)$ allow powers higher than the first to be disregarded, and hence,

$$\frac{e}{kT}x' = \frac{F(\eta')}{\left[\dfrac{Nhm}{\epsilon}\sqrt{\dfrac{\pi}{\alpha}}\,\epsilon^{-\eta'}\right]^{1/2}} - \frac{1}{2\left[\dfrac{Nhm}{\epsilon}\sqrt{\dfrac{\pi}{\alpha}}\,\epsilon^{-\eta'}\right]^{3/2}}\int_0^{\eta'}\frac{\alpha(\delta)d\eta}{\Phi(\eta)} \quad (15)$$

and

$$\frac{e}{kT}(x - x') = \frac{f(\eta)}{\left[\dfrac{Nhm}{\epsilon}\sqrt{\dfrac{\pi}{\alpha}}\,\epsilon^{-\eta'}\right]^{1/2}} - \frac{1}{2\left[\dfrac{Nhm}{\epsilon}\sqrt{\dfrac{\pi}{\alpha}}\,\epsilon^{-\eta'}\right]^{3/2}}\int_0^{\eta}\frac{\beta(\delta)d\eta}{\varphi(\eta)}, \quad (16)$$

where for convenience

$$\left.\begin{aligned}
F(\eta') &= \int_0^{\eta'}\frac{d\eta}{\left[\epsilon^{\eta} - 1 + \epsilon^{\eta}P(\sqrt{\eta}) - 2\sqrt{\dfrac{\eta}{\pi}}\right]^{1/2}}\\
f(\eta) &= \int_0^{\eta}\frac{d\eta}{\left[\epsilon^{\eta} - 1 - \epsilon^{\eta}P(\sqrt{\eta}) + 2\sqrt{\dfrac{\eta}{\pi}}\right]^{1/2}}\\
\Phi(\eta) &= \left[\epsilon^{\eta} - 1 + \epsilon^{\eta}P(\sqrt{\eta}) - 2\sqrt{\frac{\eta}{\pi}}\right]^{3/2}\\
\varphi(\eta) &= \left[\epsilon^{\eta} - 1 - \epsilon^{\eta}P(\sqrt{\eta}) + 2\sqrt{\frac{\eta}{\pi}}\right]^{3/2}
\end{aligned}\right\}. \quad (17)$$

Up to this point, the present analysis is very similar to that given by Spenke.[3] For the reasons stated above, the two methods digress hereafter.

Since the instantaneous position of the potential minimum depends upon the operating conditions as indicated above, the elimination of this variable by the addition of the two equations (15) and (16) results in a simpler expression for the cathode-anode spacing, namely:

$$\frac{e}{kT}x\left[\frac{Nhm}{\epsilon}\sqrt{\frac{\pi}{\alpha}}\,\epsilon^{-\eta'}\right]^{1/2} = F(\eta') + f(\eta) - \frac{1}{2\left[\frac{Nhm}{\epsilon}\sqrt{\frac{\pi}{\alpha}}\,\epsilon^{-\eta'}\right]}\left\{\int_0^{\eta'}\frac{\alpha(\delta)}{\Phi(\eta)}\,d\eta + \int_0^{\eta}\frac{\beta(\delta)}{\varphi(\eta)}\,d\eta\right\}. \quad (18)$$

To separate the noise or fluctuation component of the potentials from their average values, it is necessary to assume at any plane in the diode that it is possible to express both the instantaneous voltage and velocity as a steady state or average value plus a very small superimposed fluctuation component. Since this assumption does not result in any discontinuities, it is possible to express the instantaneous values of the dimensionless variable as

$$\eta = \eta_0 + \eta_1$$

and

$$\eta' = \eta_0' + \eta_1', \quad (19)$$

where both η_1 and η_1' are very small. From this assumption, it may readily be shown that the d-c. solution and the first approximations to the fluctuation component of the solution for (18) are respectively

$$\frac{ex}{kT}\left[\frac{Nhm}{\epsilon}\sqrt{\frac{\pi}{\alpha}}\,\epsilon^{-\eta_0'}\right]^{1/2} = F(\eta_0') + f(\eta_0) \quad (20)$$

and

$$-\frac{ex}{kT}\left[\frac{Nhm}{\epsilon}\sqrt{\frac{\pi}{\alpha}}\,\epsilon^{-\eta_0'}\right]^{1/2}\frac{\eta_1'}{2} = \eta_1'\frac{dF(\eta_0')}{d\eta_0'} + \eta_1\frac{df(\eta_0)}{d\eta_0} - \frac{1}{2\left[\frac{Nhm}{\epsilon}\sqrt{\frac{\pi}{\alpha}}\,\epsilon^{-\eta_0'}\right]}\left[\int_0^{\eta_0'}\frac{\alpha(\delta)}{\Phi(\eta)}\,d\eta + \int_0^{\eta_0}\frac{\beta(\delta)}{\varphi(\eta)}\,d\eta\right]. \quad (21)$$

In the last equation, the average or d-c. values of all quantities except η_1, η_1' and $\delta(u_c)$ are to be used.

To avoid the necessity of using long, awkward equations, it will be of great convenience to define several new variables.

Let

$$\left.\begin{aligned} y &= \sqrt{\alpha}\, u' \\ B &= \frac{F(\eta_0') + f(\eta_0)}{2} + \frac{dF(\eta_0')}{d\eta_0'} + \frac{df(\eta_0)}{d\eta_0} \\ C &= \frac{1}{\sqrt{\pi}} \int_0^{\eta_0'} \frac{[\sqrt{y^2 + \eta} - y]}{\Phi(\eta)} d\eta \\ D &= \frac{1}{\sqrt{\pi}} \int_0^{\eta_0} \frac{[\sqrt{y^2 + \eta} - y]}{\varphi(\eta)} d\eta \end{aligned}\right\}. \tag{22}$$

From the definition for η, given in connection with (13) and (14), it may be shown that

$$\eta_1 = \eta_1' - \frac{e}{kT} V_1, \tag{23}$$

where V_1 is the a-c. anode potential. With the definition given for $\alpha(\delta)$ and $\beta(\delta)$ in (11) and (12), and from the above relations, (21) may be rewritten as

$$\eta_1' B - \frac{eV_1}{kT}\frac{df(\eta_0)}{d\eta_0} = \frac{1}{N\epsilon^{-\eta_0'}} \left\{ \int_{\bar{u}_c}^{\infty} (C + D)\delta(u_c)du_c + \frac{2}{\sqrt{\pi}} \int_0^{\eta_0'} \int_{\bar{v}}^{\bar{u}_c} \frac{\sqrt{y^2 + \eta}\, \delta(u_c)du_c d\eta}{\Phi(\eta)} \right\}, \tag{24}$$

where

$$\bar{u}_c = \sqrt{\frac{2e}{hm} V_0'}, \qquad \bar{v} = \sqrt{\frac{2e}{hm} V_0}.$$

