Series for the Wave Function of a Radiating Dipole at the Earth's Surface

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In this paper three series expansions are derived for the wave function of a vertical dipole placed at the surface of a plane earth. Two convergent series and one asymptotic series are obtained. A remainder term for the latter series is given which enables one to set an upper limit to the amount of error obtained by stopping at any particular stage in the series.

Introduction

THE wave function above the earth of a vertical dipole placed at the surface of a plane earth is [1]

$$\Pi_1(r, z) = (k_1^2 + k_2^2) \int_0^\infty \frac{J_0(\xi r) e^{-z\sqrt{\xi^2 - k_1^2}} \xi d\xi}{k_2^2 \sqrt{\xi^2 - k_1^2} + k_1^2 \sqrt{\xi^2 - k_2^2}}, \tag{1}$$

where r and z are the horizontal and vertical distances from the dipole. k_1 and k_2 are constants depending upon the electrical properties of the air and ground, respectively.[2] We shall be concerned with the value of this function at the surface of the earth. Setting $z = 0$ gives us an integral for $\Pi_1(r, 0)$ which is the function of r to be investigated here.

Although the electric and magnetic intensities are the properties of an electromagnetic field which have the greatest physical significance, writers on this subject often deal with the wave function because of its simpler form and because in many cases of practical interest it is nearly proportional to the electric intensity. However, the electromagnetic field may be obtained from the wave function by differentiation. If the real parts of $He^{-i\omega t}$ and $Ee^{-i\omega t}$ represent the electric and magnetic intensities the field above the earth produced by the dipole is

$$H_r = H_z = 0, \qquad H_\varphi = -\frac{\partial \Pi_1(r, z)}{\partial r},$$

$$E_r = \frac{ic^2}{\omega} \frac{\partial^2 \Pi_1(r, z)}{\partial r \partial z}, \qquad E_\varphi = 0, \qquad E_z = -\frac{ic^2}{\omega} \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \Pi_1(r, z)}{\partial r} \right).$$

[1] A. Sommerfeld, *Ann. der Physik*, vol. 28, pp. 665–736, No. 4 (1909).
[2] The symbols used here are defined in a list at the end of the paper.

From these expressions it will be observed that if we obtain expressions for $\Pi_1(r, 0)$ we shall be able to compute the field at the earth's surface except for the radial component E_r, which is small compared to E_z.

Statement of Results

The asymptotic expression for $\Pi_1(r, 0)$ is

$$\Pi_1(r, 0) = -\frac{1}{(1-\tau^2)r}\left[e^{ik_1 r} \sum_{n=1}^{N} \frac{n!\,P_n(k_2/s)}{(i\tau s r)^n} + R_{1N} - \tau^2 R_{20} \right], \quad (9)$$

where R_{1N} and R_{20} satisfy the inequalities

$$|R_{1N}| < \left| \frac{(N+1)!\, e^{ik_1 r}\sqrt{\csc\theta}}{[r(k_1 - s)\sin\theta]^{N+1}} \right|, \qquad |R_{20}| < \left| \frac{e^{ik_2 r}}{rk_2 - rs} \right|,$$

$\theta = \pi/2 - \arg(k_1 - s)$ being an angle slightly greater than $\pi/2$.

The convergent series for $\Pi_1(r, 0)$ are [3]

$$\Pi_1(r, 0) = \frac{1}{(1-\tau^2)r}\sqrt{\frac{\pi\tau}{2}}\left[e^{ik_1 r}\sum_{n=0}^{\infty}(-is\tau r)^n P_{-1/2}^{1/2-n}(k_1/s) - \tau e^{ik_2 r}\sum_{n=0}^{\infty}\left(\frac{sr}{i\tau}\right)^n P_{-1/2}^{1/2-n}(k_2/s) \right] \quad (14)$$

and

$$\Pi_1(r, 0) = \frac{1}{(1-\tau^2)r}\left[\sum_{n=0}^{\infty}\frac{(ik_1 r)^n}{n!} F(1, -n/2; 1/2; s^2/k_2^2) - \tau^2 \sum_{n=0}^{\infty}\frac{(ik_2 r)^n}{n!} F(1, -n/2; 1/2; s^2/k_1^2) \right]. \quad (19)$$

The quantities τ and s are defined by $\tau = k_1/k_2$ and $1/s^2 = 1/k_1^2 + 1/k_2^2$, and the numbers on the right are the equation numbers in the text. W. H. Wise [4] has obtained series which are equivalent to those appearing in (9) and (14).

