Irregularities in Broad-Band Wire Transmission Circuits

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The effects of inhomogeneities along the length of a wire transmission circuit are considered, affecting its use as a broad-band transmission medium. These inhomogeneities give rise to reflections of the transmitted energy which in turn cause irregularities in the measured sending or receiving end impedance of the circuit in its overall attenuation, and in its envelope delay. The irregularities comprise departures of the characteristic from the average, in an ensemble of lines, or departures from a smooth curve of the characteristic of a single line when this is plotted as a function of frequency. These irregularities are investigated quantitatively.

WIRE transmission circuits in their elementary conception are considered as perfectly uniform or homogeneous from end to end. Actually, of course, they are manufactured in comparatively short pieces and joined end to end, and there is a finite tolerance in the deviation of the characteristics of one piece from those of the next and also from one part of the same piece to another. A real transmission circuit therefore has a large number of irregularities scattered along its length which reflect wavelets back and forth when it is used for the propagation of a signal wave. When a cable pair, coaxial conductor, or similar medium is used for broad-band transmission it is important to know how these irregularities influence the transmission characteristics of the medium.

The transmission characteristics which will be studied are the impedance, the attenuation, the sinuosity of the attenuation (to be defined), and the delay distortion. The derivations for the first two characteristics parallel substantially those published by Didlaukis and Kaden (ENT, vol. 14, p. 13, Jan., 1937). They are set forth here for completeness of presentation because the steps in them illustrate the more complicated steps in the derivation of the last two characteristics.

When the characteristic impedance changes from point to point, its variation from the average characteristic impedance for the whole length of conductor forms the irregularities which produce reflections. Assume that successive discrete elementary pieces of the circuit are homogeneous throughout their length, that the lengths of these elementary pieces are equal throughout the length of the whole circuit, and that there is no correlation between the deviations from average characteristic impedance of any two elementary pieces. This represents a first approximation to the problem. It is fairly accurate for pairs in ordinary cable in which the outstanding irregularities are deviations, from the average, between whole reel lengths; and in which the lengths of the successive spliced pieces (reel lengths) are at least roughly the same.

There are irregularities in some coaxial conductors in which the impedance change is gradual rather than abrupt from one element to the next, and in which the elements can vary in length along the line. For these cases the approximation is a little over-simplified. However, although this somewhat affects the echo wavelets as computed from the impedance deviations along the line, Didlaukis and Kaden, as referred to above, have shown that it does not affect the ratio between the echo wavelets, suitably averaged, reaching the receiving end and those, similarly averaged, returning to the sending end.

With the above assumptions there will be some correlation between the reflections at the two ends of an elementary length. If, for example, this length happens to be high in characteristic impedance the reflection at one end will tend greatly to be the negative of that at the other end. For this reason we are going to break up the reflection into two parts, at a point between any two successive elementary lengths of circuit—one part from one length of the circuit to an infinitesimal length of cable of average characteristics inserted between the two elementary lengths—and the other from this infinitesimal piece to the next elementary length of circuit. There is then 100 per cent correlation between the reflections at the two ends of a given elementary length (one being exactly the negative of the other); but there is no correlation between the reflections from any one elementary length to its adjacent infinitesimal piece of average cable, and the reflections from any other elementary length to its adjacent piece. This same treatment is used in the calculation of certain types of "reflection" crosstalk.

The departure in characteristic impedance in the usual transmitting circuit in the higher frequency range, where the irregularities are most important, results essentially from deviations in the two primary constants of capacitance and inductance, each per unit length. There is a certain correlation between these, inasmuch as the capacitance deviation is contributed to both by differences in the dielectric constant of the insulation and by differences in the geometrical size, shape, and relative arrangement of the conductors; and the inductance deviation is contributed to by the latter alone. If there were no deviation in dielectric constant there would be no deviation in velocity of propagation (phase or envelope), which (at the higher frequencies) is inversely proportional to the square root of the product of the capacitance by the inductance. Consequently the portion of the fractional deviation in capacitance which is due to geometrical deviations correlates with an equal and opposite fractional deviation in inductance. Since in practice the contribution from the geometrical deviation is apt to be dominating, that due to the variation in dielectric constant will be neglected and the above correlation assumed as 100 per cent.

The standard deviation of the capacitance of the successive elementary lengths, as a fraction of the average capacitance, will be designated as δ.

The secondary constant of the line most affected by these irregularities is the sending end (or similarly receiving end) impedance. If we consider a large ensemble of lines of infinite length of similar manufacture (and equal average characteristics and δ) but in which the individual irregularities are uncorrelated, then the sending end impedances of these lines, measured at a given frequency, also form an ensemble. The standard deviation of the real parts in this latter is $\sqrt{\overline{\Delta K_r^2}}$, and that of the imaginary parts $\sqrt{\overline{\Delta K_i^2}}$.

In general, the departure in the impedance of one individual line from the average will vary with frequency; and perhaps over a moderate frequency range a sizeable sample can be collected which is fairly typical of the ensemble of the departures at a fixed frequency in the interval. If this is the case, and if at the same time the average impedance varies smoothly and slowly with frequency, and the standard deviation of the ensemble of departures also varies smoothly and slowly with frequency, then the standard deviation of the sample of departures over the moderate frequency interval is substantially equal to that of the ensemble of departures at a fixed frequency in this interval. (It is clear that this disregards exceptional lines in the ensemble, characterized by regularity in the array of their capacitance deviations, for which these conditions do not hold.) Under the circumstances where this observation is valid it makes it possible to correlate measurements on a single line, provided it is not too exceptional, with theory deduced for an ensemble.

