A Ladder Network Theorem

By JOHN RIORDAN

The theorem of this paper gives four-terminal representation of ladder networks satisfying a prescribed condition on the side impedances, in terms of the three parameters specifying the network connected as a transducer, the driving-point impedance between short-circuited transducer terminal pairs, and an impedance ratio involving the side impedances only. This mode of representation has a special advantage in applications to electric railway networks in that the transducer parameters which alone involve the ladder shunt impedances (under the stated conditions) may be calculated in a relatively simple fashion, and extensive networks reduced to manageable form. The theorem is stated and proved, and its applications are sketched in some detail.

THE theorem of this paper gives a four-terminal representation of ladder networks satisfying a certain condition with respect to the side impedances. Ladder networks appearing in transmission and filter theory generally are connected as transducers, that is, such that the entry and exit terminals on the ladder sides are associated in pairs; the networks are two-terminal pairs. As is well known, passive transducers may be completely specified by three parameters (as is the case for three-terminal networks, with which transducers are similar in some, though not all, respects), the choice of which has been the occasion for much study and ingenuity.[1] The present theorem does not assume transducer connection and is thus quite distinct from earlier work; indeed it arose outside the communication field in the problem of the calculation of short-circuit currents and network current distribution of electric railway networks, where at present it seems to have chief application.

This paper gives a statement of the theorem, an indication of its applications, and finally its proof.

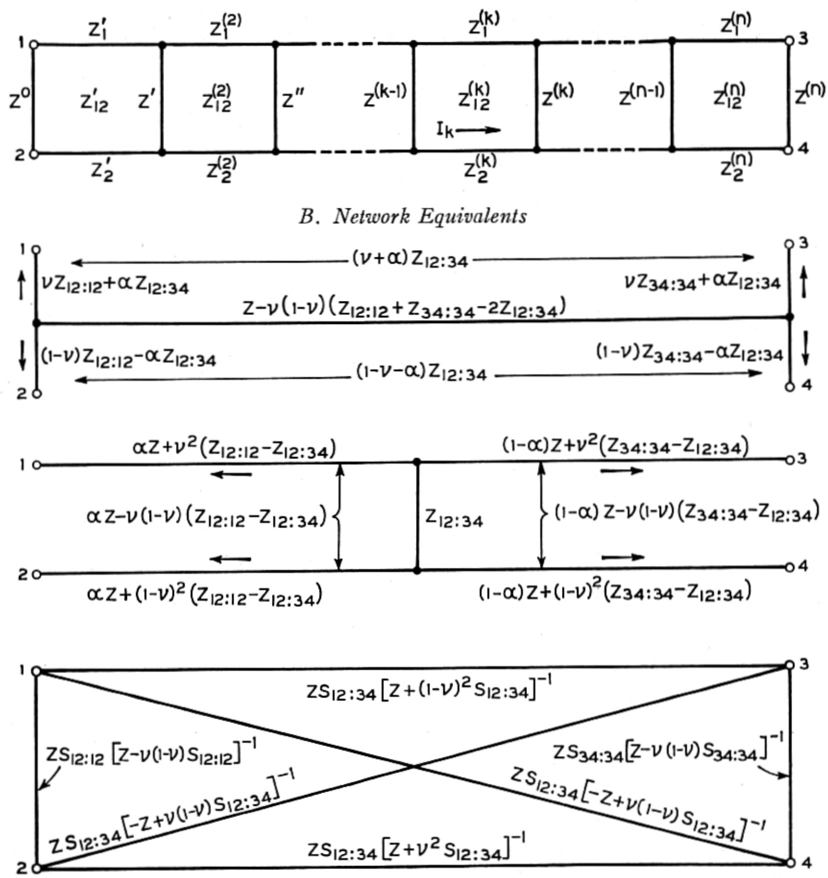
The Theorem

A ladder network, composed of any number of arbitrary shunt impedances forming sections whose side impedances $Z_1^{(k)}$, $Z_2^{(k)}$ *and* $Z_{12}^{(k)}$, $k = 1$,

[1] Five types of equivalent networks by which a transducer may be replaced, including T, π, transformer and artificial line networks, and their interrelations are given on Table I of "Cisoidal Oscillations" by G. A. Campbell, *Trans. A. I. E. E.*, **30**, pp. 873–909 (1911). The most significant addition to the table would appear to be the image impedance representation due to O. J. Zobel.

2 ⋯, are such that $[Z_1^{(k)} - Z_{12}^{(k)}][Z_2^{(k)} - Z_{12}^{(k)}]^{-1}$ *is a constant, may be completely specified by its three transducer parameters (with transducer terminal pairs each made up of adjacent terminals on opposite ladder sides*[2]*), the driving-point impedance between short-circuited transducer*

A. Network Diagram



Notation

$$\nu = \nu_k = [Z_1^{(k)} - Z_{12}^{(k)}][Z_1^{(k)} + Z_2^{(k)} - 2Z_{12}^{(k)}]^{-1}$$

$$S_{12:12} = (Z_{12:12}Z_{34:34} - Z^2_{12:34})/Z_{34:34}$$

$$Z = \sum_1^n \frac{Z_1^{(k)}Z_2^{(k)} - (Z_{12}^{(k)})^2}{Z_1^{(k)} + Z_2^{(k)} - 2Z_{12}^{(k)}}$$

$$S_{34:34} = (Z_{12:12}Z_{34:34} - Z^2_{12:34})/Z_{12:12}$$

α = Arbitrary Constant

$$S_{12:34} = (Z_{12:12}Z_{34:34} - Z^2_{12:34})/Z_{12:34}$$

Fig. 1—A sample ladder network illustrating notation; and some network equivalents.