Procedure

The results given here depend upon a transformation of the integral obtained by setting $z = 0$ in equation (1). This integral can be expressed in the following way as has been shown by B. van der Pol: [5]

$$\Pi_1(r, 0) = -\frac{\tau}{1-\tau^2}\int_{k_1/s}^{k_2/s} \frac{e^{isrw}}{r}\, d(w^2 - 1)^{-1/2}, \quad (2)$$

[3] The Legendre functions are discussed by E. W. Hobson, "Th. of Spherical and Ellipsoidal Harmonics." Hypergeometric functions are discussed in Chap. XIV, "Modern Analysis," by Whittaker and Watson.

[4] W. H. Wise, *Proc. I.R.E.*, vol. 19, pp. 1684–1689, September 1931.

[5] *Jahrbuch der drahtlosen Telegraphie Zeitschr. f. Hochfrequenz Techn.*, *37* (1931), p. 152.

which becomes, after integration by parts,

$$= +\frac{\tau}{1-\tau^2}\left\{\left[\frac{-e^{isrw}}{r\sqrt{w^2-1}}\right]_{k_1/s}^{k_2/s} + is\int_{k_1/s}^{k_2/s}\frac{e^{isrw}dw}{\sqrt{w^2-1}}\right\}. \qquad (3)$$

The path of integration is the straight line in the complex w plane joining the points k_1/s and k_2/s. Arg $(w-1)$ and arg $(w+1)$ are taken to be zero at the point this contour crosses the real axis. The Argand diagram for a typical case is shown in Fig. 1. From the definitions of k_1, k_2, and s it follows that $|s| < |k_1| < |k_2|$, and $0 = \arg k_1 < \arg s < \arg k_2 < \pi/4$.

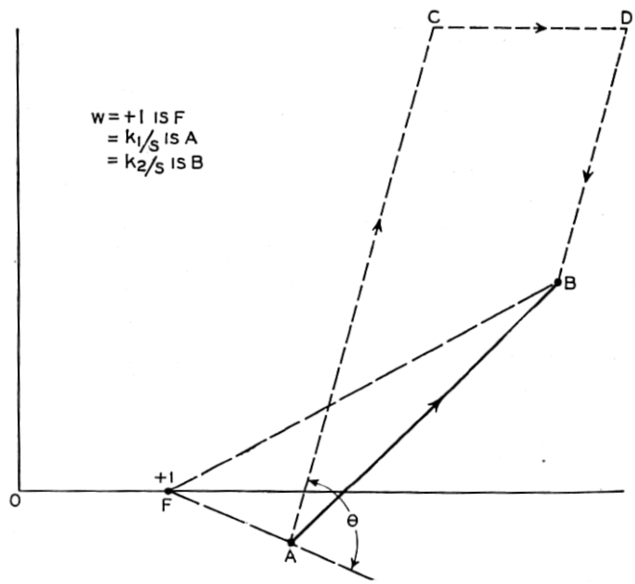

Fig. 1—Paths of integration in the w plane.

Asymptotic Expansion

To obtain an asymptotic expansion for $\Pi_1(r, 0)$ we deform the linear path joining A and B into the path $ACDB$ as is shown in Fig. 1. The lines AC and BD are both inclined to the real axis at the angle arg (is^*) where s^* is the conjugate of s. This is the direction in which the exponential term e^{isrw} decreases most rapidly since along it the variable part of the exponent is real and negative.[6] The section CD may be displaced to infinity where its contribution to the value of the integral becomes zero because of this exponential decrease.

[6] To show this for the line AC we set $w = k_1/s + is^*u$. As w goes from A to C u is real and increases from zero. The exponent then becomes $isrw = ik_1r - |s|^2ru$ since $ss^* = |s|^2$.

The integral $\Pi_1(r, 0)$ is then composed of two components consisting of the integrals along AC and DB, respectively, and we may write

$$\Pi_1(r, 0) = -\frac{\tau}{(1-\tau^2)r}[I(k_1) - I(k_2)], \tag{4}$$

where

$$I(k) = \int_{k/s}^{\infty is*} e^{isrw} d(w^2-1)^{-1/2}. \tag{5}$$

We integrate (5) by parts N times and find

$$I(k) = \left[- e^{isrw} \sum_{n=1}^{N} \frac{(-)^n}{(isr)^n} \frac{d^n}{dw^n}(w^2-1)^{-1/2} \right]_{k/s}^{\infty is*} + (-)^N \int_{k/s}^{\infty is*} \frac{e^{isrw}}{(isr)^N} \frac{d^{N+1}}{dw^{N+1}} (w^2-1)^{-1/2} dw.$$