The irregularities in the transmission line will also affect its attenuation. If again we consider an ensemble of lines and measure the attenuation of each at a given frequency these attenuations will also form an ensemble.

It will be found in this case, as will be demonstrated further below, that the average attenuation is a little higher than that of a single completely smooth line having throughout its length a characteristic impedance equal to the average of that for the irregular line. This rise varies slowly with frequency. The standard deviation of the attenuation will also include not only the effect of the reflections which we have been considering but in addition one caused by the fact that the attenuations of the successive elementary pieces are not alike, and hence their sum, aside from any reflections, will also show a distribution. This additional contribution will vary only very slowly with frequency. The standard deviation will be $\sqrt{\overline{\Delta\Lambda_1^2} + \overline{\Delta\Lambda_2^2}}$ where Λ represents the losses in the total line, the subscript 1 indicates the contribution due to the reflections, and the subscript 2 that due to the distribution of the individual attenuations.

The same observation may be made about the attenuation that was made about the terminal impedance, as regards measurements made at one frequency on an ensemble of lines and measurements over a range of frequencies on one line; except that the contribution to the deviation caused by the distribution of individual attenuations varies so slowly with frequency that on each individual line it will look like a displacement from the average attenuation, over the whole frequency range. For the purposes of the present paper only the contributions from the reflections will be computed.

When this information on irregularities is being used by a designer of equalizers he is interested in two characteristics: first, how far each attenuation curve for a number of lines will be displaced as a whole from the average; and second, how "wiggly" each individual curve is likely to be. While the observations above give the general amplitude of the latter they do not tell how closely together in frequency the individual "wiggles" are likely to come. To express this, the term "sinuosity" has been defined as the standard deviation of the difference in attenuation (for the ensemble of lines) at two frequencies separated by a given interval Δf. By the previous observations this can be extended to the attenuation differences for successive frequencies separated by the interval Δf, for a range of frequencies in a single line.

When the transmission line is used for certain types of communication, notably for telephotography or television, it is important to equalize it accurately for envelope delay as well as attenuation. The envelope delay is defined as

$$T = d\beta/d\omega \tag{1}$$

where β is the phase shift through the line and ω is 2π times the frequency. For an ensemble of lines, the envelope delay at a given frequency will also form an ensemble, the standard deviation of which will be $\sqrt{\overline{\Delta T^2}}$. By the observations which have already been made the same standard deviation also holds for the envelope delay departures over a range of frequencies on one line.

Let Fig. 1 represent a line of the type we have been discussing. The successive η's represent the reflection coefficients between successive elementary pieces of line. As mentioned before, to avoid correlation, each η is broken up as shown into two h's, representing reflections between the elementary pieces and infinitesimal lengths of average line.

The main signal transmission will flow as shown by the arrow a in Fig. 1. In addition there will be single reflections as shown by the arrow b. Following the assumptions we have set up, this really consists of two reflections from infinitesimally separated points. Further there will be double reflections, that is reflections of reflections, as shown by c. Here again each reflection point, according to our assumptions, consists of two infinitesimally separated ones. There will be a variety of double reflections according to the number of elementary lengths between reflection points. Finally there will be triple, quadruple and higher order reflections which are not shown. The wave amplitude after reflection is cut down by the reflection coefficient. Consequently, even though there are more of them, the total of any given higher order reflections can always be made smaller than that of lower order reflections by a small enough reflection coefficient. We will here study only small reflection coefficients and therefore neglect all reflections of higher order than needed to give a finite result. For effects on the impedance this means neglect of all but first-order reflections. For the other effects studied it means neglect of all but first- and second-order reflections.

The reflection coefficient between two successive impedances (one being $\overline{K}$), is, approximately

$$h = \Delta K/(2\overline{K}). \tag{2}$$

Following our earlier assumptions, namely that the principal cause of impedance departures lies in geometrical irregularities, and that these may be expressed in terms of capacitance departures,

$$\frac{\Delta K}{\overline{K}} = \frac{\Delta C}{\overline{C}}, \qquad \text{or} \qquad h = \frac{\Delta C}{2\overline{C}}, \qquad \text{or} \qquad \sqrt{\overline{h^2}} = \delta/2. \tag{3}$$

Consequently the reflection coefficients are real, namely, they introduce no phase shifts other than 0 or π in the reflections.

The irregularities in sending-end impedance have been computed in Appendix I from the single reflections of the type b in Fig. 1. The