Before (24) can be used, the relation between η_1' and the a-c. anode current, and also the expression for the a-c. plate impedance must be obtained. From the general relation given in (1), the instantaneous current per unit area is

$$I = e\int_{u_c'}^{\infty} n(u_c)du_c = e\int_{u_c'}^{\infty} n_0(u_c)du_c + e\int_{u_c'}^{\infty} \delta(u_c)du_c = Ne\epsilon^{-\eta'} + e\int_{u_c'}^{\infty} \delta(u_c)du_c.$$

Since the instantaneous voltage of the potential minimum may be expressed as a sum of a d-c. component plus a small a-c. fluctuation, the d-c. and first order fluctuation plate current are respectively

$$I_0 = Ne\epsilon^{-\eta_0'}, \tag{25}$$

$$I_1 = -\eta_1' Ne\epsilon^{-\eta_0'} + e\int_{\bar{u}}^{\infty} \delta(u_c)du_c. \tag{26}$$

Whence,

$$\eta_1' = \frac{e}{I_0}\int_{\bar{u}_c}^{\infty} \delta(u_c)du_c - \frac{I_1}{I_0}. \tag{27}$$

This is the desired relation between η_1' and the a-c. anode current.

To find the other desired relation, it is first observed that by (20) and (25), the d-c. voltage-current relation is

$$\frac{ex}{kT}\left[\frac{hm}{\epsilon e}\sqrt{\frac{\pi}{\alpha}}\right]^{1/2} I_0^{1/2} = F(\eta_0') + f(\eta_0). \tag{28}$$

This is identical with that obtained by Fry and Langmuir. The values of $F(\eta_0')$ and $f(\eta_0)$ have been tabulated by Langmuir.[7]

From this d-c. relation, it may readily be shown that the a-c. plate impedance of the planar diode is

$$r_p = \frac{\dfrac{F(\eta_0') + f(\eta_0)}{2} + \dfrac{dF(\eta_0')}{d\eta_0'} + \dfrac{df(\eta_0)}{d\eta_0}}{\dfrac{eI_0}{kT}\dfrac{df(\eta_0)}{d\eta_0}} = \frac{BkT}{eI_0}\frac{1}{\dfrac{df(\eta_0)}{d\eta_0}}. \tag{29}$$

Since the noise generator voltage,

$$E = I_1 r_p + V_1,$$

the substitution of (27) and (29) reduces (24) to the following relations:

$$\begin{aligned}\frac{e}{kT}(I_1 r_p + V_1)\frac{df(\eta_0)}{d\eta_0} &= \frac{eE}{kT}\frac{df(\eta_0)}{d\eta_0} = \frac{e}{I_0}\left\{\int_{\bar{u}_c}^{\infty}(B - C - D)\delta(u_c)du_c\right.\\ &\left.\quad - \frac{2}{\sqrt{\pi}}\int_0^{\eta_0'}\int_{-}^{\bar{u}_c}\frac{\sqrt{y^2+\eta}\,\delta(u_c)du_c d\eta}{\Phi(\eta)}\right\}.\end{aligned} \tag{30}$$

By the additional symbols,

$$\left.\begin{aligned} H(u_c, \eta_0, \eta_0') &\equiv \frac{kT}{I_0\dfrac{df(\eta_0)}{d\eta_0}}(B - C - D)\\ G(u_c, \eta, \eta_0) &\equiv -\frac{2}{\sqrt{\pi}}\frac{kT}{I_0\dfrac{df(\eta_0)}{d\eta_0}}\frac{\sqrt{y^2+\eta}}{\Phi(\eta)}\end{aligned}\right\}, \tag{31}$$

the noise generator voltage may be expressed in the condensed form,

$$E = \int_{\bar{u}_c}^{\infty} H\delta(u_c)du_c + \int_0^{\eta_0'}\int_{\bar{v}}^{\bar{u}_c} G\delta(u_c)du_c d\eta. \tag{32}$$

Unfortunately (32) cannot be integrated because the specific value of the instantaneous deviation in the rate of electron emission is unknown. Moreover, as shown by Fry [10] there exists no frequency spectrum for this deviation. The reason is because there is no way of foretelling at a particular instant just when the next electron is going to be emitted from a thermionic cathode. It does not follow, however, that Fourier methods are powerless, as the following argument will show. Imagine that the emission in a thermionic system has been going on for a long time, and place a recorder in the system which makes an oscillograph record of the voltage produced across the tube by the fluctuating current. Let the record be made over a long period of time. Then, it is a perfectly possible thing to analyze the record so obtained into a Fourier spectrum. The result will give no information that the Fourier spectrum which would be obtained on the oscillograph during an ensuing time period of equal length would be the same as the one which has been secured during the past. However, when the mean square value of the Fourier spectrum for the recorded interval is computed, it is found that the mean square value of any two records so obtained is the same when they are both produced by random emission. Moreover, the mean square value within a specified frequency interval is also the same in the two records. These facts result from the random character of the events producing the records, as may be seen even more clearly by examination of the mathematical steps in the following equations, particularly as given in the progression from (37) to (38). It follows, then, that one is justified in concluding that the mean square response of an electrical system to a random excitation can be calculated by the Fourier series method, and that the result so obtained applies equally well either to systems which have been measured in the past, or to those which will be measured in the future, provided only that they both are similar in their configuration and external operating conditions.

To obtain the Fourier expression for the emission deviation, it will be assumed that this function repeats itself after a very long period of time. To find this Fourier Series for the instantaneous deviation from the average rate of emission of electrons with x-directed velocities between u_c and $u_c + du_c$, the very long period of time, L, is divided into P equal intervals of length, τ, where τ is assumed to be mathematically small. The exact number of electrons emitted during any of these intervals of time with velocities between u_c and $u_c + du_c$ per unit area of the cathode is denoted by $n_m(u_c)\tau$. The average rate of emission of electrons in this velocity class may then be defined as follows:

$$n_0(u_c) \equiv \frac{1}{P} \sum_{m=1}^{P} n_m(u_c). \tag{33}$$

During each interval, the deviation from the average rate of emission is

$$\delta_m(u_c) = n_m(u_c) - n_0(u_c). \tag{34}$$

The mean square of the instantaneous deviation for all P intervals is then

$$\overline{\delta^2(u_c)} = \frac{1}{P} \sum_{m=1}^{P} [n_m(u_c) - n_0(u_c)]^2. \tag{35}$$

The value of the mean square emission deviation may be found from the following considerations. In previous studies of pure shot noise, it was assumed that the electrons are emitted from the cathode independently of one another, that is, the probability of an electron being emitted during a very small interval of time depends only upon the average rate of emission and the length of the time interval. The classical theoretical equation developed for the shot noise from this assumption was found experimentally to be correct to well within experimental errors. Hence, the same assumption may be made in this analysis of tube noise as the presence of the space charge near the cathode can have but a negligible effect upon the rate of emission of electrons from the cathode. Thus, since the electrons in any velocity class are also emitted independently of one another, the application of probability theory shows that the mean square deviation in the total number of electrons emitted with velocities between u_c and $u_c + du_c$ in a time, τ, is equal to the average number of electrons of this velocity class emitted in the same time, τ.