The derivatives may be expressed in terms of Legendre polynomials by means of the relation

$$(-)^n \frac{d^n}{dw^n}(w^2-1)^{-1/2} = n!(w^2-1)^{-n/2-1/2} P_n\left(\frac{w}{\sqrt{w^2-1}}\right).$$

When the limits in the integrated portion are inserted and the definition of s used we see that

$$I(k_1) = \frac{k_2 e^{ik_1 r}}{k_1} \sum_{n=1}^{N} \left(\frac{k_2}{ik_1 sr}\right)^n n! P_n(k_2/s) + R_{1N}\frac{k_2}{k_1}, \tag{6}$$

where

$$R_{1N} = -\frac{k_1(N+1)!}{k_2(isr)^N} \int_{k_1/s}^{\infty is*} P_{N+1}\left[\frac{w}{\sqrt{w^2-1}}\right] \frac{e^{isrw}}{(w^2-1)^{N/2+1}} dw. \tag{7}$$

An inequality for R_{1N} may be obtained by using [3]

$$|P_{N+1}(t)| \leqslant |t + \sqrt{t^2-1}|^{N+1}$$

which holds for all values of t in the t plane cut from -1 to $+1$, if $\arg \sqrt{t^2-1} = 0$ when t is real and greater than $+1$. For then the absolute value of the Legendre polynomial in the integrand is seen to be less than $|(w+1)/(w-1)|^{(N+1)/2}$ when $R(w) > 0$, and R_{1N} may be compared with an integral having $|e^{isrw}|$ and powers of the factors $|w+1|$ and $|w-1|$ in the integrand. On the path AC we

[3] E. W. Hobson, loc. cit., p. 60.

have $|w - 1| \geq |(k_1/s - 1) \sin \theta|$ where

$$\theta = \arg is^* - \arg\left(\frac{k_1}{s} - 1\right) = \frac{\pi}{2} - \arg (k_1 - s) > \frac{\pi}{2}\cdot$$

Similarly we have $|w + 1| \geq |k_1/s + 1|$. These inequalities enable us to deal with the integral of $|e^{isrw}|$ which may be integrated to show that

$$|R_{1N}| < \left|\frac{(N+1)!e^{ik_1r}\sqrt{\csc \theta}}{[r(k_1 - s)\sin \theta]^{N+1}}\right|\cdot \tag{8}$$

By interchanging k_1 and k_2 in (6) and (7) we obtain expressions for $I(k_2)$ and R_{2N}. An inequality for $|R_{2N}|$ is obtained from (8) by setting $\theta = \pi/2$ and interchanging k_1 and k_2. By combining these expressions in accordance with equation (4) we obtain an asymptotic expansion for $\Pi_1(r, 0)$.

In general, $I(k_2)$ is negligible in comparison with $I(k_1)$ because k_2 has a positive imaginary part which causes e^{ik_2r} to decrease rapidly. Since $I(k_2) = R_{20}$, R_{20} being the remainder after zero terms, we may obtain an inequality for $I(k_2)$ by setting $N = 0$, $\theta = \pi/2$, and interchanging k_1 and k_2 in (8). Then from (4) we have the result

$$\Pi_1(r, 0) = -\frac{1}{(1-\tau^2)r}\left[e^{ik_1r}\sum_{n=1}^{N}\frac{n!P_n(k_2/s)}{(i\tau sr)^n} + R_{1N} - \tau^2 R_{20}\right], \tag{9}$$

where R_{1N} satisfies the inequality (8) and $|R_{20}| < |e^{ik_2r}/(rk_2 - rs)|$.

Series for $\Pi_1(r, 0)$ in Ascending Powers of r

Put

$$K(k_1) = e^{ik_1r} - i\frac{k_1sr}{k_2}\int_1^{k_1/s}\frac{e^{isrw}dw}{\sqrt{w^2 - 1}} \tag{10}$$

and define $K(k_2)$ as being obtained from (10) by interchanging k_1 and k_2. By referring to equation (3) we see that I may be written in the form

$$\Pi_1(r, 0) = \frac{1}{(1-\tau^2)r}[K(k_1) - \tau^2 K(k_2)]. \tag{11}$$

We write

$$K(k_1) = e^{ik_1r}\left[1 - \frac{ik_1sr}{k_2}\int_1^{k_1/s}\frac{e^{irs(w-k_1/s)}dw}{\sqrt{w^2-1}}\right]$$

$$= e^{ik_1r}\left[1 - \frac{ik_1sr}{k_2}\sum_{n=1}^{\infty}\frac{(irs)^{n-1}}{(n-1)!}\int_1^{k_1/s}\frac{(w - k_1/s)^{n-1}dw}{\sqrt{w^2-1}}\right], \tag{12}$$

the infinite series being uniformly convergent.