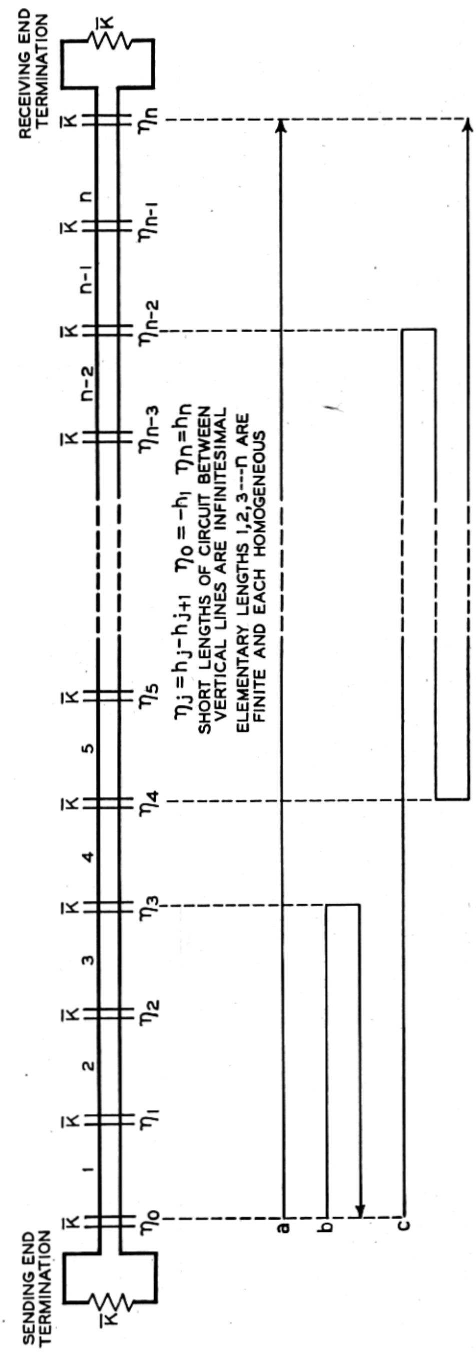

FIG. 1—Inhomogeneous line divided into elementary segments.

final simplified result is

$$\frac{\sqrt{\overline{\Delta K_r^2}}}{\overline{K}} = \frac{\sqrt{\overline{\Delta K_i^2}}}{\overline{K}} = \frac{|\phi|\delta}{2\sqrt{\epsilon}}, \tag{4}$$

where ϕ is the phase shift in radians in two elementary lengths, ϵ is the attenuation in nepers of two elementary lengths, and δ is, as mentioned before, the standard deviation in C measured as a fraction of $\overline{C}$. It will be noted that as a consequence of the single reflections, the irregularities in impedance vary as the first power of δ.

The irregularities in attenuation have been computed in Appendix II from the double reflections of the type c in Fig. 1. It is found, as mentioned before, that there is a net rise in average attenuation caused by the reflections, equal, in nepers, to

$$\left(\epsilon + \frac{\phi^2}{2}\right)\frac{n\delta^2}{4}, \tag{5}$$

where n is the number of elementary lengths in the total line. Considering the factor in parentheses in the expression above, although the term ϵ is not usually wholly negligible compared with the term $\phi^2/2$, nevertheless the latter is dominating and sets the order of magnitude of the factor. If the ϵ is disregarded, the expression can easily be put in terms of the impedance irregularities, giving

$$\left[\frac{\sqrt{\overline{\Delta K_r^2}}}{\overline{K}}\right]^2 \Lambda, \tag{6}$$

where Λ as before represents the loss in the total line.

The standard deviation in the loss in nepers, when finally simplified, is, for the reflections,

$$\sqrt{\overline{\Delta\Lambda_1^2}} = \frac{\phi^2\delta^2\sqrt{n}}{8\sqrt{\epsilon}}. \tag{7}$$

Expressed in terms of the impedance irregularities, this amounts to

$$\sqrt{\overline{\Delta\Lambda_1^2}} = \left[\frac{\sqrt{\overline{\Delta K_r^2}}}{\overline{K}}\right]^2 \sqrt{\frac{\Lambda}{2}}. \tag{8}$$

It will be noted that these irregularities in the attenuation vary with the square of δ, or the square of the impedance irregularities. This is a consequence of the double reflections, and will continue to hold for the sinuosity and irregularities in envelope delay. It will also be noted that in this form the equation is independent of ϵ, ϕ, and n. It is in this case that Didlaukis and Kaden found that the result is independent of whether the reflection points are sharp and equally spaced or not.

The sinuosity has been computed in Appendix III. When finally simplified and measured in nepers, it amounts to

$$\sqrt{\overline{(\Delta\Lambda_1 - \overline{\Delta\Lambda_1})^2}} = \frac{\phi^2\delta^2\sqrt{n}}{8\sqrt{2}\epsilon^{3/2}}\frac{d\phi}{df}\Delta f. \tag{9}$$

Expressed in terms of the impedance irregularities this amounts to

$$\sqrt{\overline{(\Delta\Lambda_1 - \overline{\Delta\Lambda_1})^2}} = \left[\frac{\sqrt{\overline{\Delta K_r^2}}}{\overline{K}}\right]^2 \frac{\pi T}{\sqrt{\Lambda}}\Delta f, \tag{10}$$

where T is, as mentioned before, the envelope delay of the whole line, in seconds.

In computing the above it is only the components of the echoes which are in phase (or π radians out of phase) with the main transmission which affect the results. If the echo components at right angles to the main transmission are considered, they will give phase shifts in the resultant signal wave. Further, an echo component whose ratio to the main transmission is x will, when π radians out of phase with it, give a loss of x nepers; and when at right angles to it, a phase shift of x radians. Now the distribution of echo components in phase (or π radians out of phase) with the main transmission is substantially the same as that of components at right angles to it. Consequently the sinuosity is also numerically equal to the standard deviation of the difference in phase shifts at two frequencies separated by the given interval Δf. Therefore if the interval is called $\Delta\omega/2\pi$ and the resulting numerical value of the sinuosity is divided by $\Delta\omega$ it will give the standard deviation of the envelope delay. This is

$$\sqrt{\overline{(T - \overline{T})^2}} = \left[\frac{\sqrt{\overline{\Delta K_r^2}}}{\overline{K}}\right]^2 \frac{T}{2\sqrt{\Lambda}}. \tag{11}$$

The quantity which has been used in considering the suitability of a line from a delay standpoint for transmitting pictorial signals is its envelope delay distortion, or maximum departure in delay each way from a fixed average in the frequency band studied. If we make the usual assumption that the maximum departure ordinarily met (strictly speaking, except in about 3 cases out of 1000) is three times the standard deviation, then the delay distortion contributed by the irregularities is $\pm$ 3 times the expression given in equation (11).