That is,

$$\frac{1}{P} \sum_{m=1}^{P} \tau^2 \delta_m^2(u_c) = \tau n_0(u_c),$$

or

$$\sum_{m=1}^{P} \delta_m^2(u_c) = \frac{L}{\tau^2} n_0(u_c). \tag{36}$$

Since the period of time, τ, is mathematically small, it can be assumed that the instantaneous deviation from the average rate of emission is constant in each of the P intervals and equal to $\delta_m(u_c)$. If it is assumed that the function representing the instantaneous deviation repeats itself after a very long period of time, L, the Fourier series for the P square-top pulses comprising the instantaneous deviation function may be shown to be:

$$\delta(u_c) = \sum_{l=-\infty}^{\infty} \sum_{m=1}^{P} \frac{\tau\delta_m(u_c)}{L} \left[\frac{e^{-il\omega\tau} - 1}{-il\omega\tau} \right] e^{il\omega(t-t_m)}, \tag{37}$$

where $\omega = 2\pi/L$, and $t_m = m\tau$.

The mean square deviation may be found by squaring the above expression and averaging the result over the long period of time L. The result is

$$\overline{\delta^2(u_c)} = 2\sum_{l=1}^{\infty}\sum_{k=1}^{P}\sum_{m=1}^{P} \frac{\tau^2}{L^2}\delta_m(u_c)\delta_k(u_c) \left[\frac{e^{-il\omega\tau} - 1}{-il\omega\tau} \right] \times \left[\frac{e^{+il\omega\tau} - 1}{+il\omega\tau} \right] e^{-il\omega(t_k-t_m)}.$$

Since the time-average of the instantaneous emission deviation must be zero over the period, L, the contribution to the mean square from the double summation with respect to k and m is zero, unless $m = k$.

Thus

$$\overline{\delta^2(u_c)} = 4\sum_{l=1}^{\infty} \frac{\tau^2}{L^2} \left[\frac{1 - \cos l\omega\tau}{l^2\omega^2\tau^2} \right] \sum_{m=1}^{P} \delta_m{}^2(u_c). \tag{38}$$

From (36), this equation reduces to

$$\overline{\delta^2(u_c)} = \frac{4}{L}\sum_{l=1}^{\infty} \left[\frac{1 - \cos l\omega\tau}{l^2\omega^2\tau^2} \right] n_0(u_c). \tag{39}$$

The contribution to the mean square of the instantaneous deviation in the electron emission, from the frequencies between f and $f + df$ is given by

$$\overline{\delta_f{}^2(u_c)} = \frac{4}{L} \sum_{l=2\pi f/\omega}^{l=2\pi(f+df)/\omega} \left[\frac{1 - \cos l\omega\tau}{l^2\omega^2\tau^2} \right] n_0(u_c). \tag{40}$$

The limit of this expression as the length of the periodicity, L, is made infinite, may readily be shown to be

$$\overline{\delta_f{}^2(u_c)} = 2n_0(u_c)df. \tag{41}$$

It is now possible to proceed to find the mean square value of the noise generator voltage given in (32) with the aid of (37) and (41). From (37), the noise generator voltage may be expressed as

$$E = \sum_{l=-\infty}^{\infty} \sum_{m=1}^{P} \frac{\tau}{L} \left[\frac{e^{-il\omega\tau} - 1}{-il\omega\tau} \right] e^{il\omega(t-t_m)} \times \left\{ \int_{\bar{u}}^{\infty} H(u_c)\delta_m(u_c)du_c + \int_0^{\eta_0'} \int_{\sqrt{\alpha(\eta-\eta_0')}}^{\bar{u}_c} G\delta_m(u_c)du_c d\eta \right\}. \tag{42}$$

The mean square of this equation for the noise generator voltage may be obtained by finding the average of the square of the expression over a very long period of time; that is

$$\overline{E^2} = 4\sum_{l=1}^{\infty}\sum_{k=1}^{P}\sum_{m=1}^{P}\frac{\tau^2}{L^2}\left[\frac{1-\cos l\omega\tau}{l^2\omega^2\tau^2}\right]e^{il\omega(t_m-t_k)}$$

$$\times\left\{\int_{x=\bar{u}_c}^{\infty}\int_{y=\bar{u}_c}^{\infty}H(x)H(y)\,\delta_m(x)\,\delta_k(y)\,dx\,dy\right.$$

$$+\,2\int_{\eta=0}^{\eta_0'}\int_{y=\bar{u}_c}^{\infty}\int_{x=\sqrt{\alpha(\eta-\eta_0')}}^{\bar{u}_c}G(\eta,x)H(y)\,\delta_m(x)\,\delta_k(y)\,dx\,dy\,d\eta$$

$$+\int_{\eta=0}^{\eta_0'}\int_{z=0}^{\eta_0'}\int_{y=\sqrt{\alpha(z-\eta_0')}}^{\bar{u}_c}\int_{x=\sqrt{\alpha(\eta-\eta_0')}}^{\bar{u}_c}G(\eta,x)G(z,y)$$

$$\left.\times\,\delta_m(x)\,\delta_k(y)\,dx\,dy\,dz\,d\eta\right\}. \qquad (43)$$

In the above equation, the contribution to the mean square noise generator voltage from the summation with respect to m, for a fixed value of k, is zero unless $m = k$, since the long time average of the emission deviation must be zero. Furthermore, since the electrons are emitted independently of one another

$$\sum_{m=1}^{P}\delta_m(x)\,\delta_m(y) = 0, \quad \text{unless} \quad x = y.$$

From these considerations, the contribution to the mean square generator voltage from the second integral in (43) is zero since x and y have no common value. The contribution from the last integral is a bit more difficult to obtain. However, from (41), the contribution to the mean square noise generator voltage from the frequencies between f and $f + df$ can be shown to be

$$\overline{E_f^2} = 2df\left\{\int_{\bar{u}_c}^{\infty}H^2(u_c)n_0(u_c)\,du_c\right.$$

$$+\int_{\eta=0}^{\eta_0'}\int_{z=0}^{\eta}\int_{u_c=\sqrt{\alpha(\eta-\eta_0')}}^{\bar{u}_c}G(\eta,u_c)G(z,u_c)n_0(u_c)\,du_c\,dz\,d\eta$$

$$\left.+\int_{\eta=0}^{\eta_0'}\int_{z=\eta}^{\eta_0'}\int_{u_c=\sqrt{\alpha(z-\eta_0')}}^{\bar{u}_c}G(\eta,u_c)G(z,u_c)n_0(u_c)\,du_c\,dz\,d\eta\right\}. \qquad (44)$$