[2] The theorem also holds when the terminals of each pair are non-adjacent, that is, for terminal pairs 1, 4 and 3, 2 of Fig. 1*A*; this result is of no importance in the railway applications.

terminal pairs, and the constant

$$[Z_1^{(k)} - Z_{12}^{(k)}][Z_2^{(k)} - Z_{12}^{(k)}]^{-1}.$$

The current in any branch of the network for any condition of energization of the network terminals is a linear function of the currents in the same branch for energization at sending and receiving transducer terminals.

It should be observed that the result stated is independent of the number or values of the shunt impedances (except as they are included in the transducer parameters); hence in the diagram on Fig. 1*A* illustrating the ladder network in question, any of the shunt impedances may be allowed to vanish or become infinite, and their number $n + 1$ may be increased or decreased at pleasure provided that one shunt remains (this excludes the trivial case in which, the sides being completely insulated from each other, the network degenerates to a pair of single impedances).

When the impedances of the sides are linearly extended impedances, as is the case in electric railway applications, the section impedances may be written:

$$Z_1^{(k)} = s_k z_1,$$
$$Z_2^{(k)} = s_k z_2,$$
$$Z_{12}^{(k)} = s_k z_{12},$$

where z_1, z_2 and z_{12} are self and mutual impedances of the sides per unit length. The condition, $[Z_1^{(k)} - Z_{12}^{(k)}][Z_2^{(k)} - Z_{12}^{(k)}]^{-1} = \text{const.}$, is replaced by the condition that the shunt impedances connect corresponding points on the sides.

Since a four-terminal network requires six independent quantities for its specification, the conditions (a) that the network be of ladder type and (b) that the given section impedance ratio be constant may be regarded, at least intuitively, as replacing two (or more) of the measurable impedances at the terminals.[3]

With the sending and receiving transducer terminal pairs short-circuited, and with the side impedances satisfying the given condition, no current flows in any of the shunt impedances and the driving-point impedance required for the theorem is

$$Z = \sum_{k=1}^{n} \frac{Z_1^{(k)} Z_2^{(k)} - Z_{12}^{(k)2}}{Z_1^{(k)} + Z_2^{(k)} - 2Z_{12}^{(k)}} = \frac{\sum Z_1^{(k)} \sum Z_2^{(k)} - (\sum Z_{12}^{(k)})^2}{\sum Z_1^{(k)} + \sum Z_2^{(k)} - 2\sum Z_{12}^{(k)}}, \qquad (1)$$

[3] It is interesting to observe that, if short circuits between terminals are permitted, there are 64 measurable impedances for a four-terminal network. The network may be specified by any six of these which are independent; hence the number of ways of specifying the network is something less than the number of combinations of 64 things taken 6 at a time, which equals 74,974,368. The number of non-independent sets which make up this large total appears at the moment to be the smaller part, and possibly a very small part indeed. These remarks are inspired by Mr. R. M. Foster.

where the summations in the last expression extend over all the sections; this impedance then is simply the parallel impedance of the sides taken in their entirety.

The current in branch k of line 2, designated by I_k on Fig. 1A, for any condition of energization is expressed in terms of the currents in the same branch and in the same direction for unit current supplied between terminals 1 and 2, and 3 and 4 [terminals 3, 4 (1, 2) open, respectively], designated by $i_{k:12}$ and $i_{k:34}$, respectively, by the following equation:

$$I_k = \nu(I_3 + I_4) - i_{k:12}[\nu I_1 - (1 - \nu)I_2] - i_{k:34}[\nu I_3 - (1 - \nu)I_4], \quad (2)$$

where I_1, I_2, I_3 and I_4 are currents flowing out of the network from the respective terminals, and ν is the current in side 2 for unit current between short-circuited transducer terminals, as given on Fig. 1B. Thus I_k is a linear function of currents $i_{k:12}$ and $i_{k:34}$, as stated in the second half of the theorem.

Three types of networks completely equivalent to any ladder network satisfying the condition of the theorem are shown on Fig. 1B. The transducer impedances employed in the representation by these networks are the driving-point impedances between transducer terminals 1 and 2, and 3 and 4 [terminals 3, 4 (1, 2) open, respectively] and the corresponding transfer impedance between the ends of the transducer. These impedances are designated $Z_{12:12}$, $Z_{34:34}$ and $Z_{12:34}$, following a notation for Neumann integrals used by G. A. Campbell.[4] For present purposes the notation has the advantage of putting into evidence the terminals between which current is supplied and the terminals between which voltage is measured; thus $Z_{12:12}$ may be read as the voltage drop from 1 to 2 for unit current from 1 to 2 (terminals 3, 4 open), $Z_{12:34}$ the voltage drop from 3 to 4 for unit current from 1 to 2 under the same conditions.[5] By the reciprocity theorem $Z_{12:34} = Z_{34:12}$.

[4] "Mutual Impedance of Grounded Circuits," *Bell System Technical Journal*, **2**, 1–30 (Oct. 1923).