Expressed in more usual units, the results given in equations (6), (8), (10), and (11) are repeated here.

$$\text{Rise in average attenuation (db)} = \left[\frac{\sqrt{\overline{\Delta K_r^2}}}{\overline{K}}\right]^2 \alpha L, \qquad (6')$$

$$\text{Standard deviation in attenuation (db)} = \left[\frac{\sqrt{\overline{\Delta K_r^2}}}{\overline{K}}\right]^2 \sqrt{4.343\alpha L}, \qquad (8')$$

$$\text{Sinuosity (db per kilocycle)} = 0.0256 \left[\frac{\sqrt{\overline{\Delta K_r^2}}}{\overline{K}}\right]^2 \frac{\pi\tau\sqrt{L}}{\sqrt{\alpha}}, \qquad (10')$$

$$\text{Delay distortion (microseconds)} = \pm\, 4.42 \left[\frac{\sqrt{\overline{\Delta K_r^2}}}{\overline{K}}\right]^2 \frac{\tau\sqrt{L}}{\sqrt{\alpha}}, \qquad (11')$$

where L = length of the line in miles,
α = attenuation of the line in db per mile,
τ = envelope delay of the line in microseconds per mile.

In order to convey a notion as to possible orders of magnitude of these effects of irregularities, and how they vary with changes in the parameters, a few calculations have been tabulated below for some hypothetical lines.

$\frac{\sqrt{\overline{\Delta K_r^2}}}{\overline{K}}$	Circuit Length, Miles	Attenuation, db per Mile	Rise in Average Loss, db	Standard Deviation in Loss, db	Sinuosity, db for Interval of 1 Kc.	Delay Distortion, Micro-Seconds
1 per cent	100	5	0.05	0.005	0.2×10^{-3}	±0.01
		10	0.10	0.007	0.15 "	±0.01
	1000	5	0.5	0.015	0.7 "	±0.04
		10	1.0	0.02	0.5 "	±0.03
2 per cent	100	5	0.2	0.02	0.9 "	±0.05
		10	0.4	0.03	0.6 "	±0.03
	1000	5	2.0	0.06	3. "	±0.15
		10	4.0	0.08	2. "	±0.1

Note: τ = 6 micro-seconds per mile.

Appendix I

Impedance

In Fig. 1 the circuit is divided into n homogeneous elementary lengths. For a current of unit value traveling down the circuit at the junction of the kth and $(k + 1)$th elementary lengths, the reflected wave is

$$h_k - h_{k+1}, \tag{1}$$

where h_k denotes the reflection coefficient (assumed to be a real number) between the impedance of the kth elementary length and the average impedance.

However, if the current starts with unit value at the sending end, then the wave has to be multiplied by the factor $e^{-kP/2}$ in reaching the point of reflection, where P is the propagation constant per two elementary lengths. In returning to the sending end the reflected wave is again multiplied by a like amount so that its value on arrival there becomes

$$(h_k - h_{k+1})e^{-kP}. \tag{2}$$

The totality of echoes returning to the sending end is

$$E_b = -h_1 + \sum_{k=1}^{n} (h_k - h_{k+1})e^{-kP} = \sum_{k=1}^{n} h_k(e^{-kP} - e^{-kP+P}). \tag{3}$$

Let

$$e^{-P} = e^{-\epsilon+i\phi} = Be^{i\phi}. \tag{4}$$

When n is large, it is permissible to use the assumption that k has ∞ for its upper limit in the above summation. The real part of E_b is accordingly

$$E_{br} = \sum_{k=1}^{\infty} h_k[B^k \cos k\phi - B^{k-1} \cos (k-1)\phi]. \tag{5}$$

By the same method as described for the more complicated case in Equation 15, Appendix II:

$$\overline{E_{br}^2} = \overline{h^2} \sum_{k=1}^{\infty} [B^{2k} \cos^2 k\phi - 2B^{2k-1} \cos k\phi \cos \{(k-1)\phi\} + B^{2k-2} \cos^2 \{(k-1)\phi\}]. \tag{6}$$

This series may next be evaluated, giving:

$$\overline{E_{br}^2} = \frac{\overline{h^2}}{2}\left(\frac{1 - 2B\cos\phi + B^2}{1 - B^2} + \frac{1 - B^2}{1 + 2B\cos\phi + B^2}\right). \tag{7}$$

In a similar manner it follows for E_{bi}, the imaginary part of E_b, that

$$\overline{E_{bi}^2} = \frac{\overline{h^2}}{2}\left(\frac{1 - 2B\cos\phi + B^2}{1 - B^2} - \frac{1 - B^2}{1 + 2B\cos\phi + B^2}\right). \tag{8}$$

Then, replacing ϵ and neglecting higher-order terms in ϕ and ϵ, which are small, and putting $\overline{h^2} = \delta^2/4$, equations (7) and (8) become

$$\overline{E_{br}^2} = \overline{E_{bi}^2} = \frac{\phi^2\delta^2}{16\epsilon}. \tag{9}$$