In terms of the variable $y = \sqrt{\alpha}u'$, the average rate of emission may readily be shown to be

$$n_0(u_c)\,du_c = \frac{2I_0}{e}\,y\epsilon^{-y^2}dy.$$

From this value of the average rate of emission, and from the definition of H and G in (31), and since the plate impedance of the planar diode is given by (28), the final expression for the mean square noise generator voltage may be expressed as follows:

$$\overline{E_f^2} = 4kr_p(\lambda T)df, \tag{45}$$

where

$$\lambda = \frac{1}{B\dfrac{df(\eta_0)}{d\eta_0}}\left\{\int_0^\infty y(B - C - D)^2 e^{-y^2} dy + \frac{4}{\pi}\int_{z=0}^{\eta_0'}\int_{x=0}^{z}\int_{y=\sqrt{-x}}^{0} \frac{y\sqrt{y^2+x}\,\sqrt{y^2+z}\; e^{-y^2}}{\Phi(x)\Phi(z)} dxdydz + \frac{4}{\pi}\int_{z=0}^{\eta_0'}\int_{x=z}^{\eta_0'}\int_{y=\sqrt{-z}}^{0} \frac{y\sqrt{y^2+x}\,\sqrt{y^2+z}\; \epsilon^{-y^2}}{\Phi(x)\Phi(z)} dxdydz\right\}, \tag{46}$$

$$\left.\begin{aligned}
\eta_0 &= \frac{e}{kT}(V' - V_p), \qquad \eta_0' = \frac{e}{kT}V' \\
B &= \frac{F(\eta_0') + f(\eta_0)}{2} + \frac{dF(\eta_0')}{d\eta_0'} + \frac{df(\eta_0)}{d\eta_0}, \qquad r_p = \frac{B}{\dfrac{eI_0}{kT}\dfrac{df(\eta_0)}{d\eta_0}} \\
F(\eta_0') &= \int_0^{\eta_0'} \frac{dx}{\left[\epsilon^x - 1 + \epsilon^x P(\sqrt{x}) - 2\sqrt{\dfrac{x}{\pi}}\right]^{1/2}}, \\
f(\eta_0) &= \int_0^{\eta_0} \frac{dx}{\left[\epsilon^x - 1 - \epsilon^x P(\sqrt{x}) + 2\sqrt{\dfrac{x}{\pi}}\right]^{1/2}} \\
C &= \frac{1}{\sqrt{\pi}}\int_0^{\eta_0'} \frac{[\sqrt{y^2+x} - y]}{\Phi(x)} dx, \\
D &= \frac{1}{\sqrt{\pi}}\int_0^{\eta_0} \frac{[\sqrt{y^2+x} - y]dx}{\left[\epsilon^{-x} - 1 - \epsilon^x P(\sqrt{x}) + 2\sqrt{\dfrac{x}{\pi}}\right]^{3/2}} \\
\Phi(x) &= \left[\epsilon^x - 1 + \epsilon^x P(\sqrt{x}) - 2\sqrt{\dfrac{x}{\pi}}\right]^{3/2}, \\
P(x) &= \frac{2}{\sqrt{\pi}}\int_0^x \epsilon^{-x^2} dx.
\end{aligned}\right\} \tag{47}$$

Part II—General Discussion

The analysis in Part I shows that as soon as a potential minimum exists, the tube noise in a planar diode is equivalent to the thermal noise of the plate resistance at an effective temperature which is a function of that of the cathode.

In general, the effective value of the diode plate resistance temperature for any operating condition is very difficult to obtain because of the complexity of the final noise equations (45) and (46). However, the limiting value of the ratio of the effective plate resistance temperature to that of the cathode, denoted by "λ" in (45), may be evaluated very readily for certain limiting conditions.

One encounters the first of these conditions when the plate potential and cathode emission are such that the potential minimum has moved just up to the cathode, and is in fact on the point of disappearing. This condition is secured by decreasing the space charge to values less than are required for the formation of a potential minimum away from the cathode. In the equations, it is represented by letting the quantity η_0' approach zero, where η_0' is the natural logarithm of the ratio of the saturation current to the anode current. For this set of operating conditions, all the electrons emitted from the cathode will go to the anode, and hence the condition is appropriate to the study of pure shot noise.

A second condition is obtained when the plate potential is equal in value to the potential of the minimum. Physically, this condition means that the minimum has moved just up to the anode, and requires a negative value for the plate potential. Mathematically, it is represented by a zero value for the quantity η_0, where η_0 is equal to the difference between η_0' and $(e/kT)V_p$. For negative plate voltages greater in magnitude than that of the potential minimum, all electrons having an initial kinetic energy greater than eV_p will reach the anode regardless of the presence of the space charge existing between the two electrodes. For these conditions, the diode becomes a temperature limited current device.

A third limiting condition occurs when the plate potential is large in magnitude compared with that of the potential minimum referred to the cathode. In this condition a potential minimum still exists. It is represented in the mathematics by letting the quantity η_0 become large. This condition represents the normal operating condition for the diode.

As the space charge is decreased, making η_0' very small, from (47), the diode plate impedance becomes very large through the action of $dF(\eta_0')/d\eta_0'$ which becomes infinite as η_0' approaches zero. As all other quantities involved remain finite, the mean square noise current for a very small space charge is

$$\overline{I_1^2} = \frac{\bar{E}_f^2}{(R_p + Z)^2} = \frac{4kTdf \dfrac{\dfrac{df(\eta_0)}{d\eta_0}}{B\dfrac{kT}{eI_0} B\dfrac{df(\eta_0)}{d\eta_0}}} {} B^2 \int_0^\infty y\epsilon^{-y^2} dy \rightarrow 2eI_0 df. \quad (48)$$

Thus, as the potential minimum voltage is reduced to zero, the tube noise as given by (45) reduces to the well known shot effect equation.

For some space charge at the cathode, the value of λ in (45) has definite limiting values for both very low and for very large plate voltages. For a very small value of η_0, that is for negative plate voltage, the value of B defined in (47) is very large because $df(\eta_0)/d\eta_0$ becomes infinite as η_0 is decreased to zero. Thus as $\eta_0 \rightarrow 0$

$$\lambda \rightarrow \frac{1}{B^2} B^2 \int_0^\infty y\epsilon^{-y^2} dy = \frac{1}{2}. \quad (49)$$

Hence, for any value of space charge, the effective plate resistance temperature for negative plate voltages is one-half of the cathode temperature, under the restriction that no potential minimum exists between the cathode and anode.

Since the diode is usually operated with a positive plate voltage, the value of the effective plate resistance temperature for a large value of η_0 is of more interest. For η_0' not equal to zero, and a large value of plate voltage, it can be readily shown that the values of $f(\eta_0)$ and of D are much larger than any other quantities involved in the equation for λ. After a bit of mathematical operation, it may be shown that the limiting values for $f(\eta_0)$ and D are

$$D = \frac{\pi^{1/4}}{2^{3/2}} \left[\frac{4}{3}\eta_0^{3/4} + 3\sqrt{\pi}\eta_0^{1/4} + \cdots - (4y\eta_0^{1/4} + \cdots) \right]$$

$$f(\eta_0) = \frac{\pi^{1/4}}{\sqrt{2}} \left[\frac{4}{3}\eta_0^{3/4} + \sqrt{\pi}\eta_0^{1/4} + \cdots \right].$$

From these relations, the limiting value of λ for a large plate voltage is given by

$$\lambda = 3\int_0^\infty y \left[\sqrt{2}y - \sqrt{\frac{\pi}{4}} \right]^2 \epsilon^{-y^2} dy = 3\left(1 - \frac{\pi}{4}\right) = 0.644. \quad (50)$$

Thus, for any value of space charge, as long as a potential minimum exists, a sufficiently large value of plate voltage may always be found for which the effective plate resistance temperature is 0.644 times the cathode temperature.

