An Optical Harmonic Analyzer *

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An instrument which makes a Fourier Series Analysis of a function by optical means has recently been completed. The function to be analyzed is supplied in the form of a variation in the density or in the width of the transparent portion of a photographic film. The analysis is performed by a direct evaluation of the integrals which form the coefficients in a Fourier Series, and the results are theoretically exact in the sense that the measurement of each harmonic is independent of the other harmonics which may be present in the function. The operation of the instrument is largely automatic, and is rapid enough so that 30 harmonics can be measured in about a minute and a half.[1]

A PERIODIC function can be represented for all values of the variable by a Fourier Series. A function which is not periodic can be so represented between any finite limits, although the series may be entirely unlike the function beyond these limits. If a function is approximately periodic, the Fourier Series representing adjacent portions of it will generally be approximately alike.

Although in general an infinite number of terms is required to represent a function exactly, it is common experience that a great many functions of practical interest can be closely approximated by a series of from ten to thirty terms.[2]

Principle of Operation

The principle of this analyzer was suggested by E. C. Wente of these Laboratories.[3] It may be outlined as follows.

The Fourier Series expansion of a function is given by either of the following equivalent expressions.[4]

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[1] For comparison, analysis to 30 harmonics on the Henrici type instrument requires five or six hours. A resonance analyzer, such as the vibrating reed type, can complete an analysis in a few seconds, but the phases will not be given, and if the function is provided in graphical form it must be converted into an electrical or acoustic wave form repeated enough times for the resonant elements to reach a steady state response.

[2] A description of a number of the more important methods of harmonic analysis, together with a bibliography, is contained in "Sound Analysis," H. H. Hall, *Jour. Acous. Soc. Amer.*, vol. 8, pp. 257–262, April 1937.

[3] U. S. Patent No. 2,098,326.

[4] The expressions in this form apply when the fundamental period is 2π. There is no loss of generality, as the scale of abscissa can always be so chosen as to conform

$$f(x) = a_0 + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx \tag{1}$$

$$= c_0 + \sum_1^\infty c_n \cos (nx - \phi_n). \tag{2}$$

Comparison of (1) and (2) gives the following relations between the coefficients in the two forms of the expression:

$$a_n = c_n \cos \phi_n, \qquad b_n = c_n \sin \phi_n. \tag{3}$$

$$c_n^2 = a_n^2 + b_n^2, \qquad \phi_n = \tan^{-1} \frac{b_n}{a_n}. \tag{4}$$

The form (2) giving amplitude and phase angle of the harmonics is generally more useful, but most methods of analysis give the coefficients in form (1) necessitating the computation of the amplitude and phase angle from the relations (4). One of the advantages of the optical analyzer is that it will give either set of coefficients directly.

The coefficients in the Fourier Series can be determined from the following expressions: [5]

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx. \tag{5}$$

$$c_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos (nx - \phi_n) dx, \qquad \int_0^{2\pi} f(x) \sin (nx - \phi_n) dx = 0. \tag{6}$$

$$a_0 = c_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x) dx. \tag{7}$$

We will now describe two methods by which a function can be represented on a photographic film. In a *variable area record* the film, which is elsewhere opaque, contains a transparent portion whose width at any point is proportional to the function. Such a record is shown in the upper part of Fig. 1. In a *variable density record* the

to this requirement. In fact this selection of a proper scale corresponds directly to a necessary adjustment of the analyzer.

[5] The expressions given in (6) can be derived from the more familiar expressions (5) as follows. From (3)

$$a_n \cos \phi_n + b_n \sin \phi_n = c_n(\cos^2 \phi_n + \sin^2 \phi_n) = c_n,$$
$$b_n \cos \phi_n - a_n \sin \phi_n = 0.$$

Substituting values of a_n and b_n given by (5) in these expressions leads at once to the expressions (6).

function is represented by gradations in the density of the film such that the light transmission at any point is proportional to the function, the density being uniform in a direction perpendicular to the axis. Such a record is shown schematically in the lower part of Fig. 1. With either type of representation of a function $g(x)$, it will be seen that the amount of light transmitted through a narrow vertical strip of width dx is proportional to $g(x)dx$. If two or more such records are superimposed, the light transmitted through all of them will be proportional to the product of the recorded functions, provided not more than one of the records is of the variable area type.