The echo E_b affects the measured impedance. If unit voltage is impressed in series with the line, and a network having impedance $\overline{K}$, the current flowing, not counting the echoes, is $1/2\overline{K}$. The echo current is then $(E_b/1)(1/2\overline{K})$, and the total current

$$\frac{1 + E_b}{2\overline{K}}. \tag{10}$$

The measured impedance is

$$\frac{2\overline{K}}{1 + E_b} \tag{11}$$

and the part due to the line is

$$K_L = \frac{2\overline{K}}{1 + E_b} - \overline{K} = \overline{K}(1 - 2E_b) \text{ approximately,} \tag{12}$$

$$K_L - \overline{K} = -2E_b\overline{K}, \tag{13}$$

$$(K_{Lr} - \overline{K}_r) = -2E_{br}\overline{K}, \tag{14}$$

$$(K_{Li} - \overline{K}_i) = -2E_{bi}\overline{K}. \tag{15}$$

For $\overline{K}$, the real part only is to be used as it is assumed that the imaginary part is negligible in comparison with it. Where departures from $\overline{K}$ are considered, however, this imaginary part may not be negligible in comparison with the departures.

$$\overline{\Delta K_r^2} = 4\overline{E_{br}^2}(\overline{K})^2 = \frac{\phi^2\delta^2\overline{K}^2}{4\epsilon}, \tag{16}$$

$$\overline{\Delta K_i^2} = 4\overline{E_{bi}^2}(\overline{K})^2 = \frac{\phi^2\delta^2\overline{K}^2}{4\epsilon}; \tag{17}$$

$$\therefore \frac{\sqrt{\overline{\Delta K_r^2}}}{\overline{K}} = \frac{\sqrt{\overline{\Delta K_i^2}}}{\overline{K}} = \frac{|\phi|\delta}{2\sqrt{\epsilon}}. \tag{18}$$

Appendix II

Attenuation

The following is a derivation of the standard deviation of the real part of the echo currents (which are received in phase with the direct transmission) over a circuit such as has been assumed in Appendix I. Accordingly, the reflected wave at the junction of the kth and $(k+1)$th homogeneous elementary lengths, for a current of unit value traveling down the circuit at this point, is:

$$h_k - h_{k+1}. \tag{1}$$

This wave returns toward the sending end and in turn suffers partial reflections. Consider this secondary reflection at the point between the jth and $(j+1)$th lengths where $j \leq k$. The wave arriving at the point in question is

$$(h_k - h_{k+1})e^{-P(k-j)/2}. \tag{2}$$

The fraction of this wave which is reflected back again is

$$-(h_j - h_{j+1}), \tag{3}$$

so that the wave which starts back from this point in the same direction as the original wave is:

$$-(h_j - h_{j+1})(h_k - h_{k+1})e^{-P(k-j)/2}. \tag{4}$$

In traveling to the junction of the kth and $(k+1)$th lengths it is again multiplied by $e^{-P(k-j)/2}$ so that the echo which is joined to the unit wave is therefore given by

$$-(h_j - h_{j+1})(h_k - h_{k+1})e^{-P(k-j)}. \tag{5}$$

If $m = k - j$, this echo is

$$-(h_j - h_{j+1})(h_{j+m} - h_{j+m+1})e^{-mP} \qquad \text{when} \qquad m > 0. \tag{6}$$

The sum of all the echoes for a given value of $m > 0$ is:

$$-e^{-mP}\sum_{j=0}^{n-m}(h_j - h_{j+1})(h_{j+m} - h_{j+m+1}) = -e^{-mP}H_m. \tag{7}$$

When $m = 0$, a slightly different treatment is necessary. Let the circuit be represented as in Fig. 1.

A unit current traveling down the circuit will suffer a reflection loss at each junction so that the current passing through the junction is $(1 - \eta_j)$ times the current entering. The ratio of the current received to the current that would be obtained without reflection loss is

$$\frac{I}{I_0} = (1 - \eta_0)(1 - \eta_1)(1 - \eta_2)(1 - \eta_3) \cdots (1 - \eta_n), \tag{8}$$

where the double reflected echoes of the previous type ($m > 0$) are omitted. The echo which is joined to the unit wave when $m = 0$ is

$$\frac{\Delta I}{I_0} = \frac{I - I_0}{I_0}. \tag{9}$$

$$\text{Log}_e \frac{I}{I_0} = \text{Log}_e \frac{I_0 + \Delta I}{I_0} = \frac{\Delta I}{I_0}, \qquad \text{when } \Delta I \text{ is small.} \tag{10}$$

Since

$$\frac{\Delta I}{I_0} = \text{Log}_e \prod_{j=0}^{n} (1 - \eta_j) = \sum_{j=0}^{n} \text{Log}_e (1 - \eta_j) \tag{11}$$

and

$$\text{Log}_e (1 - \eta) = -\eta - \eta^2/2 - \eta^3/3 \cdots, \tag{12}$$

therefore the echo is given as follows in nepers:

$$-\sum_{j=0}^{n} (\eta_j + \eta_j^2/2 + \cdots)$$
$$= -[-h_1 + h_1 - h_2 + h_2 - h_3 + h_3 \cdots - h_n + h_n]$$
$$- \frac{1}{2} \sum_{j=0}^{n} (h_j - h_{j+1})^2. \tag{13}$$

The first term is zero. The sum of all the echoes is

$$-\left\{\frac{1}{2}\sum_{j=0}^{n} (h_j - h_{j+1})^2\right\} - \sum_{m=1}^{n} e^{-mP} H_m$$
$$= -\left\{\frac{1}{2}\sum_{j=0}^{n} (h_j - h_{j+1})^2\right\} - \sum_{m=1}^{n} H_m B^m e^{im\phi}. \tag{14}$$

The in-phase component of these echoes is

$$E_{cr} = -\left\{\frac{1}{2}\sum_{j=0}^{n} (h_j - h_{j+1})^2\right\} - \sum_{m=1}^{n} H_m B^m \cos m\phi, \tag{15}$$

assuming h's may be taken as real and as having a symmetrical distribution curve about zero, the square of whose standard deviation may be denoted by $\overline{h^2}$.