From Hobson's contour integral definition [3] of $P_n^m(t)$ it can be shown that if $R(m) < 1/2$

$$P^m_{-1/2}(t) = \sqrt{\frac{2}{\pi}}\,\frac{e^{-\pi i(m+1/2)}(t^2 - 1)^{m/2}}{\Gamma(1/2 - m)}\int_1^t \frac{(w - t)^{-m-1/2}}{\sqrt{w^2 - 1}}\,dw,$$

where arg $(w - 1) = \varphi$, arg $(w - t) = -\pi + \varphi$, where φ is the angle measured counter-clockwise from the positive direction of the real axis to the line directed from $w = 1$ to $w = t$. Setting $m = +1/2 - n$ where n is a positive integer we obtain

$$\int_1^t \frac{(w - t)^{n-1}}{\sqrt{w^2 - 1}}\,dw = (-)^{n-1}(n - 1)!\sqrt{\frac{\pi}{2}}\,(t^2 - 1)^{n/2-1/4}P^{1/2-n}_{-1/2}(t).$$

Thus Equation (12) becomes

$$\begin{aligned} K(k_1) &= e^{ik_1r}\left[1 + \sum_{n=1}^{\infty}\left(\frac{srk_1}{ik_2}\right)^n\cdot\sqrt{\frac{\pi k_1}{2k_2}}\,P^{1/2-n}_{-1/2}(k_1/s)\right] \\ &= e^{ik_1r}\sqrt{\frac{\pi k_1}{2k_2}}\sum_{n=0}^{\infty}\left(\frac{srk_1}{ik_2}\right)^n P^{1/2-n}_{-1/2}(k_1/s), \end{aligned} \qquad (13)$$

where in passing from the first to the second line we have set $n = 0$ in [3]

$$P^{1/2-n}_{-1/2}(t) = \frac{1}{\Gamma(1/2 + n)}\left(\frac{t - 1}{t + 1}\right)^{n/2-1/4} F\left(1/2,\, 1/2;\, n + 1/2;\, \frac{1 - t}{2}\right),$$

and have summed the resulting series to show that $P^{1/2}_{-1/2}(k_1/s) = \sqrt{2/\pi\tau}$. The function $K(k_2)$ may be obtained from (13) by interchanging k_1 and k_2.

Combining (13) and (11), and using $\tau = k_1/k_2$ gives the convergent series for $\Pi_1(r, 0)$ given in the statement of results as equation (14).

Another Power Series for I

Here we obtain an expression for I somewhat similar to the one obtained in the previous section. The first step is to deform the contour joining the points A and B ($w = k_1/s$ and $w = k_2/s$). The deformation is carried out in two steps shown in Figs. 2a and 2b, respectively.

In Fig. 2(a) the contour joining A to B has been pulled around the point $+1$ and looped over itself. The point H is destined to move

[3] E. W. Hobson, loc. cit., p. 188.

over to B and G is to move over to A. This deformation of the contour does not alter the value of the integral as long as we pay attention to the arguments of $w - 1$ and $w + 1$. In Fig. 2(*b*) the deformation is almost completed; all that remains is for G to coincide with A and H to coincide with B.

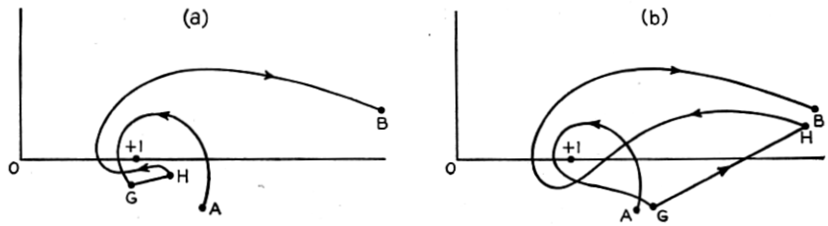

Fig. 2—Deformation of contour in w plane.

Using this deformation of the contour we may write equation (2) as follows:

$$\Pi_1(r, 0) = +\frac{\tau}{1 - \tau^2}\left[\int_A^G + \int_G^H + \int_H^B \frac{e^{isrw}w\,dw}{r(w^2 - 1)^{3/2}}\right]$$
$$= \frac{\tau}{(1 - \tau^2)r}\left[\int_{k_1/s}^{(1+)} - \int_{k_1/s}^{k_2/s} - \int_{k_2/s}^{(1+)} \frac{e^{isrw}w\,dw}{(w^2 - 1)^{3/2}}\right]$$

with the understanding that $\arg w - 1$ and $\arg w + 1$ have their principal values at the beginning of each integration. Upon referring to (2) we see that the middle integral is $- \Pi_1(r, 0)$ and hence