[5] Further, the subscripts may be handled algebraically to give results following from the superposition theorem. For this purpose the numbers in each part of the two-part subscript are taken as separated by a minus sign and the colon is taken as a sign of multiplication; thus:

$$Z_{12:12} = Z_{(1-2)(1-2)} = Z_{11} + Z_{22} - 2Z_{12},$$

the last expression being formed by writing out the indicated product and separating the terms. The equation expresses the fact that the impedance of a circuit may be subdivided into the self-impedances of sides (real or fictional) associated with its terminals minus twice their mutual impedance. Moreover, any additional subscripts desired may be intercalated by adding and subtracting the same numeral; the expansion of bracketed terms then gives a relation between circuit impedances; thus:

$$\begin{aligned} Z_{(1-2)(1-2)} &= Z_{[(1-3)+(3-2)][(1-3)+(3-2)]} \\ &= Z_{(13+32)(13+32)} \\ &= Z_{13:13} + Z_{32:32} + 2Z_{13:32}. \end{aligned}$$

The first two equivalent networks are of the H type;[6] as seven impedances are shown on each, whereas only six are required for complete representation, an arbitrary constant α has been introduced so that the mutual impedance of the uprights may be varied at pleasure. Thus in the first H network, the condition $\alpha + \nu = 0$ puts all the mutual impedance between the uprights below the crossbar; the condition $1 - \nu - \alpha = 0$ puts it all above. The same type of shift may be made in the second H network.

The third equivalent is the network of direct impedances ("Cisoidal Oscillations," loc. cit. designation (b)); these are expressed in terms of the transducer parameters with opposite pairs of terminals short-circuited, which following Campbell are denoted by S's. Thus $S_{12:12}$ is the driving-point impedance between terminals 1 and 2 with terminals 3 and 4 short-circuited; $S_{12:34}$ is the ratio of current from 3 to 4 to voltage from 1 to 2 with 1, 2 energized and 3, 4 short-circuited, or its reciprocity theorem equivalent.

These three equivalents correspond respectively to transformer, T and π transducer equivalent networks. For the first H type the transducer condition that currents into terminals 1 and 2, and 3 and 4, shall be equal and opposite entails zero current in the H crossbar, which may be removed, leaving a transformer connection. For the second H type the transducer condition allows grouping the impedances of branches 1 and 2 and their mutual impedance, and of 3 and 4 and their mutual impedance, into single branches, say, branches 1 and 3, which gives the T equivalent network. The reduction of the direct impedance network is not so immediate.

Periodic Ladder Networks

When the network is periodic, the transducer impedances and current distribution may be expressed completely in terms of the

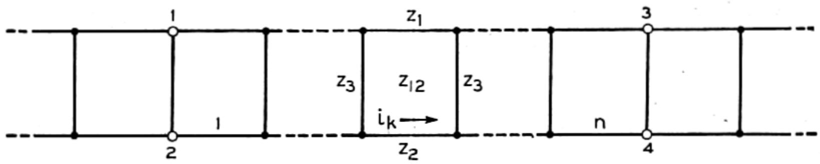
The justification of the operation lies in the fact that, as regards the current half of the subscript, a unit current from 1 to 2 is equivalent by the superposition theorem to unit currents 1 to 3 and 3 to 2 and similarly the voltage 1 to 2 for unit current 1 to 2 is the same as the sum of voltages 1 to 3 and 3 to 2. Thus the notation is a shorthand for application of the superposition theorem. Its use is illustrated further in the course of the proof of the theorem.

[6] This is a form of equivalent network falling under designation (c) of the list of equivalents for an arbitrary number of terminals given by G. A. Campbell ("Cisoidal Oscillations," p. 889, loc. cit.), which is described as branches radiating from a common concealed point, one to each of the terminals, with mutual impedances between pairs. This is not a unique representation since the number of elements is redundant, and the set of mutual impedances may be given values appropriate for particular purposes provided that the self-impedances are adjusted correspondingly. In the present application the mutual impedances of branches to terminals 1 and 4, and 2 and 3, have been set at zero and the mutual impedances of branches 1 and 2 and 3 and 4 in the first H diagram, and of branches 1 and 3, 2 and 4 in the second, have been eliminated in forming the cross bar of the H.

section impedances, giving a certain concreteness to the application of the theorem, which may be valuable. Partly on this account, and partly because periodic iterative impedances are in themselves useful in applications, a particular type of periodic network is considered in this section.

The network considered is infinite in extent, with section series impedances z_1, z_2 and z_{12} and shunt impedance z_3, as shown on Fig. 2*A*.

A. Network Diagram



B. Network Impedances $Z_{ab:cd}$

cd \ ab	12	13	14	24
12	K			
13	$\nu(K-T)$	$Z+2\nu^2(K-T)$		
14	$T+\nu(K-T)$	$Z-\nu(1-2\nu)(K-T)$	$Z+K-2\nu(1-\nu)(K-T)$	
23	$-K+\nu(K-T)$	$Z-\nu(1-2\nu)(K-T)$	$Z-T-2\nu(1-\nu)(K-T)$	$Z+(1-\nu)(1-2\nu)(K-T)$
24	$-(1-\nu)(K-T)$	$Z-2\nu(1-\nu)(K-T)$	$Z+(1-\nu)(1-2\nu)(K-T)$	$Z+2(1-\nu)^2(K-T)$
34	T	$-\nu(K-T)$	$(1-\nu)K+\nu T$	$(1-\nu)(K-T)$

C. Current Distribution

Unit Current Between:	i_k
1 and 2	$-\frac{K_1}{K_1+K_2}e^{-k\alpha}$
1 and 3	$\nu-\frac{K_1}{K_1+K_2}[\nu e^{-k\alpha}+\nu e^{-(n+1-k)\alpha}]$
1 and 4	$\nu-\frac{K_1}{K_1+K_2}[\nu e^{-k\alpha}-(1-\nu)e^{-(n+1-k)\alpha}]$
2 and 3	$\nu+\frac{K_1}{K_1+K_2}[(1-\nu)e^{-k\alpha}-\nu e^{-(n+1-k)\alpha}]$
2 and 4	$\nu+\frac{K_1}{K_1+K_2}[(1-\nu)e^{-k\alpha}+(1-\nu)e^{-(n+1-k)\alpha}]$
3 and 4	$\frac{K_1}{K_1+K_2}e^{-(n+1-k)\alpha}$

Fig. 2—Periodic ladder network of infinite extent; network diagram, impedances and current distribution.