We will consider the distribution curve of H_m, which also is real. The average value of a function $H(h)$ in a given distribution is equal to the integral of the product of the function by the frequency of occurrence for each value of it, divided by the integrated frequency of occurrence alone. The frequency of occurrence of individual values of the function is the same as that of the corresponding values of its argument, and hence can be written as $F(h)dh$ where $F(h)$ is the distribution function of the variable h. The average value of H_m is therefore

$$\begin{aligned}\overline{H}_m &= \int\int\cdots\int H_m F_1 F_2 \cdots F_n dh_1 dh_2 \cdots dh_n \\ &= \int\int\cdots\int \sum_{j=0}^{n-m} (h_j - h_{j+1})(h_{j+m} - h_{j+m+1}) \\ &\qquad \times F_1 F_2 \cdots F_n dh_1 dh_2 \cdots dh_n, \end{aligned} \tag{16}$$

where F_k is the distribution curve of h_k, and

$$\int_{-\infty}^{\infty} F_k dh_k = 1, \tag{17}$$

$$\int_{-\infty}^{\infty} h_k F_k dh_k = 0. \tag{18}$$

Assuming the h's all have equal distribution curves:

$$\int_{-\infty}^{\infty} h_k^2 F_k dh_k = \overline{h^2}, \tag{19}$$

except that since $h_0 = 0$ and $h_{n+1} = 0$, then

$$\int_{-\infty}^{\infty} h_0^2 F_0 dh_0 = 0, \tag{20}$$

and

$$\int_{-\infty}^{\infty} h^2{}_{n+1} F_{n+1} dh_{n+1} = 0. \tag{21}$$

Likewise

$$\int_{-\infty}^{\infty} h_k^4 F_k dh_k = \overline{h^4}, \tag{22}$$

except that

$$\int_{-\infty}^{\infty} h_0^4 F_0 dh_0 = 0, \tag{23}$$

$$\int_{-\infty}^{\infty} h^4{}_{n+1} dF_{n+1} dh_{n+1} = 0. \tag{24}$$

Considering the four products $h_j h_{j+m}$, $h_j h_{j+m+1}$, $h_{j+1} h_{j+m}$ and $h_{j+1} h_{j+m+1}$, it will be seen that they all integrate to zero by virtue of symmetry unless $m = 1$ or $m = 0$. We have

$$\overline{H}_0 = \int\int\cdots\int\sum_{j=0}^{n}(h_j^2 - 2h_jh_{j+1} + h^2{}_{j+1}) \times F_1F_2\cdots F_ndh_1dh_2\cdots dh_n = 2n\overline{h^2}, \quad (25)$$

$$\overline{H}_1 = \int\int\cdots\int\sum_{j=0}^{n-1}(h_j - h_{j+1})(h_{j+1} - h_{j+2}) \times F_1F_2\cdots F_ndh_1dh_2\cdots dh_n$$

$$= \int\int\cdots\int\sum_{j=0}^{n-1}(-h^2{}_{j+1})F_1F_2\cdots F_ndh_1dh_2\cdots dh_n = -n\overline{h^2}, \quad (26)$$

$$\overline{H}_m = 0 \quad \text{if} \quad m > 1. \quad (27)$$

The average value of E_{cr} is equal to the sum of the average values of its terms. Applying the results for $\overline{H}_0$, $\overline{H}_1$, and $\overline{H}_m$, we obtain

$$\overline{E}_{cr} = -\tfrac{1}{2}\overline{H}_0 - \overline{H}_1B\cos\phi = -[1 - B\cos\phi]n\overline{h^2}, \quad (28)$$

$$(\overline{E}_{cr})^2 = [1 - 2B\cos\phi + B^2\cos^2\phi]n^2\overline{h^2}^2. \quad (29)$$

For the mean square of E_{cr} we have:

$$\overline{E_{cr}^2} = \int\int\cdots\int\left(-\frac{1}{2}H_0 - \sum_{m=1}^{n}H_mB^m\cos m\phi\right)^2 \times F_1F_2\cdots F_ndh_1dh_2\cdots dh_n$$

$$= \int\int\cdots\int\frac{1}{4}\sum_{p=0}^{n}\sum_{q=0}^{n}(h_p - h_{p+1})^2(h_q - h_{q+1})^2 \times F_1F_2\cdots F_ndh_1dh_2\cdots dh_n$$

$$+ \int\int\cdots\int\sum_{m=1}^{n}B^m(\cos m\phi)\sum_{p=0}^{n}\sum_{q=0}^{n-m}(h_p - h_{p+1})^2 \times (h_q - h_{q+1})(h_{q+m} - h_{q+m+1})F_1F_2\cdots F_ndh_1dh_2\cdots dh_n$$

$$+ \int\int\cdots\int\sum_{r=1}^{n}\sum_{s=1}^{n}B^{r+s}(\cos r\phi)(\cos s\phi) \times \sum_{p=0}^{n-r}\sum_{q=0}^{n-s}(h_p - h_{p+1})(h_{p+r} - h_{p+r+1})(h_q - h_{q+1}) \times (h_{q+s} - h_{q+s+1})F_1F_2\cdots F_ndh_1dh_2\cdots dh_n. \quad (30)$$