It is possible to obtain a good approximation for the effective diode temperature for any operating condition by the following method. The values of "C" and "D" in (47) may be found without too much difficulty by graphical integration for several different values of y, η_0 and η_0'. From the tabulated values for $F(\eta_0')$ and $f(\eta_0)$ given by Langmuir, and from the values found for C and D, the integral,

$$S = \int_0^\infty y[B - C - D]^2 \epsilon^{-y^2} dy, \tag{51}$$

may be evaluated by mechanical means for several values of η_0 and η_0'. This gives the first integral in (46).

It was found practically impossible to calculate directly the contribution to λ from the last two integrals in (46). However, a rough approximation to them may be found indirectly by the following method: If the sum of the two integrals is denoted by Q, then (46) may be written

$$\lambda = \frac{S + Q}{B \dfrac{df(\eta_0)}{d\eta_0}},$$

or $Q = B\lambda(df(\eta_0)/d\eta_0) - S =$ function of η_0' only.

For a fixed value of η_0', the solution of the above equation for several values of η_0 should give a constant value for Q. Unfortunately, only the limiting values of λ are known. However, if the limiting value of 0.644 is substituted for λ in this equation, the calculated value of Q, for a fixed value of η_0', should approach a constant value as η_0 is increased since λ does assume the 0.644 value for η_0 sufficiently large. The limiting value of Q calculated in this manner is the desired contribution to λ from the last two integrals in (46). This method of evaluating Q cannot be very accurate since it involves the difference of two quantities of the same magnitude. However, since Q is small compared to the contribution from the first integral in (46), a large error in Q will introduce a much smaller error in the value of λ.

The values of the effective diode plate resistance temperature calculated in this manner for several different operating conditions are shown in Fig. 2. These curves indicate that the effective diode temperature is 0.644 times the cathode temperature for all practical operating conditions. The values of η_0' and η_0 may be determined from the following relations:

$$\eta_0' = \log_e \frac{I_s}{I_p}, \tag{52}$$

where I_s is the saturation current and I_p is the anode current, and

$$\eta_0 = \eta_0' - \frac{e}{kT} V_p = \eta_0' + \frac{1.16}{T} \times 10^4 V_p, \qquad (53)$$

where V_p is the anode potential, and T is the absolute cathode temperature.

For $T = 900°$ K,

$$\eta_0 = \eta_0' + 12.9 V_p. \qquad (54)$$

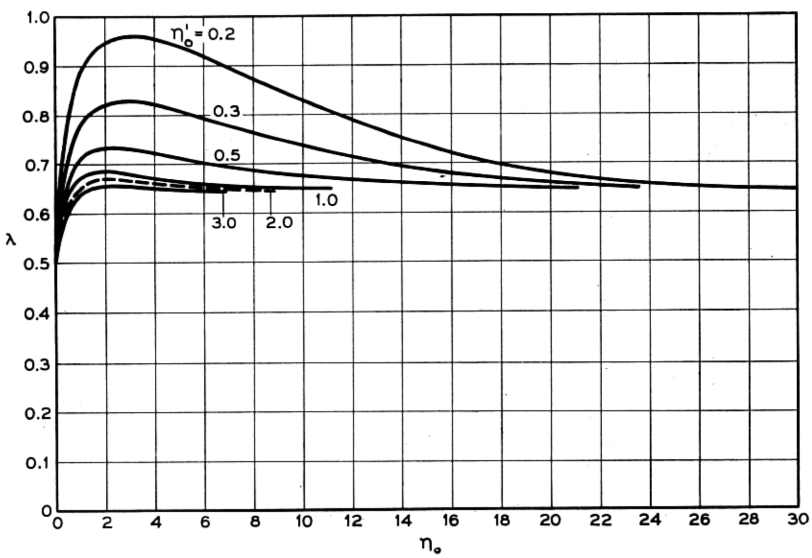

Fig. 2—Effective noise generator voltage of planar diode.

$\overline{E_f^2} = 4kr_p(\lambda T)df, \qquad \eta_0' = \log_e \frac{I_s}{I_p}, \qquad \eta_0 = \eta_0' - \frac{e}{kT} V_p.$

For $T = 900°$ K., $\eta_0 = \eta_0' + 12.9 V_p$.

Even for the small space charge condition for which the plate current is eight-tenths of the saturation current ($\eta_0' = 0.2$), the value of η_0 need be greater than about 25 only before λ assumes its limiting value. For a temperature of 900° K., as for oxide coated cathodes, this would require a plate voltage of only two volts. If the plate current were less than eight-tenths of the saturation current for very high plate voltages, then as the plate voltage is reduced, η_0' would increase. For this operating condition λ maintains its limiting value of 0.644 for all except negative values of plate voltages.

The transition between the various effective planar diode plate resistance temperatures is more clearly shown in Fig. 3. In this figure, the natural logarithm of the ratio of saturation current to the plate current is plotted as a function of the plate voltage for several constant values of the coefficient λ. These curves show for a fixed positive value of plate voltage that as the space charge is decreased toward zero, by a reduction in the ratio of the saturation current to the plate current, the value of λ for moderately large values of space charge increases but little from 0.644, and then for a very low space charge, increases very rapidly to its limiting value given by the shot noise