The infinite network is the simplest to formulate since there are no points of reflection; it is of course symmetrical with respect to the terminal pairs 1, 2 and 3, 4.

The impedance across either of these terminal pairs is the parallel impedance of the full-series and full-shunt iterative impedances (or one-half the mid-shunt iterative impedance). The full-series and full-shunt iterative impedances are given by the following formulas:

$$\textit{Full-series}\ K_1 = \tfrac{1}{2}[\sqrt{z(z + 4z_3)} + z] = K_2 + z,$$

$$\textit{Full-shunt}\ K_2 = \tfrac{1}{2}[\sqrt{z(z + 4z_3)} - z] = \frac{K_1 z_3}{K_1 + z_3}, \tag{3}$$

where, for brevity, $z = z_1 + z_2 - 2z_{12}$.

Then

$$Z_{12:12} = Z_{34:34} = K = \frac{K_1 K_2}{K_1 + K_2} = z_3 \left[1 + \frac{4z_3}{z} \right]^{-1/2}. \tag{4}$$

The voltage across lines is propagated as exp $(- k\alpha)$ where α is the section propagation constant; hence

$$Z_{12:34} = T = Ke^{-n\alpha}. \tag{5}$$

The propagation factor exp $(- \alpha)$ is defined in terms of the iterative impedances by

$$e^{-\alpha} = \frac{K_2}{K_1}. \tag{6}$$

The currents $i_{k:12}$ and $i_{k:34}$ are given by the following formulas:

$$i_{k:12} = - \frac{K_1}{K_1 + K_2} e^{-k\alpha}, \tag{7}$$

$$i_{k:34} = \frac{K_1}{K_1 + K_2} e^{-(n+1-k)\alpha}. \tag{8}$$

This completes the formulation, since the remaining quantities, ν and Z, are given immediately by

$$\nu = \frac{z_1 - z_{12}}{z_1 + z_2 - 2z_{12}},$$

$$Z = \sum_{k=1}^{n} \frac{z_1 z_2 - z_{12}^2}{z_1 + z_2 - 2z_{12}} = n \frac{z_1 z_2 - z_{12}^2}{z_1 + z_2 - 2z_{12}}.$$

Figure 2*B* shows driving-point and transfer impedances for energization between terminals, omitting certain impedances equal by symmetry. Figure 2*C* shows the corresponding *k*-section currents in side 2.

Applications to Electric Railroad Networks

A.-c. electric railroad networks in one-line diagram are predominantly of the ladder type. The series elements of sides 1 and 2 represent, for two-wire networks, impedances of sections of transmission lines and traction circuits, respectively; the shunt elements represent transformer impedances. For three-wire networks, the series elements may represent trolley-feeder (or feeder-rail) and trolley-rail impedance elements, the shunt elements autotransformer impedances.

The theorem may be used for representing portions of a network or a whole network of ladder form,[7] when the series impedances satisfy the condition of the theorem. As the circuits are linearly extended this is almost always the case except where the traction circuits change character, from two to four tracks, for example. For approximate purposes the H networks may be used even in these cases provided that the parameter ν is properly chosen. In many cases the transfer impedance $Z_{12:34}$ is negligible and a value of ν may be associated with each pair of terminals; the values for the sections immediately adjoining the terminal pairs 1–2 and 3–4 (sections 1 and n on Fig. 1A) are of dominant importance and serve for rough purposes. If the transfer impedance is not negligible a mean of these values may be sufficiently accurate.

In two-wire networks, generator circuits are connected directly to the transmission line (side 1), and the short circuits of chief interest (grounding points on the one-line diagram) are those on the traction circuits (side 2). Thus, for a single generating point the network is energized between points on sides 1 and 2, such as 1 and 4, for example; if the impedance in the generator connection is Z_g and the impedance of the short circuit is zero, the short-circuit driving-point impedance and the traction circuit currents are as follows:

$$\begin{aligned} Z_0 &= Z_g + Z_{14:14} \\ &= Z_g + Z + \nu^2 Z_{12:12} + (1-\nu)^2 Z_{34:34} + 2\nu(1-\nu)Z_{12:34}, \end{aligned} \tag{9}$$

$$I_k = [\nu + \nu i_{k:12} + (1-\nu) i_{k:34}]\frac{E}{Z_0}, \tag{10}$$

where E is the generator voltage. The impedance may be obtained immediately from either of the H networks—as the sum of the self-

[7] For multiple transmission-line two-wire networks the ladder form is obtained when all transmission lines are bussed at all generating stations and substations or when the generators, step-up transformers and substation transformers connected to each line are of similar impedances and are similarly connected. When these conditions are not met the network is of multiple-side ladder form, for the representation of which an extension of the theorem would be required. Similar remarks apply to three-wire networks.

impedances of legs 1 and 4 and of the crossbar. The current expression follows from equation (2) with $I_2 = I_3 = 0;\ -I_1 = I_4 = E/Z_0$.