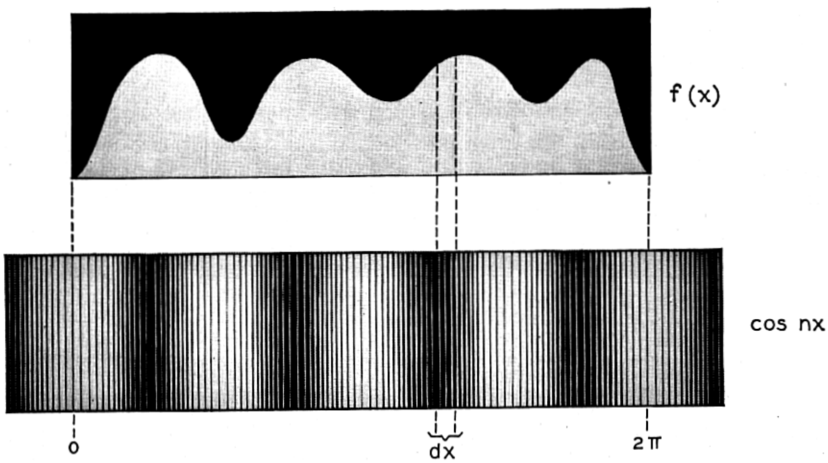

Fig. 1—Representation of $f(x)$ and cos nx on film.

The determination of a_n and b_n is now very straight-forward. Suppose we have $f(x)$ recorded on one film and cos nx on another. For illustration we will assume that $f(x)$ is recorded in variable area and cos nx in variable density, as shown in Fig. 1, although the only necessary requirement is that they shall not both be variable area. If the two films are superimposed, the amount of light transmitted through both of them between the limits zero and 2π is just the first integral in (5) and hence proportional to a_n. Similarly, if the cosine screen is moved a quarter wave-length along the axis it becomes sin nx and we have at once b_n. If the cosine screen is moved a whole wave-length along the axis, the transmitted light will go through a maximum. It will be shown below that this maximum value is proportional to c_n, and the position of the cosine screen at which it occurs is ϕ_n.

One more matter needs to be considered before we write down the expressions which describe the operation of the analyzer. Since $f(x)$

and cos nx will in general have both negative and positive values they cannot be directly represented by the transmission of light, which is essentially positive. However, the addition of a constant to each function will eliminate this difficulty, and merely results in a constant in the measured amplitude, as shown below.

The optical transmission of the film on which $f(x)$ is recorded may be written

$$A + f(x),$$

where A is a constant large enough to make the expression positive for all values of x. Similarly, the transmission of the cosine screens may be written

$$B_n[1 + M_n \cos(nx - \theta)],$$

where θ is a parameter denoting the position of the cosine screen along the x-axis, M_n is a constant somewhat less than unity, known as the modulation of the record, and B_n is a constant which is seen to be the average optical transmission of the screen.

If one or both of these records is of the variable density type, the total transmission when they are superimposed will be

$$\begin{aligned} T &= \int_0^{2\pi} B_n[A + f(x)][1 + M_n \cos(nx - \theta)]dx \\ &= \int_0^{2\pi} AB_n dx + \int_0^{2\pi} B_n f(x)dx + \int_0^{2\pi} AB_n M_n \cos(nx - \theta)dx \\ &\qquad + \int_0^{2\pi} B_n M_n f(x) \cos[(nx - \phi_n) - (\theta - \phi_n)]dx, \\ &= 2\pi B_n(A + c_0) + \pi B_n M_n c_n \cos(\theta - \phi_n). \end{aligned} \tag{8}$$