Multiplying the factors containing the h's as indicated in (30) gives terms containing $h_ah_bh_ch_d$ where the subscripts denote some integer such as the value for p, $p + 1$, $p + r$, q, $q + 1$, $q + s$, etc. When there is equality among subscripts so that the terms become $h_u^2 h_v^2$ or h_w^4 the integration gives $(\overline{h^2})^2$ or $\overline{h^4}$, respectively. However, if such equality does not exist, or if one of the subscripts is zero or $n + 1$, the integration gives zero. By integrating term by term in the manner above indicated, adding the results, and finally thereafter putting $r = m$ and $s = m$, the following result is obtained:

$$\overline{E_{cr}^2} = (1 - B\cos\phi)^2 n\overline{h^4} + \Bigg[n^2 - 1 - 2(n^2 + n - 2)B\cos\phi + 2(n-1)B^2\cos 2\phi + (n^2 + 4n - 6)B^2\cos^2\phi + \left\{ \sum_{m=2}^{n} (6n - 6m)B^{2m}\cos^2 m\phi \right\} - 8\left\{ \sum_{m=1}^{n-1} (n - m - \tfrac{1}{2})B^{2m+1}\cos\{(m+1)\phi\}\cos m\phi \right\} + 2\left\{ \sum_{m=1}^{n-2} (n - m - 1)B^{2m+2}\cos\{(m+2)\phi\}\cos m\phi \right\} \Bigg] \overline{h^2}^2. \quad (31)$$

If the distribution of the h's is assumed to be a normal distribution, then:

$$\overline{h^4} = 3(\overline{h^2})^2. \quad (32)$$

Making this substitution and subtracting $(\overline{E}_{cr})^2$ gives:

$$\overline{E_{cr}^2} - (\overline{E}_{cr})^2 = \Bigg[3n - 1 - 8(n - \tfrac{1}{2})B\cos\phi + 2(n-1)B^2\cos 2\phi + nB^2\cos^2\phi + \left\{ \sum_{m=1}^{n} (6n - 6m)B^{2m}\cos^2 m\phi \right\} - 8\left\{ \sum_{m=1}^{n-1} (n - m - \tfrac{1}{2})B^{2m+1}\left(\frac{\cos\{(2m+1)\phi\}}{2} + \frac{\cos\phi}{2} \right) \right\} + 2\left\{ \sum_{m=1}^{n-2} (n - m - 1)B^{2m+2}\left(\frac{\cos\{(2m+2)\phi\}}{2} + \frac{\cos 2\phi}{2} \right) \right\} \Bigg] \overline{h^2}^2. \quad (33)$$

When n is large, it is permissible to use the assumption that m has ∞ for its upper limit in the above summations. It is likewise permissible to neglect terms in the result which do not contain the factor n. Accordingly,

$$\overline{E_{cr}^2} - (\overline{E}_{cr})^2 = \left[-3 + B^2\cos^2\phi + 2\frac{(1 - B\cos\phi)^2}{1 - B^2} + 4\frac{(1 + B\cos\phi)}{1 + B^2 + 2B\cos\phi} \right] n\overline{h^2}^2. \quad (34)$$

The echo current which is joined to the unit received wave affects the final resultant and therefore the effective loss of the line. From equation (28), neglecting higher-order terms, the attenuation of the whole line is increased (in nepers) by

$$\left(\epsilon + \frac{\phi^2}{2}\right)\frac{n\delta^2}{4}. \tag{35}$$

The standard deviation of the attenuation (Λ, in nepers), from equation (34) and neglecting higher order terms, is

$$\sqrt{\overline{(\Lambda - \bar{\Lambda})^2}} = \frac{\phi^2\delta^2\sqrt{n}}{8\sqrt{\epsilon}}. \tag{36}$$

Appendix III

Sinuosity

The following is a derivation of the sinuosity of the attenuation, defined as the standard deviation of the difference $\Lambda(f + \Delta f) - \Lambda(f)$. Here $\Lambda(f)$ is the loss in the circuit at the frequency, f.

For practical purposes, the difference of the expression $E_{cr} - \bar{E}_{cr}$ at two discrete frequencies is

$$\lambda = \frac{d(E_{cr} - \bar{E}_{cr})}{df}\Delta f, \tag{1}$$

whose standard deviation will be derived below. From values of E_{cr} and $\bar{E}_{cr}$ given in Appendix II we obtain

$$E_{cr} - \bar{E}_{cr} = -\frac{1}{2}\left[\sum_{j=0}^{n}(h_j - h_{j+1})^2\right] - \left[\sum_{m=1}^{n} H_m B^m \cos m\phi\right] + [1 - B\cos\phi]n\overline{h^2}, \tag{2}$$

$$\begin{aligned}\lambda &= -\left[n\overline{h^2}\frac{d(B\cos\phi)}{df} + \sum_{m=1}^{n} H_m\frac{d(B^m\cos m\phi)}{df}\right]\Delta f \\ &= -\left[n\overline{h^2}\left\{B\frac{d\cos\phi}{df} + (\cos\phi)\frac{dB}{df}\right\}\right. \\ &\quad \left. + \sum_{m=1}^{n} H_m\left\{B^m\frac{d\cos m\phi}{df} + (\cos m\phi)\frac{dB^m}{df}\right\}\right]\Delta f \\ &= \left[n\overline{h^2}(BQ\sin\phi - D\cos\phi) + \sum_{m=1}^{n} mH_m \right. \\ &\quad \left. \times (B^mQ\sin m\phi - B^{m-1}D\cos m\phi)\right]\Delta f, \end{aligned} \tag{3}$$

where $Q = d\phi/df$ and $D = dB/df$.