For multiple generating points (or for multiple points of short

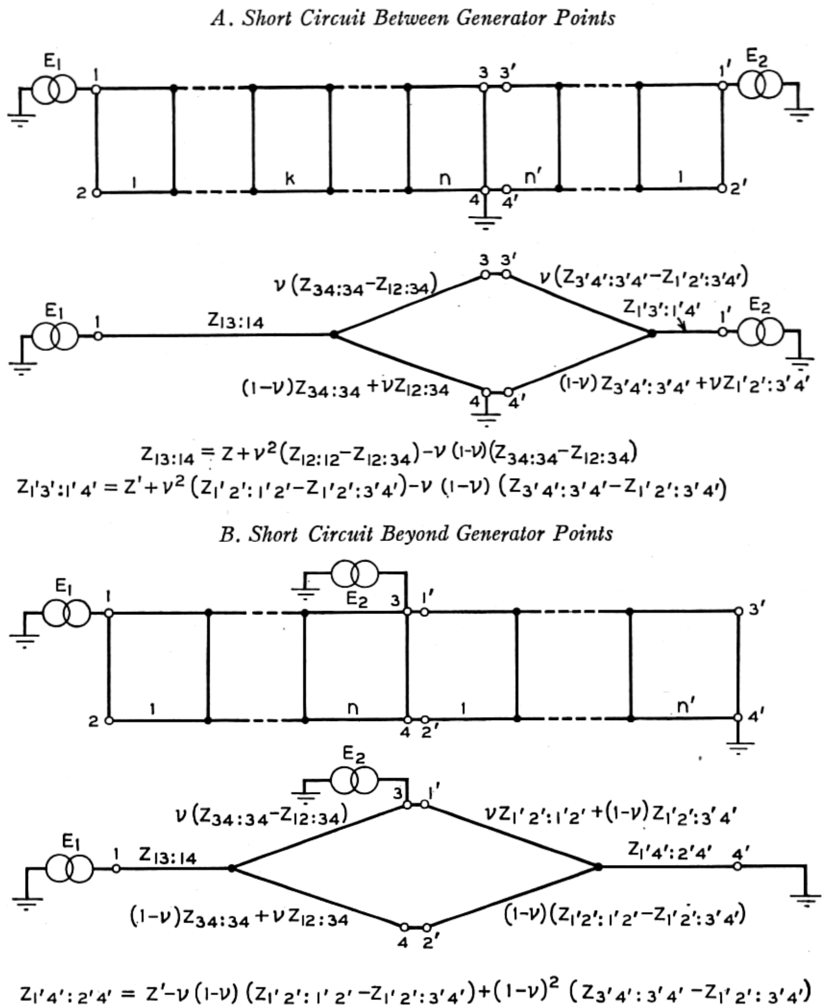


Fig. 3—Equivalent networks for electrified railways; two-wire system, two sources.

circuit) the theorem may be used to represent portions of the network. Examples for the case of two generators are shown on Fig. 3; on Fig. 3*A* the short circuit is located between generator points, on Fig. 3*B* beyond them. On Fig. 3*A* the network is supplied with duplicate pairs of terminals at the short-circuit point, separated by an infinitesimal difference; the parts of the original network thus formed are represented by Y-connected impedances which may be derived from the first H network. On Fig. $3B$ the network is broken at the intermediate generator point and similarly represented. A similar process may be followed for any number of generator points but in some cases it may be expedient to superpose additional generators; the network impedances required for superposition may be formulated in the manner followed in the proof of the theorem.

The solution of these reduced networks supplies the currents I_1, I_3, I_4; I_1', I_3', I_4', etc., from which the current in branch k of side 2 of any of the ladder sections may be found from equation (2). Thus, for example the current I_k in the kth section of the ladder network with terminals 1, 2, 3 and 4 on Figs. $3A$ and $3B$ is formulated as follows:

$$I_k = [\nu + \nu i_{k:12} - \nu i_{k:34}]I_3 + [\nu + \nu i_{k:12} + (1 - \nu)i_{k:34}]I_4, \quad (11)$$

which follows from equation (2) with $I_2 = 0$; $-I_1 = I_3 + I_4$. Similar formulas apply to the other two ladder networks on Fig. 3.

In three-wire networks, generators are usually connected to the traction network by three-winding transformers which may be represented on the network diagram by three impedances connected in star. The traction network may be represented on a trolley-feeder, trolley-rail or a feeder-rail, trolley-rail base and it is well known that the three-winding transformer equivalent impedances for the two bases are related. Using the notation shown on Figs. $4A$ and $4A'$, with primes distinguishing the feeder-rail, trolley-rail base, the relations are as follows:

$$\begin{aligned} V_{tf}Z_a &= V_{tf}Z_a' + V_{tr}Z_b', \\ V_{tf}Z_b &= V_{fr}Z_b', \\ V_{tf}^2Z_c &= V_{fr}^2Z_c' - V_{tr}V_{fr}Z_b', \end{aligned} \quad (12)$$

where V_{tr}, V_{tf} and V_{fr} are the trolley-rail, trolley-feeder and feeder-rail circuit voltages, respectively.