To obtain a_n, we take the difference in T for $\theta = 0$ and $\theta = \pi$, which is seen to be

$$2\pi B_n M_n c_n \cos\phi_n = 2\pi B_n M_n a_n. \tag{9a}$$

Similarly, to obtain b_n, we make $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, giving for the difference in T

$$2\pi B_n M_n c_n \sin\phi_n = 2\pi B_n M_n b_n. \tag{9b}$$

To obtain c_n and ϕ_n note that the maximum value of T occurs at $\theta = \phi_n$ and the minimum at $\theta = \phi_n + \pi$, which serves to determine ϕ_n. The difference between the maximum and minimum values of T is

$$2\pi B_n M_n c_n. \tag{10}$$

If the factors B_n and M_n can be made approximately constant for all the screens, the coefficients for either form of the Fourier Series are directly proportional to the change in the amount of transmitted light for specified pairs of positions of the cosine screens.

Description of the Instrument

The process which the analyzer is required to carry out consists of superimposing the function to be analyzed on a cosine screen and measuring the variation in the transmitted light when the cosine screen is moved along the x-axis. This is repeated with a different cosine screen for each harmonic which it is desired to measure.

A schematic diagram of the instrument is shown in Fig. 2. The film containing $f(x)$ is placed in a holder at A and strongly illuminated by

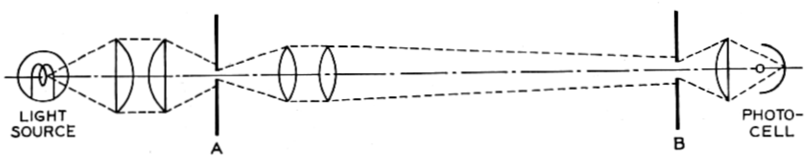

Fig. 2—Diagram of optical system.

an incandescent lamp and condensing lens. An enlarged image of $f(x)$ is formed at B on a window bounded by two knife edges .750 inch apart. Functions of different length are accommodated by adjusting the optical enlargement so that the image of the portion of $f(x)$ to be analyzed will just fill the window. The cosine screens slide in a track directly behind the window, and receive the image of $f(x)$. The transmitted light is collected by another lens and brought to a photocell.

A series of cams and levers is arranged to bring the cosine screens out of a drum shaped magazine in which they are stored into the optical path, give them the small motion required for analysis, and return them to the magazine, which is then rotated to bring the next screen into position. These operations are all automatic, and the attention of the operator is required only for the adjustment of the enlargement and focus and resetting of the cams at the beginning of each analysis. A photograph of the instrument is shown in Fig. 3.

The variations in the photocell output take place at the rate of about two cycles per second. These are recorded on a moving chart by an instrument similar to a high speed level recorder,[6] differing from it chiefly in having a linear instead of a logarithmic scale.

[6] "A High Speed Level Recorder," Wente, Bedell and Swartzel, *Jour. Acous. Soc. Amer.*, vol. 6, p. 121, January 1935.

The present instrument has been designed to take records of $f(x)$ which are from one-sixteenth to five-sixteenths of an inch long and no higher than their length. The focal length of the enlarging lens is 1.5 inches. The collecting lens is placed quite close to the cosine screens, and forms an image of the enlarging lens on the plate of the photocell. With this arrangement the patterns of both $f(x)$ and the cosine screens are well diffused on the photocell plate, so that surface variations in sensitivity of the plate are unimportant. The illumination is uniform across the field to ± 2 per cent.