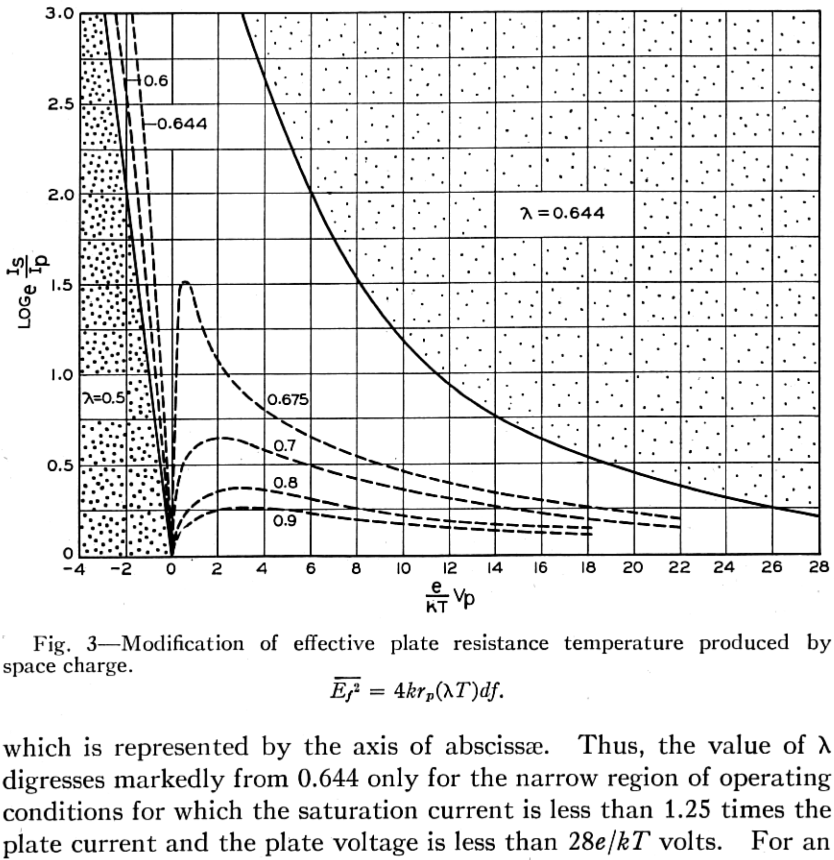

Fig. 3—Modification of effective plate resistance temperature produced by space charge.

$$\overline{E_f^2} = 4kr_p(\lambda T)df.$$

which is represented by the axis of abscissæ. Thus, the value of λ digresses markedly from 0.644 only for the narrow region of operating conditions for which the saturation current is less than 1.25 times the plate current and the plate voltage is less than $28e/kT$ volts. For an oxide coated cathode for which T is 900° K., the effective plate resistance temperature is $0.644T$ for any operating condition for which plate current is less than eight-tenths of the saturation current and the plate voltage is greater than two volts.

For a cylindrical diode, the general method of analysis used in the parallel plane case results in equations which are practically impossible to solve. The difficulty in these equations arises from the fact that tangential as well as radial initial velocities must be considered in obtaining the total anode current. Since it was shown for the planar diode that the effective temperature of the plate resistance is 0.644 times the cathode temperature for practically all operating conditions, all that is really desired in the cylindrical diode solution is the limiting value of the effective tube temperature. This may be found rather easily from a comparison of the cylindrical diode with the planar tube in the following manner.

For a very large space charge, and a high plate potential the radius of an equipotential surface near the potential minimum will be very nearly equal to that of the cathode. Hence, for these operating conditions, the planar diode equations may be applied to this region of the cylindrical diode. In the planar tube, it was shown that for $\eta_0' > 3$, η_0 had to be of the order of unity to obtain the limiting value of 0.644 for λ. If the space charge and plate potential are sufficiently large in the cylindrical diode, the radius of the equipotential surface for which η_0 is greater than unity will practically be equal to that of the cathode. The cylindrical diode may then be divided into two parts, a planar diode between the cathode and the equipotential surface for which $\eta_0 > 1$, and a cylindrical diode formed from the remainder of the tube. In any diode, the only source of noise energy is the cathode from which the noise power is transferred to the anode and external circuit through the mechanism of the initial electronic velocities. Furthermore, the same total noise power must be transferred across any equipotential surface between the cathode and anode. In the planar portion of the cylindrical diode as described above, the total noise power crossing any equipotential surface was shown to be $2.576kTdf$. This same noise power must be transferred across any other equipotential surface in the cylindrical diode. Hence, the effective plate resistance temperature for the cylindrical electrode tube must also be 0.644 times its cathode temperature. From this line of reasoning, it may be shown that the limiting value of the effective temperature for any shape diode is the same as that for the planar tube with the same cathode temperature.

From the experimental data given in his paper, Pearson definitely recognized that the limiting value of the diode plate resistance temperature should be between 0.59 and 0.65 of that of the cathode.[2] The writer understands that North and Thompson of the R.C.A. in an unpublished paper have obtained the same general result for the effect of space charge upon shot noise in diodes.

In a diode, the tube noise may be expressed equally well and with equal correctness either as a modified shot noise or as a thermal resistance noise. In this paper, the thermal resistance viewpoint was taken for two reasons. First, the coefficient "λ," used in the thermal resistance noise equation

$$\overline{E_f^2} = 4kr_p(\lambda T)df,$$

is practically always a constant equal to 0.644, whereas, the factor, "F," used by Schottky and Spenke in their modified shot noise equation

$$\overline{I_1^2} = 2eF^2I_0df$$

is always a function of the operating condition. That is, for the operating conditions for which λ is a constant, F has the following value:

$$F = \frac{1.39}{\left[\log\frac{I_s}{I_p} - \frac{e}{kT}V_p\right]^{1/2}}.$$

The second reason for the selection of the thermal resistance noise relation is that power from the motion of the atoms in the cathode is actually transferred to the plate electrode and external circuit through the mechanism of the initial electron velocities. Hence, the tube noise in a diode with space charge is very similar to a thermal resistance noise.

Part III—Effect of Transit Time

The analysis, in Part I, while giving the correct results for all operating conditions in the ordinary frequency range, is extremely long and cumbersome. It shows, however, that only the limiting values of the effective temperature of the plate resistance are required for most practical cases, and therefore it points the way to make simplifying assumptions which result in a much shorter analysis, and moreover, which allow the analysis to be extended to frequencies so high that electron transit time phenomena become of importance.

Thus the final noise equation in Part I shows that for moderately high anode potentials and for the usual excess of cathode emission, a very good approximation may be had by a consideration of the current-voltage relations existing in the β-region between potential minimum and anode without the necessity of encumbering the analysis by including the α-region between potential minimum and cathode. Moreover, for a large anode potential, the terminal velocities of the electrons at the plate are very large in comparison with their initial velocities for practically all of the electrons. This means that the transit time for the various electrons is practically the same for all of them which leave the cathode within a particular very short time interval, even though the initial velocities of the various electrons are statistically distributed among them. It results that the various individual velocities of the electrons in the β-region may be replaced by an average value, which at the potential minimum may be defined as follows:

$$\bar{u} = \frac{\int_{u_c'}^{\infty} u' n(u_c) du_c}{\int_{u_c'}^{\infty} n(u_c) du_c} . \tag{55}$$

Physically, the meaning of this expression is the average velocity of these electrons which cross a plane in the β-region close to the potential minimum in a unit of time. Inasmuch as the unit of time may be taken to be very small, it follows that (55) expresses the effective instantaneous value of the initial velocity which may, and does, fluctuate as time goes on.

On the basis of an equation of the form

$$I = \rho u - \epsilon \frac{\partial^2 V}{\partial t \, \partial x} \tag{56}$$

the planar diode has been extensively investigated by a number of workers and it has been shown [8] that the relation between current and voltage is completely specified as soon as two boundary conditions are given. These may be the initial velocity and acceleration, or they may equally well be the initial velocity and conduction current ρu. However, the analysis based on (56) applies strictly to the case where all of the charge moves with the same velocity and hence contains a certain approximation when electrons are considered whose velocities have a certain dispersion around some mean value. The error will be small until frequencies are considered which are so high that a large proportion of the electrons which left the cathode in a time interval which is very short compared with the period of the high frequency arrive at the anode in a time interval which is not small compared with the high frequency period. Normally this means that the error is small even for frequencies so high that the majority of the electrons require several cycles to make their transit from potential minimum to anode.