From Figs. $4B$ and $4B'$ showing the reduced networks for trolley-rail short-circuits on the two bases for a single source feed, it is apparent that the impedances involved in the equivalent networks must be similarly related. The relations are found to be as follows:

$$\begin{aligned} V_{tf}^2Z_{12:12} &= V_{fr}^2Z'_{12:12}, \\ V_{tf}^2Z_{34:34} &= V_{fr}^2Z'_{34:34}, \\ V_{tf}^2Z_{12:34} &= V_{fr}^2Z'_{12:34}, \\ V_{fr}(1 - \nu) &= V_{tf}(1 - \nu'), \\ Z &= Z', \\ i_{k:12} &= i'_{k:12}, \\ i_{k:34} &= i'_{k:34}. \end{aligned} \quad (13)$$

TROLLEY-FEEDER, TROLLEY-RAIL BASE

A. Actual Network Connections

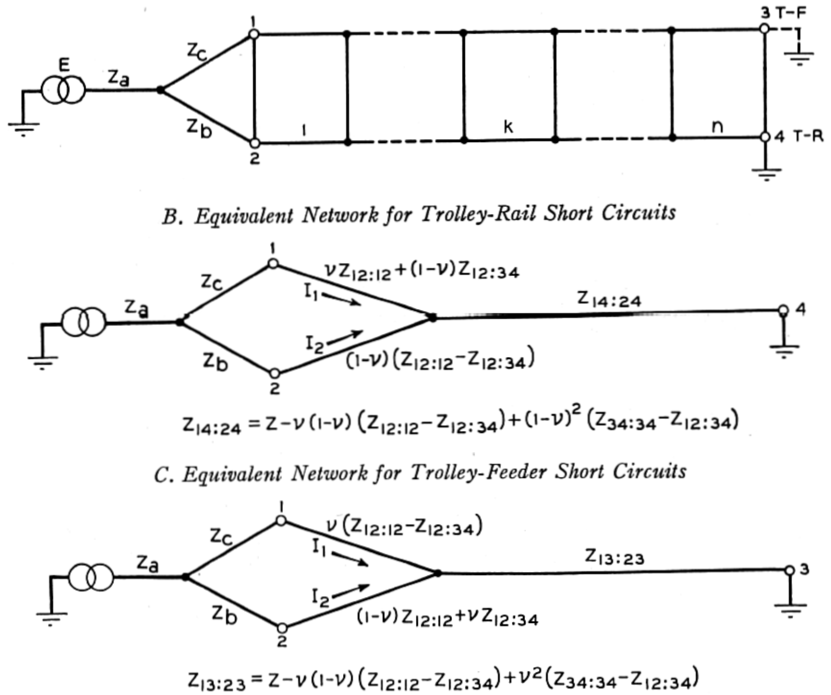


Fig. 4—Equivalent networks for electrified railways; three-wire system, single source, trolley-feeder, trolley-rail and feeder-rail, trolley-rail bases.

Thus the complete set of short-circuit currents (trolley-rail, trolley-feeder and feeder-rail short circuits) may be made from a single determination of the transducer impedances and current distributions on either of the two bases, whenever the theorem is applicable.

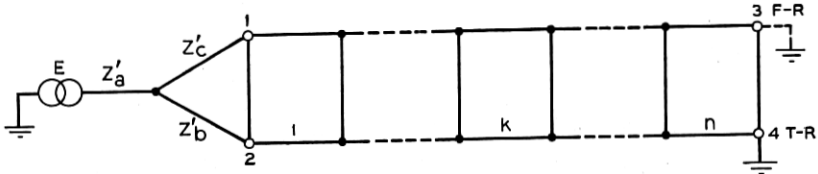
For multiple generator three-wire systems, and for three-wire systems with auxiliary transmission lines, the theorem may be used to represent portions of the network, possibly broken as in the two-wire cases illustrated above, four-terminal representation being necessary in general. The application follows the lines indicated above.

PROOF OF THEOREM

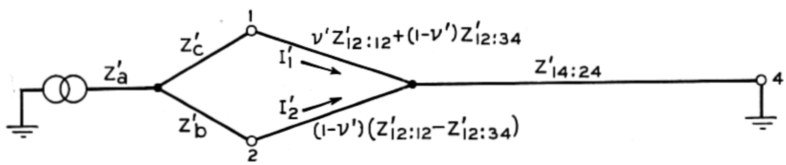
For energization between terminals 1 and 2, the sum of the currents in sides 1 and 2 at any point on the ladder is zero; the current $i_{k:12}$ may be taken as flowing in a mesh made up of the k-section sides and its terminating shunt impedances. For unit current supplied, the

FEEDER-RAIL, TROLLEY-RAIL BASE

A′. Actual Network Connections

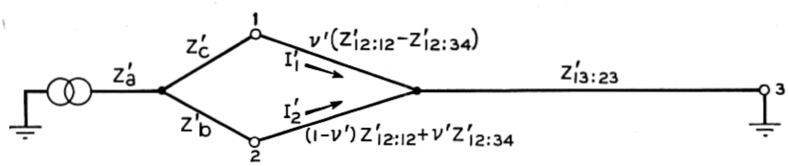


B′. Equivalent Network for Trolley-Rail Short Circuits



$$Z'_{14:24} = Z' - \nu'(1-\nu')(Z'_{12:12} - Z'_{12:34}) + (1-\nu')^2(Z'_{34:34} - Z'_{12:34})$$

C′. Equivalent Network for Feeder-Rail Short Circuits



$$Z'_{13:23} = Z' - \nu'(1-\nu')(Z'_{12:12} - Z'_{12:34}) + (\nu')^2(Z'_{34:34} - Z'_{12:34})$$

Fig. 4—*Continued from page 346.*

driving-point impedance between terminals 1, 2 and the transfer impedance to terminals 3, 4 for the network shown on Fig. 1*A* may be formulated immediately as:

$$\begin{aligned} Z_{12:12} &= (1 + i_{1:12})Z^0, \\ Z_{12:34} &= -\, i_{n:12}Z^{(n)}. \end{aligned} \tag{14}$$

The positive sense for currents $i_{1:12}$ and $i_{n:12}$ is taken as indicated on Fig. 1*A*, namely, in the direction from terminal 2 to terminal 4 on side 2.