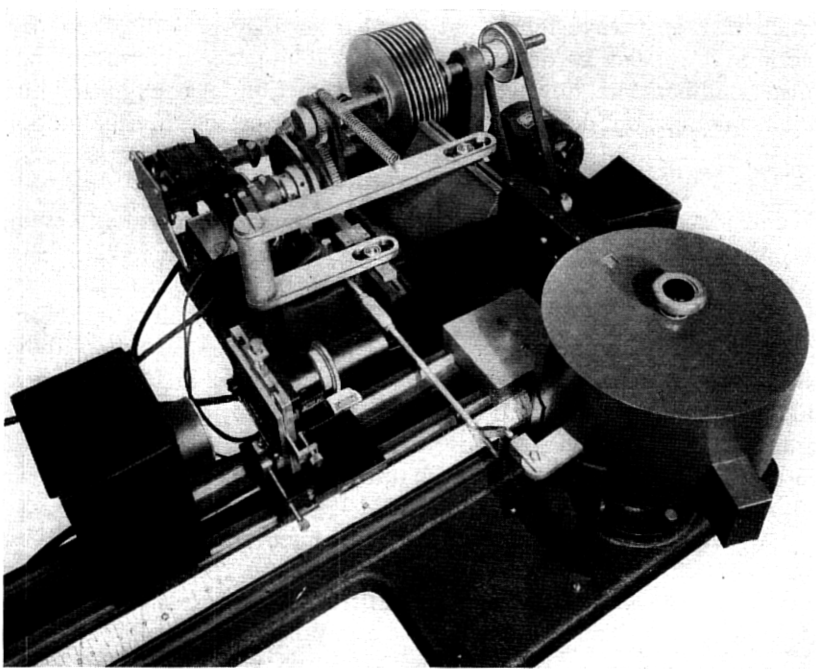

Fig. 3—The optical harmonic analyzer.

The cosine screens were made on photographic plates, by printing from variable density motion picture sound track negatives containing records of pure single frequencies. The pattern thus produced is about 1×2 inches. The increase in width of the track from about one-tenth of an inch in the negative to one inch on the plate was secured by making a "contact" print with negative and plate slightly separated, and moving the plate sideways under the negative while printing.

The important requirements for the screens are good wave form, uniformity in modulation and average transmission, and accuracy in wave-length. In the present instrument it was found possible to keep the harmonic content of the screens down to 5 per cent. The modulation varied from 79 per cent to 94 per cent in different screens, and the average transmission from 20 per cent to 24 per cent. Variations in the wave-length of the screens amounted to about 1 per cent.[7]

It is convenient though not necessary to have good wave form in the screens. When readings are made in pairs as described above, the effects of even order harmonics in the cosine screens cancel out. Moreover, G. R. Stibitz has shown [8] that the cosine screens may have practically any wave form (not even necessarily periodic), and correction factors can be derived for them. The process of correction is rather cumbersome, however, since the correction for each harmonic is not a constant, but depends on other harmonics present in the function.

Uses of the Analyzer

This instrument was designed particularly to accommodate sound records on film as used in commercial motion picture work. However, functions from any other source can be analyzed equally well if they are reproduced with the proper dimensions on film. Provision is made for measurement of the first 30 harmonics. As stated above, the function must be between 1/16 and 5/16 inch in length and no higher than its length. At the speeds customarily used for recording sound on film this corresponds to a fundamental frequency of from 65 to 310 cycles per second, or 1950 to 9300 cycles per second for the 30th harmonic.

The smallest harmonic which the instrument will indicate is about 2 per cent of the peak which it can accommodate. In connection with this statement, it should be remembered, however, that for many functions the largest harmonic in the analysis is considerably less than the peak amplitude of the function, which reduces the effective amplitude range.

An interesting check on the operation of the analyzer can be obtained by making an analysis of a simple geometric wave form. For example, a single cycle of a saw-toothed wave can easily be formed by placing a straight edge obliquely across the sound track slot of the analyzer. Such a wave form is shown at the top of Fig. 4. It is known that this wave form can be resolved into a series of har-

[7] Since a 1 per cent error in wave-length amounts to an error of about one-third of a wave in the total length of a screen of the 30th order, errors of this magnitude are quite objectionable in the higher order screens, although unimportant in the low orders.

[8] Unpublished work.