$$\Pi_1(r, 0) = \frac{\tau}{2r(1 - \tau^2)}\left[L(k_1) - L(k_2)\right], \tag{15}$$

where

$$L(k_1) = \int_{k_1/s}^{(1+)} \frac{e^{isrw}w\,dw}{(w^2 - 1)^{3/2}} = \sum_{n=0}^{\infty} \frac{(isr)^n}{n!} \int_{k_1/s}^{(1+)} \frac{w^{n+1}dw}{(w^2 - 1)^{3/2}} \tag{16}$$

and $L(k_2)$ is obtained from $L(k_1)$ by interchanging k_1 and k_2.

Let $w^2 - 1 = \tau^2(1 - t)$, or $sw = k_1\sqrt{1 - ts^2/k_2^2}$, then

$$\int_{k_1/s}^{(1+)} \frac{w^{n+1}dw}{(w^2 - 1)^{3/2}} = -\frac{k_2}{2k_1}\left(\frac{k_1}{s}\right)^n \int_0^{(1+)} \frac{[(1 - (s^2t/k_2^2)]^{n/2}}{(1 - t)^{3/2}}\,dt$$
$$= 2\frac{k_2}{k_1}\left(\frac{k_1}{s}\right)^n F(1, -n/2; 1/2; s^2/k_2^2), \tag{17}$$

where it is understood that at the initial point of the contour arg $(1 - t) = 0$, arg $(1 - (s^2/k_2^2)t) = 0$. This may be verified by expanding the numerator of the integrand and using

$$\int_0^{(1+)} t^m(1 - t)^\nu dt = (1 - e^{2\pi i\nu})\frac{\Gamma(m+1)\Gamma(\nu+1)}{\Gamma(m+\nu+2)},$$

where m is a positive integer or zero, with $\nu = -3/2$.

Expression (16) now becomes

$$L(k_1) = 2\frac{k_2}{k_1}\sum_{n=0}^{\infty}\frac{(ik_1r)^n}{n!}F(1, -n/2; 1/2; s^2/k_2^2) \tag{18}$$

and the series converges for all finite values of r since the series integrated termwise in Equation (16) is uniformly convergent.

We obtain the series for $\Pi_1(r, 0)$ given in statement of results as equation (19) by putting (18) and the corresponding expression for $L(k_2)$ in equation (15).

Notation

The following symbols are used. C.G.S. electromagnetic units are used throughout the paper.

c = velocity of light, 3×10^{10} cm./sec.

$F(a, b; c; x)$ = The hypergeometric function

$$1 + \frac{ab}{1!c}x + \frac{a(a+1)b(b+1)}{2!c(c+1)}x^2 + \cdots.$$

$J_0(\xi r)$ = Bessel function of the first kind, zero order.

$k_1 = \omega/c$.

$k_2 = \sqrt{\epsilon\omega^2 + i4\pi\sigma\omega}$. The real and imaginary parts are positive.

$P_n(t)$, $P_{-1/2}^{1/2-n}(t)$ = Legendre's polynomial, and associated Legendre's function of the first kind.

R_{1N}, R_{20} = Remainder terms in asymptotic series.

r = horizontal distance of representative point from dipole.

$s = k_1k_2/\sqrt{k_1^2 + k_2^2}$ or $1/s^2 = 1/k_1^2 + 1/k_2^2$. The real and imaginary parts of s are positive.

s^* = The complex conjugate of s.

t = time in the introduction, otherwise a complex variable.

w = complex variable.

z = height of representative point above ground.

ϵ = dielectric constant of the ground in e.m.u. The dielectric constant of air in e.m.u. is $1/c^2$. If the dielectric constant in e.s.u. is ϵ', then $\epsilon = \epsilon'/c^2$. The dielectric constant of air in e.s.u. is 1.

$\Pi_1(r, z)$ = Wave function for $z \geqq 0$ for a vertical unit dipole centered at the interface between air and ground. The wave function for a unit dipole wholly in air is obtained by multiplying the wave function given here by $2/(1 + \tau^2)$. By a unit dipole is meant the system obtained by letting the length l of a conductor approach zero while the current in the conductor approaches infinity in such a way that Il = unity, where the current equals the real part of $Ie^{-i\omega t}$ and does not vary with position along the conductor.

$\Pi_1(r, 0)$ = Value of wave function at earth's surface.

σ = conductivity of the ground in e.m.u. If the conductivity is σ' mhos per meter cube then $\sigma = 10^{-11}\sigma'$.

$\tau = k_1/k_2$.

ω = angular velocity, radians/sec.