$$\overline{\lambda^2} = \int\int \cdots \int \lambda^2 F_1 F_2 F_3 \cdots F_n dh_1 dh_2 dh_3 \cdots dh_n$$

$$= \int\int \cdots \int n^2 \overline{h^2}^2 (BQ \sin \phi - D \cos \phi)^2 (\Delta f)^2 \times F_1 F_2 \cdots F_n dh_1 dh_2 \cdots dh_n$$

$$+ \int\int \cdots \int 2n\overline{h^2}(BQ \sin \phi - D \cos \phi) \times \left[\sum_{m=1}^{n} mH_m (B^m Q \sin m\phi - B^{m-1} D \cos m\phi) \right] \times (\Delta f)^2 F_1 F_2 \cdots F_n dh_1 dh_2 \cdots dh_n$$

$$+ \int\int \cdots \int \sum_{r=1}^{n} \sum_{s=1}^{n} rs(B^r Q \sin r\phi - B^{r-1} D \cos r\phi) \times (B^s Q \sin s\phi - B^{s-1} D \cos s\phi) \left[\sum_{p=0}^{n-r} \sum_{q=0}^{n-s} (h_p - h_{p+1}) \times (h_{p+r} - h_{p+r+1})(h_q - h_{q+1})(h_{q+s} - h_{q+s+1}) \right] (\Delta f)^2 \times F_1 F_2 \cdots F_n dh_1 dh_2 \cdots dh_n. \quad (4)$$

By methods similar to those employed in Appendix II it follows that

$$\overline{\lambda^2} = \Bigg[(BQ \sin \phi - D \cos \phi)^2 \overline{h^4} + \Bigg[-2(BQ \sin \phi - D \cos \phi)^2 + \frac{(B^2Q^2 + D^2)(3 - 8B \cos \phi + \{6B^2 - 2B^4\} \cos^2 \phi + B^4)}{(1 - B^2)^3} - \left(Q^2 - \frac{D^2}{B^2} \right) \left(1 - \frac{1 + 6B^2 - 3B^4}{(1 + 2B \cos \phi + B^2)^3} - \frac{6B(1 + B^2) \cos \phi + 6B^2(1 + B^2) \cos^2 \phi + 4B^3 \cos^3 \phi}{(1 + 2B \cos \phi + B^2)^3} \right) - 2BQD \frac{\{6(1 + B^2) \cos \phi + 4B \cos^2 \phi + 8B\} \sin \phi}{(1 + 2B \cos \phi + B^2)^3} \Bigg] \times (\overline{h^2})^2 \Bigg] n(\Delta f)^2. \quad (5)$$

When the distribution of the h's is normal, this expression can be simplified by noting that

$$\overline{h^4} = 3\overline{h^2}^2. \tag{6}$$

The sinuosity may be obtained from $\overline{\lambda^2}$ as follows:

$$\Delta\Lambda - \overline{\Delta\Lambda} = \Lambda(f + \Delta f) - \Lambda(f) - \{\overline{\Lambda(f + \Delta f) - \Lambda(f)}\} \tag{7}$$

$$= \Lambda(f + \Delta f) - \overline{\Lambda(f + \Delta f)} - \Lambda(f) + \overline{\Lambda(f)} \tag{8}$$

$$= E_{cr}(f + \Delta f) - \overline{E_{cr}(f + \Delta f)} - E_{cr}(f) + \overline{E_{cr}(f)}. \tag{9}$$

Consequently,

$$\sqrt{\overline{(\Delta\Lambda - \overline{\Delta\Lambda})^2}} = \sqrt{\overline{\lambda^2}}. \tag{10}$$

Therefore the sinuosity, expressed in nepers, is

$$\sqrt{\overline{(\Delta\Lambda - \overline{\Delta\Lambda})^2}} = S\delta^2\sqrt{n}, \tag{11}$$

where, in accordance with equations (5) and (6):

$$S = \frac{1}{4}\Bigg[(BQ \sin\phi - D\cos\phi)^2 + \frac{(B^2Q^2 + D^2)(3 - 8B\cos\phi + \{6B^2 - 2B^4\}\cos^2\phi + B^4)}{(1 - B^2)^3} - \left(Q^2 - \frac{D^2}{B^2}\right)\left(1 - \frac{1 + 6B^2 - 3B^4}{(1 + 2B\cos\phi + B^2)^3} - \frac{6B(1 + B^2)\cos\phi + 6B^2(1 + B^2)\cos^2\phi + 4B^3\cos^3\phi}{(1 + 2B\cos\phi + B^2)^3}\right) - 2BQD\,\frac{\{6(1 + B^2)\cos\phi + 4B\cos^2\phi + 8B\}\sin\phi}{(1 + 2B\cos\phi + B^2)^3}\Bigg]^{\frac{1}{2}}(\Delta f) \tag{12}$$

and

$$\delta^2 = 4\overline{h^2}. \tag{13}$$

By expanding S in powers of ϵ and ϕ, and neglecting those higher than needed to give a finite result, it is found that

$$S = \frac{\phi^2\sqrt{Q^2 + D^2}}{8\epsilon\sqrt{2\epsilon}}(\Delta f). \tag{14}$$

In general, D is negligible compared to Q and the sinuosity is

$$\sqrt{\overline{(\Delta\Lambda - \overline{\Delta\Lambda})^2}} = \frac{Q\phi^2\delta^2\sqrt{n}}{8\epsilon\sqrt{2\epsilon}}(\Delta f). \tag{15}$$