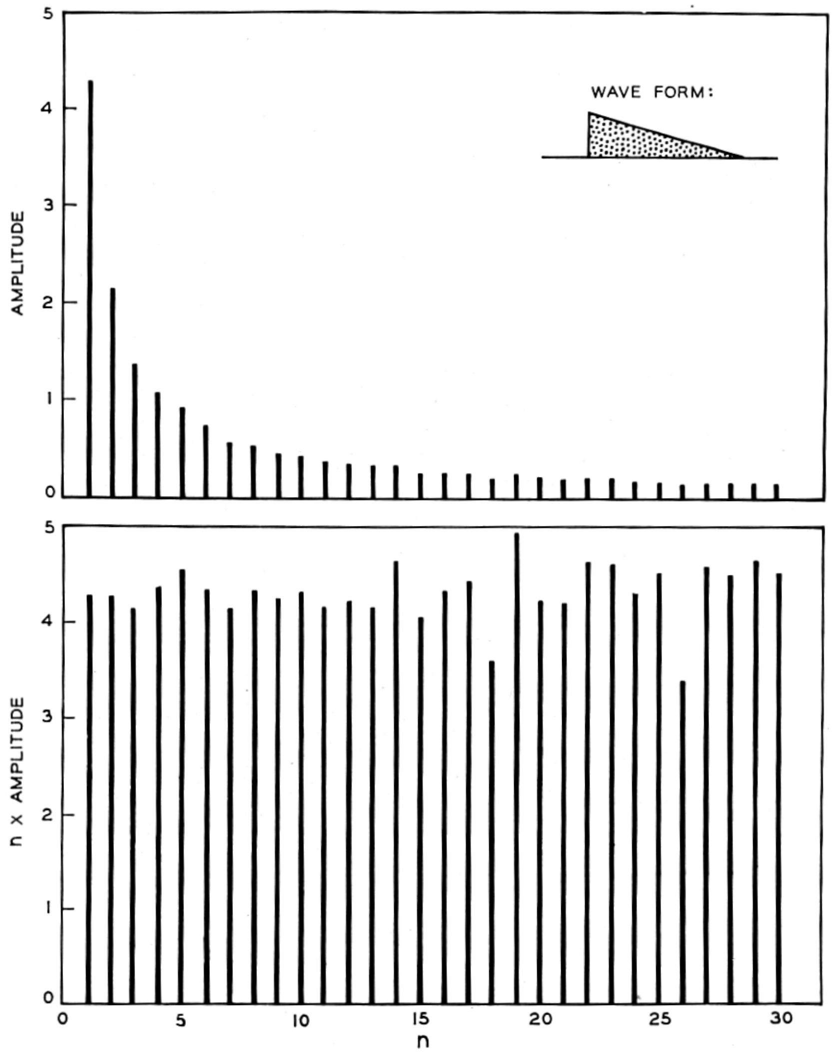

Fig. 4—Analysis of saw-toothed wave.

monics whose amplitude falls off as $1/n$ where n is the order of the harmonic. The values obtained with this analyzer are shown in the upper graph in the figure. In the lower graph each harmonic has been multiplied by n, which should make all the ordinates equal if the analysis were exactly correct.

The use of the analyzer for the sounds of speech is illustrated in Fig. 5, which shows the analyses of portions of two vowel sounds made