From the voltage equation around the loop formed from sides 1 and 2 in their entirety and the terminal shunt impedances, the difference of these impedances may be expressed by:

$$Z_{12:12} - Z_{12:34} = -\sum_{k=1}^{n} (Z_1^{(k)} + Z_2^{(k)} - 2Z_{12}^{(k)})i_{k:12}. \tag{15}$$

The transfer impedances with respect to the side terminals 1, 3 and 2, 4 are formulated as:

$$\begin{aligned} Z_{12:13} &= -\sum_{k=1}^{n} (Z_1^{(k)} - Z_{12}^{(k)}) i_{k:12}, \\ Z_{12:24} &= \sum_{k=1}^{n} (Z_2^{(k)} - Z_{12}^{(k)}) i_{k:12}. \end{aligned} \tag{16}$$

From the condition $[Z_1^{(k)} - Z_{12}^{(k)}][Z_2^{(k)} - Z_{12}^{(k)}]^{-1} = \text{const.}$, a constant ν may be defined such that:

$$\begin{aligned} \nu = \nu_k &= [Z_1^{(k)} - Z_{12}^{(k)}][Z_1^{(k)} + Z_2^{(k)} - 2Z_{12}^{(k)}]^{-1}, \\ 1 - \nu &= [Z_2^{(k)} - Z_{12}^{(k)}][Z_1^{(k)} + Z_2^{(k)} - 2Z_{12}^{(k)}]^{-1}, \end{aligned}$$

and equations (16) may be combined with (15) to give:

$$\begin{aligned} Z_{12:13} &= \nu Z_{12:12} - \nu Z_{12:34}, \\ Z_{12:24} &= -(1 - \nu) Z_{12:12} + (1 - \nu) Z_{12:34}. \end{aligned} \tag{17}$$

The remaining transfer impedances follow by superposition; thus

$$\begin{aligned} Z_{12:14} &= Z_{12:13} + Z_{12:34} = \nu Z_{12:12} + (1 - \nu) Z_{12:34}, \\ Z_{12:23} &= Z_{12:13} - Z_{12:12} = -(1 - \nu) Z_{12:12} - \nu Z_{12:34}. \end{aligned} \tag{18}$$

It may be observed that

$$Z_{12:24} = Z_{12:13} + Z_{12:34} - Z_{12:12}. \tag{19}$$

Similarly

$$\begin{aligned} Z_{34:13} &= -\nu Z_{34:34} + \nu Z_{12:34}, \\ Z_{34:14} &= (1 - \nu) Z_{34:34} + \nu Z_{12:34}, \\ Z_{34:23} &= -\nu Z_{34:34} - (1 - \nu) Z_{12:34}, \\ Z_{34:24} &= (1 - \nu) Z_{34:34} - (1 - \nu) Z_{12:34}. \end{aligned} \tag{20}$$

These impedances, together with $Z_{34:34}$ and $Z_{34:12}$, form a set of 12 impedances of which only five are independent. There are three independent impedances determined by energization at each pair of terminals, including $Z_{12:34}$ and $Z_{34:12}$, which are equal by the reciprocity theorem; one independent set, for example, is $Z_{12:12}$, $Z_{12:13}$, $Z_{12:34}$, $Z_{34:34}$, $Z_{34:13}$. Consequently the network may be completely specified by the addition of a single impedance; for the set illustrated, the impedance required is $Z_{13:13}$.

This impedance may be formulated as:

$$\begin{aligned} Z_{13:13} &= \sum Z_1^{(k)} - \sum (Z_1^{(k)} - Z_{12}^{(k)}) i_{k:13} \\ &= \sum Z_1^{(k)} - \nu \sum (Z_1^{(k)} + Z_2^{(k)} - 2Z_{12}^{(k)}) i_{k:13}, \end{aligned} \tag{21}$$

where the summations extend as above from 1 to n.

Writing the equation around the loop used in deriving equation (15) it is found that:

$$\begin{aligned} \sum (Z_1^{(k)} + Z_2^{(k)} - 2Z_{12}^{(k)}) i_{k:13} \\ &= \sum (Z_1^{(k)} - Z_{12}^{(k)}) - Z_{13:12} + Z_{13:34} \\ &= \sum (Z_1^{(k)} - Z_{12}^{(k)}) - \nu (Z_{12:12} + Z_{34:34} - 2Z_{12:34}), \end{aligned} \tag{22}$$

the last step being made by use of the reciprocity theorem and the formulas already developed. Thus, finally:

$$Z_{13:13} = Z + \nu^2 (Z_{12:12} + Z_{34:34} - 2Z_{12:34}), \tag{23}$$

where

$$Z = \sum_{k=1}^{n} \frac{Z_1^{(k)} Z_2^{(k)} - (Z_{12}^{(k)})^2}{Z_1^{(k)} + Z_2^{(k)} - 2Z_{12}^{(k)}}.$$

As already mentioned, Z is the impedance between short-circuited terminals 1, 2 and 3, 4; this may be verified in a number of ways.