It is convenient to write the resulting equations in terms of d-c. and first order a-c. values where the initial values of d-c. velocity and acceleration are given, but initial values of a-c. velocity and conduction current are employed. The first order a-c. relation derived by Llewellyn may be written in the form

$$-\frac{e}{hm}V_1 = \frac{e}{hm\epsilon}I_1 A + \frac{e}{hm\epsilon}q_a B + \mu_a C, \tag{57}$$

where q_a and μ_a are the initial values of fluctuation conduction current and velocity, respectively, while A, B and C are defined by:

$$\left.\begin{aligned} A &= \frac{1}{\omega^4}\left[-i\omega^3 x + \frac{eI_0}{hm\epsilon}(2 - 2e^{-i\theta} - i\theta - i\theta e^{-i\theta})\right] \\ B &= -\frac{1}{i\omega^3}[a_a(i\theta e^{-1\theta} + e^{-i\theta} - 1) + u_a i\omega(e^{-i\theta} - 1)] \\ C &= \frac{eI_0}{hm\epsilon\omega^2}[i\theta e^{-i\theta} + e^{-i\theta} - 1] \end{aligned}\right\}, \tag{58}$$

in which η is the transit angle, $\omega\tau$, the transit time being τ, and I_0 is the d-c. current.

In the application of these relations to noise analysis, the initial values of velocity, acceleration, and conduction current must be taken at a point in the β-region beyond the potential minimum, but just as close to it as possible without encountering conditions where electrons may be moving toward the cathode, for the equations apply only to cases where the electrons are moving in one direction only. The initial point is, however, located so near to the potential minimum that the d-c. acceleration in (58) may be taken as zero. When this is the case, it may be shown that the initial conduction current is equal to the total current. In other words, the initial value of displacement current is zero. Under such conditions (57) and (58) reduce to the following expression for the a-c. anode potential in terms of the a-c. component of current and initial velocity:

$$V = \frac{I_1}{\omega^4\epsilon}\left[\frac{eI_0}{hm\epsilon}\left(\frac{i\theta}{6} + i\theta + 2e^{-i\theta} + i\theta e^{-i\theta} - 2\right) + \omega^2 u_a(i\theta + e^{-i\theta} - 1)\right] + \frac{\mu_a I_0}{\omega^2\epsilon}[i\theta e^{-i\theta} + e^{-i\theta} - 1]. \tag{59}$$

The term multiplying the a-c. current I_1 in the above equation is the internal high-frequency impedance z of the planar diode. The last term may therefore be identified with an internal emf. When the initial velocity μ, is expressed in terms of the fluctuation of electron velocity, the term gives the equivalent noise generator, E. Thus

$$E = \frac{\mu_a I_0}{\omega^2 \epsilon} (i\theta e^{-i\theta} + e^{-i\theta} - 1) \tag{60}$$

and the mean-square value of the noise emf. (at a frequency ω) is given by:

$$\overline{E^2} = \frac{\overline{\mu_a{}^2} I_0{}^2}{\omega^4 \epsilon^2} |i\theta^{-i\theta} + e^{-i\theta} - 1|^2. \tag{61}$$

The problem is now reduced to finding the mean square value of initial velocity fluctuation, $\overline{\mu_a{}^2}$, which corresponds to electrons crossing the potential minimum. This may be done by going to (55) which gives the effective value of the instantaneous initial velocity and separating all quantities, including the lower integration limits into d-c. and a-c. components. Thus

$$\left.\begin{aligned} n(u_c) &= \underline{n_0}(u_c) + \delta(u_c) \\ u_c' &= \overline{u_c'} + \delta u_c' \\ u' &= \overline{u'} + \delta u' \\ \bar{u} &= u_a + \mu_a \end{aligned}\right\}. \tag{62}$$

The result may be expanded in series form and products of the δ's may be disregarded inasmuch as the a-c. components are small in comparison with the d-c. The indicated operations have as a result

$$u_a = \sqrt{\frac{\pi k T}{2hm}} \tag{63}$$

and

$$\mu_a = \frac{e}{I_0} \int_{\bar{u}_c}^{\infty} (u' - u_a) \delta(u_c) du_c. \tag{64}$$

The Fourier analysis may be applied to this in the way outlined in connection with (37) and (41) in Part I and gives the mean-square value of velocity fluctuation corresponding to a frequency interval df as follows:

$$\overline{\mu_f{}^2} = \frac{2e^2}{I_0{}^2} df \int_{\bar{u}_c}^{\infty} (u' - u_a)^2 u_c(u_c) du_c = \frac{4ekT}{I_0 hm} df \left(1 - \frac{\pi}{4}\right). \tag{65}$$

This may be substituted in (62) giving for the effective noise emf. in the frequency range df

$$\overline{E_f{}^2} = 4kTdf \left[\frac{eI_0 \tau^4}{hm\epsilon^2}\right] \left[1 - \frac{\pi}{4}\right] \left[\frac{1}{\theta^4}\right] \times [\theta^2 + 2 - 2(\cos\theta + \theta \sin\theta)]. \tag{66}$$

The initial average velocity is small so that the low-frequency plate impedance may be written

$$r_p = \frac{eI_0\tau^4}{12hm\epsilon^2}. \tag{67}$$

Thus for any transit angle, the mean square noise generator voltage is given by

$$\left.\begin{aligned} \overline{E_f^2} &= 12\left(1 - \frac{\pi}{4}\right) kr_pTdf\left\{\frac{4}{\theta^4}\left[2 + \theta^2 - 2(\cos\theta + \theta\sin\theta)\right]\right\} \\ &= 4Sk(0.644T)r_pdf \\ \text{where} \\ S &= \frac{4}{\theta^4}\left[2 + \theta^2 - 2(\cos\theta + \theta\sin\theta)\right] \end{aligned}\right\}. \tag{68}$$

For low transit angles, this expression reduces to

$$\overline{E_f^2} = 12\left(1 - \frac{\pi}{4}\right) kr_pTdf, \tag{69}$$

which is precisely the limiting value obtained by the much longer, but more rigorous analysis.

It must be understood that (68) is an approximation since the transit time effect in the region between cathode and potential minimum was entirely neglected, and because the validity of the average velocity concept does fail at the very high frequencies.

Some knowledge of the extent of the operating conditions for which the above equations are good approximations may be obtained from the d-c. current-voltage relation. For the boundary conditions assumed, the low frequency current equation derived from the general solution given by Llewellyn reduces to

$$I = \frac{-2.33(V - V')^{3/2}}{10^6(x - x')^2}\left[1 + 2.66\sqrt{\frac{kT}{e(V' - V)}}\right]. \tag{70}$$

This equation was shown by Langmuir to be a very good approximation for the plate current for most operating conditions and fails only for very low values of plate voltages. Thus, it may be concluded that (68) is a good approximation for all operating conditions except for very low plate voltages and a small space charge.