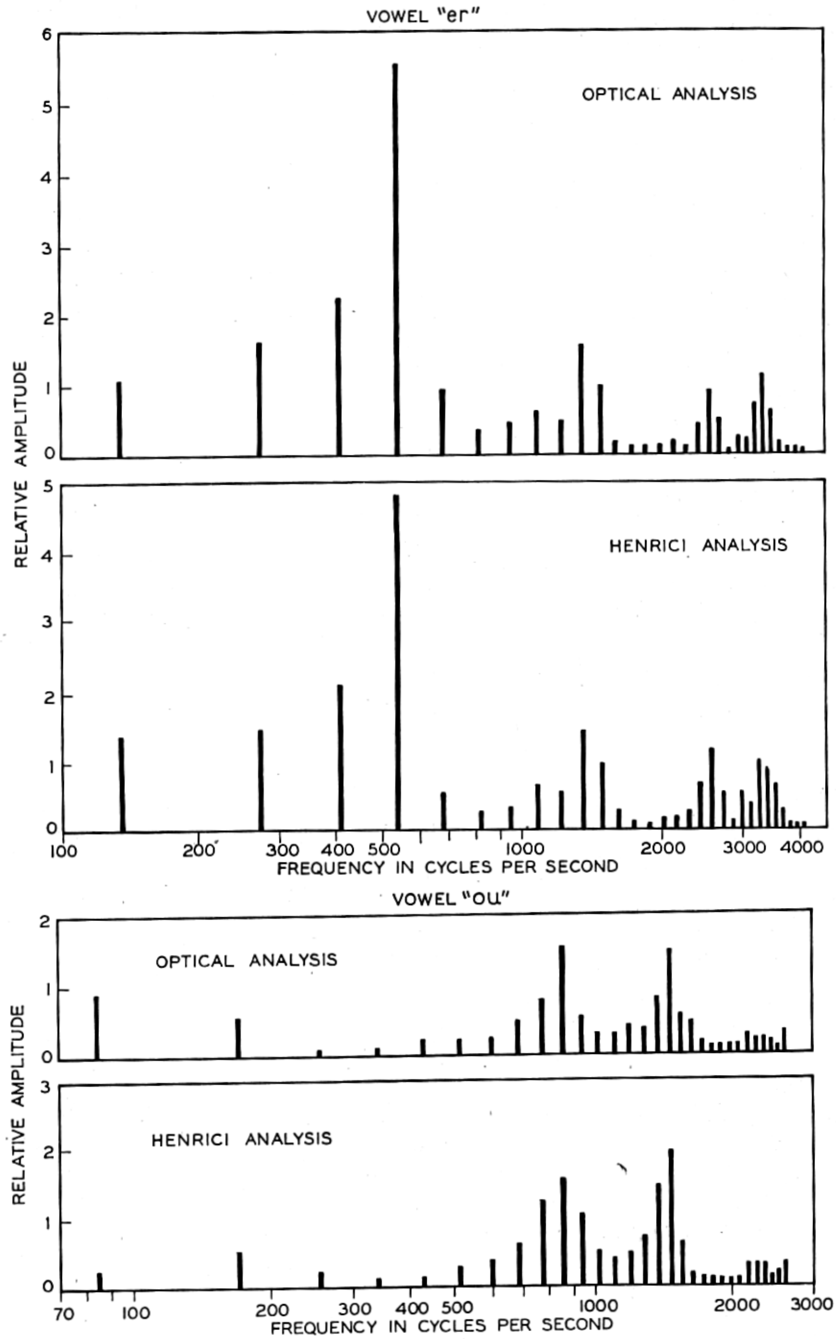

Fig. 5—Analyses of spoken vowel sounds.

with the optical analyzer. Each is compared with an analysis of the identical wave form made on a Henrici type analyzer at the State University of Iowa.[9] The first sound is a portion of the *er* in *father*. There is a very prominent fourth harmonic, indicating a strong resonance in the voice at 530 cycles. Other smaller peaks occur at 1400, 2650 and 3500 cycles. The second sound is a portion of the diphthong *ou* in *out*. It shows two peaks of about equal magnitude, with a suggestion of a third smaller one. The general features of the analyses by the two methods are seen to be in good agreement. A series of such analyses throughout the course of a spoken sound furnishes a fairly complete description of the changes in resonance, amplitude, and fundamental frequency which are taking place. Because of its high speed of operation and convenient application to records of speech on film, the present form of the optical analyzer is especially adapted to such a study of the characteristics of connected speech.

[9] "The Henrici Harmonic Analyzer," D. C. Miller, *Jour. Franklin Inst.*, vol. 185, pp. 285–322 (1916).