The remaining impedances follow by superposition, which can be carried out formally through the impedance notation in the manner suggested. There are 21 driving-point and transfer impedances between terminals which can be displayed in a triangular array similar to that shown on Fig. 2*B*. The additional measurable impedances at the terminals arise as follows: 36 from short-circuiting two terminals, 4 from short-circuiting three terminals and 3 from short-circuiting terminals in pairs.

Equation (22) may also be written in terms of currents $i_{k:12}$ and $i_{k:34}$, since $Z_{13:12}$ and $Z_{13:34}$ may be expressed in terms of the latter; this suggests the following relation:

$$i_{k:13} = \nu + \nu i_{k:12} - \nu i_{k:34}. \tag{24}$$

The relation is verified by substituting into the mesh equations for currents $i_{k:13}$; the typical equation is as follows:

$$\begin{aligned} - Z^{(k-1)} i_{k-1:13} + [Z_1^{(k)} + Z_2^{(k)} - 2Z_{12}^{(k)} + Z^{(k-1)} + Z^{(k)}] i_{k:13} \\ - Z^{(k)} i_{k+1:13} = Z_1^{(k)} - Z_{12}^{(k)}. \end{aligned}$$

The remaining current relations then follow by superposition, as follows:

$$\begin{aligned} i_{k:14} &= i_{k:13} + i_{k:34} = \nu + \nu i_{k:12} + (1 - \nu) i_{k:34}, \\ i_{k:23} &= i_{k:13} - i_{k:12} = \nu - (1 - \nu) i_{k:12} - \nu i_{k:34}, \\ i_{k:24} &= i_{k:14} - i_{k:12} = \nu - (1 - \nu) i_{k:12} + (1 - \nu) i_{k:34}. \end{aligned} \tag{25}$$

It will be observed that only three of the six currents $i_{k:12}$, $i_{k:13}$, $i_{k:14}$, $i_{k:23}$, $i_{k:24}$ and $i_{k:34}$ are independent; one independent set is $i_{k:12}$, $i_{k:34}$ and $i_{k:13}$. Hence any arbitrary set of currents I_1, I_2, I_3 and I_4 flowing out of the network at the terminals may be resolved into three flows, such as those illustrated in the independent set above, which leads to equation (2).

The first H network may be obtained in the following manner. The value of $Z_{12:13}$, namely, $\nu Z_{12:12} - \nu Z_{12:34}$, in conjunction with the condition $Z_1 + Z_2 = Z_{12:12}$, Z_1 and Z_2 being the impedances of branches to terminals 1 and 2, respectively, suggests the following values of branch self and mutual impedances:

$$\begin{aligned} Z_1 &= \nu Z_{12:12}, \\ Z_2 &= (1 - \nu) Z_{12:12}, \\ Z_{13} &= \nu Z_{12:34}. \end{aligned}$$

The value of Z_{13} is verified by inspection of $Z_{13:34}$ if

$$Z_3 = \nu Z_{34:34}.$$

Similarly, by inspection of $Z_{12:14}$ and $Z_{34:32}$, the values of Z_{24} and Z_4 may be tentatively set at

$$\begin{aligned} Z_{24} &= (1 - \nu) Z_{12:34}, \\ Z_4 &= (1 - \nu) Z_{34:34}. \end{aligned}$$

The impedance of the crossbar, say Z_0, may be found from any of the impedances $Z_{13:13}$, $Z_{14:14}$, $Z_{23:23}$, $Z_{24:24}$; e.g.,

$$\begin{aligned} Z_0 &= Z_{13:13} - (Z_1 + Z_3 - 2Z_{13}) \\ &= Z - \nu(1 - \nu)(Z_{12:12} + Z_{34:34} - 2Z_{12:34}). \end{aligned}$$

But the presence of seven elements, as already mentioned, suggests an arbitrariness which may be put into evidence by adding $\alpha Z_{12:34}$ to Z_1, which entails a similar addition to Z_3 and Z_{13}, and a similar subtraction from Z_2, Z_4 and Z_{24}.

Similar considerations apply to the derivation of the second H network.

The direct impedances may be found, in a well-known manner, by energizing the network between one terminal and the other three short-circuited terminals or by applying a formula due to G. A. Campbell.[8]

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[8] "Direct Capacity Measurement," *Bell System Technical Journal*, I, 1, pp. 18–38 (July, 1922). The formula, given on page 34, requires modification only to the extent of substituting impedances for capacities; for four terminals the direct impedance, D_{ij}, between terminals i and j is given by:

$$D_{ij} = \frac{\Delta}{2\Delta_{ij}},$$

where Δ_{ij} is the co-factor of the element in row i, column j of the determinant:

$$\Delta = \begin{vmatrix} 0 & Z_{12} & Z_{13} & Z_{14} & 1 \\ Z_{12} & 0 & Z_{23} & Z_{24} & 1 \\ Z_{13} & Z_{23} & 0 & Z_{34} & 1 \\ Z_{14} & Z_{24} & Z_{34} & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}.$$

The elements of the determinant are the driving-point impedances between terminals indicated by the subscripts (all other terminals open), namely, $Z_{12:12}$, $Z_{13:13}$, etc., written for brevity Z_{12}, Z_{13}, etc.