The plot of (68) given in Fig. 4 shows that the magnitude of the mean square noise generator voltage decreases by five per cent only for transit angles as large as one radian.

The effect of transit time on the pure shot noise for a low space charge density and a high plate voltage may be obtained quite readily from (57) and (58). Since for a very small space charge, I_0 and u_a are small, and a_a large, the equations then reduce to the following expression:

$$-V_1 = \frac{I_1}{i\omega\epsilon}\frac{a_a\tau^2}{2} + \frac{q_a}{i\omega\epsilon}a_a\tau^2\left[\frac{1}{-\theta^2}(i\theta e^{-i\theta} + e^{-i\theta} - 1)\right]. \quad (71)$$

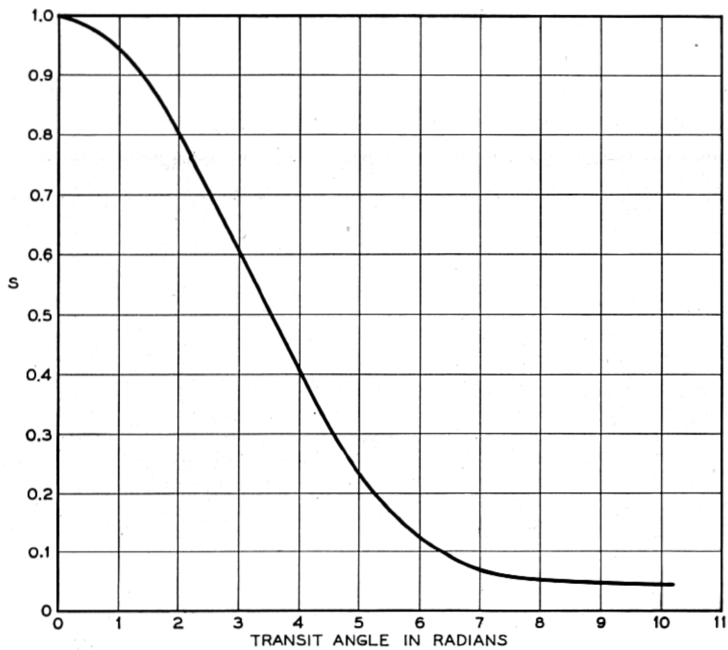

Fig. 4—Effect of transit time on both thermal and shot tube noise.

$$\overline{E_f^2} = 4Sk(0.644T)r_p df,$$
$$\overline{I_f^2} = 2eSI_0 df.$$

For these operating conditions, the transit time in terms of the d-c. acceleration and the electrode spacing is given by

$$x = \frac{a_a\tau^2}{2}.$$

In terms of the external circuit impedance Z_f

$$V_1 = I_1 Z_f, \quad (72)$$

so that (71) and (72) combine to give

$$I_1 = -\frac{q_a}{1 + \dfrac{Z_f i\omega\epsilon}{x}}\left[\frac{2}{i\theta^2}(i\theta e^{-i\theta} + e^{-i\theta} - 1)\right]. \tag{73}$$

The mean square shot noise current is thus given by

$$\overline{I_1^2} = \frac{\overline{q_a^2}}{\left|1 + \dfrac{Z_f i\omega\epsilon}{x}\right|^2}\left|\frac{2}{\theta^2}(i\theta e^{-i\theta} + e^{-i\theta} - 1)\right|^2. \tag{74}$$

The value of the mean square a-c. conduction current at the cathode to be substituted in the above equation may be derived as follows:

The total current emitted from the filament was defined as

$$I = e\int_0^\infty n(u_c)du_c = e\int_0^\infty n_0(u_c)du_c + e\int_0^\infty \delta(u_c)du_c. \tag{75}$$

Hence

$$q_a = e\int_0^\infty \delta(u_c)du_c. \tag{76}$$

From (37) and (41), the contribution to the mean square of this conduction current from the frequencies between f and $f + df$ is

$$\overline{q_a^2} = 2e^2 df\int_0^\infty n_0(u_c)du_c = 2eI_0 df. \tag{77}$$

With this result, the effect of transit time on the shot noise current is given by

$$\overline{I_f^2} = \frac{2eI_0 df}{\left|1 + \dfrac{Z_f i\omega\epsilon}{x}\right|^2}\left\{\frac{4}{\theta^2}[2 + \theta^2 - 2(\cos\theta + \theta\sin\theta)]\right\}, \tag{78}$$

where Z_f is the impedance of the external circuit at the frequency f and $x/i\omega\epsilon$ is the capacitive reactance of the diode at the same frequency. Thus the shot noise current is modified by transit time in precisely the same manner as the noise generator voltage for the thermal tube noise.

The effect of transit time upon the shot noise, as indicated in (78), is identical with that obtained by Spenke for the same operating condition of low space charge and high anode potential.[4] Spenke derives this result through a clever application of a Fourier Series in which account was taken of the effect of transit time upon the wave shape of the current induced in the anode by the electron moving from cathode to the plate. The advantage of the method of average velocities used in this paper is that the effect of transit time in both the thermal tube noise and the shot noise may be found.

It is noteworthy that Ballantine in 1928 derived an expression for the effect of transit time upon the pure shot noise which is identical to that obtained in this paper.[9]

In conclusion, the writer wishes to express his appreciation to F. B. Llewellyn whose supervision and numerous suggestions made possible this paper.

References

1. F. B. Llewellyn, "A Study of Noise in Vacuum Tubes and Attached Circuits," *Proc. I.R.E.*, vol. 18, pp. 243–265 (1930).
2. G. L. Pearson, "Shot Effect and Thermal Agitation in a Space Charge Limited Current," *Physics*, vol. 6, pp. 6–9 (1935).
3. W. Schottky and E. Spenke, "Die Raumladungsschwächung des Schroteffektes," *Wissenschaftliche Vëroffentlichungen aus den Siemens-Werken*, vol. 16, pp. 1–41, Aug. 6, 1937.
4. E. Spenke, "Die Frequenzabhangigkiet der Schroteffektes," *Wissenschaftliche Vëroffentlichungen aus den Siemens-Werken*, vol. 16, pp. 127–136, Oct. 8, 1937.
5. L. H. Germer, "Distribution of Initial Velocities Among Thermonic Electrons," *Phys. Rev.*, vol. 25, 795 (1925).
6. T. C. Fry, "Thermonic Current Between Parallel Plane Electrodes," *Phys. Rev.*, vol. 17, 441 (1921).
7. I. Langmuir, "Effect of Space Charge and Initial Velocities," *Phys. Rev.*, vol. 21, 419 (1923).
8. F. B. Llewellyn, "Operation of Ultra High Frequency Vacuum Tubes," *B.S.T.J.*, Oct. 1935.
9. S. Ballantine, "Schrot-Effect in High Frequency Circuits," *J.F.I.*, vol. 206, 159 (Aug. 1928).
10. T. C. Fry, "The Theory of the Schroteffekt," *J.F.I.*, vol. 199, No. 2 (Feb. 1925).