New Results in the Calculation of Modulation Products

By W. R. BENNETT

A new method of computing modulation products by means of multiple Fourier series is described. The method is used to obtain for the problem of modulation of a two-frequency wave by a rectifier a solution which is considerably simpler than any hitherto known.

THE problem of computing modulation products has long been recognized as being of fundamental importance in communication engineering. Heretofore certain quite fundamental modulation problems have been attacked by methods which are difficult to justify from the standpoint of mathematical rigor and some of the solutions obtained have been in the form of complicated infinite series that are not easy to use in practical computations. In this paper these problems are solved by means of a new method which is mathematically sound and which yields results in a form well suited for purposes of computation.

The analysis here given applies specifically to the case of two frequencies applied to a modulator of the "cut off" type; i.e., a modulator which operates by virtue of its being insensitive to input changes throughout a particular range of values. A simple rectifying characteristic forms a convenient basis of approximation for study of such modulators, and hence we consider in detail methods of calculating modulation in rectifiers when two frequencies are applied. Applications to certain other types of modulation problems and to the case of more than two applied frequencies are discussed briefly at the close.

Half Wave Linear Rectifier—Two Applied Frequencies

We shall define a half wave linear rectifier as a device which delivers no output when the applied voltage is negative and delivers an output wave proportional to the applied voltage when the applied voltage is positive. We may take the constant of proportionality as unity since its only effect is to multiply the entire solution by a constant. Assume the input voltage $e(t)$ to be specified by

$$e(t) = P \cos (pt + \theta_p) + Q \cos (qt + \theta_q). \tag{1}$$

The output wave will then consist of the positive lobes of the above function with the negative lobes replaced by zero intervals. It is convenient to represent the amplitude ratio Q/P by k, and without loss of generality to take

$$P > 0 \text{ and } 0 \leq k \leq 1. \tag{2}$$

The problem we now consider is the resolution of the output wave into sinusoidal waves, a complete solution requiring the determination of the frequencies present, their amplitudes, and their phase relations.

The method of solution used employs the auxiliary function of two independent variables $f(x, y)$ defined by

$$\left.\begin{aligned} f(x, y) &= P(\cos x + k \cos y), & \cos x + k \cos y \geq 0, \\ &= 0, & \cos x + k \cos y < 0. \end{aligned}\right\} \tag{3}$$

It is clear that the function $f(x, y)$ may be represented by a surface which does not pass below the xy-plane and which coincides with the xy-plane throughout certain regions which are bounded by the multi-branched curve,

$$\cos x + k \cos y = 0. \tag{4}$$

If either x or y is increased or decreased by any multiple of 2π, the value of $f(x, y)$ is unchanged. Hence $f(x, y)$ is a periodic function of x and y, and if its value is known for every point in the rectangle bounded by $y = \pm\pi$, $x = \pm\pi$ say, the value of the function may be determined for any point in the entire xy-plane.

From the above considerations we are led to investigate the expansion of $f(x, y)$ in a double Fourier series in x and y. We may readily verify that the function satisfies any one of several sets of sufficient conditions [1] to make such an expansion valid. We may write the expansion thus:

$$f(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [A_{\pm mn} \cos(mx \pm ny) + B_{\pm mn} \sin(mx \pm ny)], \tag{5}$$

with the summation to be extended over both the upper and lower of the ambiguous signs except when m or n is zero, in which case one value only is taken (it is immaterial which one); when m and n are both zero, we divide the coefficient A_{00} by two in order that all the A-coefficients may be expressed by the same formula. Determining the coefficients by the usual method of multiplying both sides of (5) by the factor the coefficient of which is to be found and integrating both sides throughout the rectangle bounded by $x = \pm\pi$, $y = \pm\pi$, we obtain:

[1] Hobson, "Theory of Functions of a Real Variable," Vol. 2, p. 710.

$$\left.\begin{aligned} A_{\pm mn} &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x, y) \cos(mx \pm ny)dydx, \\ B_{\pm mn} &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x, y) \sin(mx \pm ny)dydx. \end{aligned}\right\} \tag{6}$$

We now return to our original problem of representing the positive lobes of a two-frequency wave as a sum of sinusoidal components. We may apply the double Fourier series expansion of $f(x, y)$, which must hold for all values of x and y, to the special case in which x and y are linear functions of the time. If we let

$$\left.\begin{aligned} x &= pt + \theta_p, \\ y &= qt + \theta_q, \end{aligned}\right\} \tag{7}$$

the function $f(x, y)$ represents the rectified two-frequency wave as a function of time. The values of x and y which are used lie on the straight line,

$$y = \frac{q}{p}x + \theta_q - \frac{q}{p}\theta_p, \tag{8}$$

which is obtained by eliminating t from (7). A representation of $f(x, y)$ valid for the entire xy-plane must of course hold for values of x and y on this straight line. Hence we may substitute the values of x and y given by (7) directly into the double Fourier series (5), and the result will evidently be an expression for the rectifier output in terms of discrete frequencies of the type $(mp \pm nq)/2\pi$. The phase angle of the typical component is $m\theta_p \pm n\theta_q$ and the amplitude is expressed by (6).

The solution is thereby reduced to the evaluation of the definite double integrals of (6). Three different methods of reducing these integrals have been investigated, and it appears that each has certain peculiar advantages and points of interest. We shall consider them separately.

I. Straightforward Geometric Method

In this method, which yields remarkably simple results in a direct manner, we determine the boundaries of the region throughout which $f(x, y)$ vanishes and substitute appropriate limits in the integrals to exclude this region from the area of integration. When this exclusion has been accomplished, $f(x, y)$ may be replaced in the integral by $\cos x + k \cos y$. The boundary between zero and non-zero values of $f(x, y)$ is the curve (4), which has two branches crossing the rectangle over which the integration is performed. The non-zero values of $f(x, y)$ lie in the shaded region of Fig. 1. From the symmetry of the region about the x and y axes we deduce at once that the sine coefficients, $B_{\pm mn}$, must vanish and that the cosine coefficients, $A_{\pm mn}$, may be obtained by integrating throughout one quadrant only and multi-

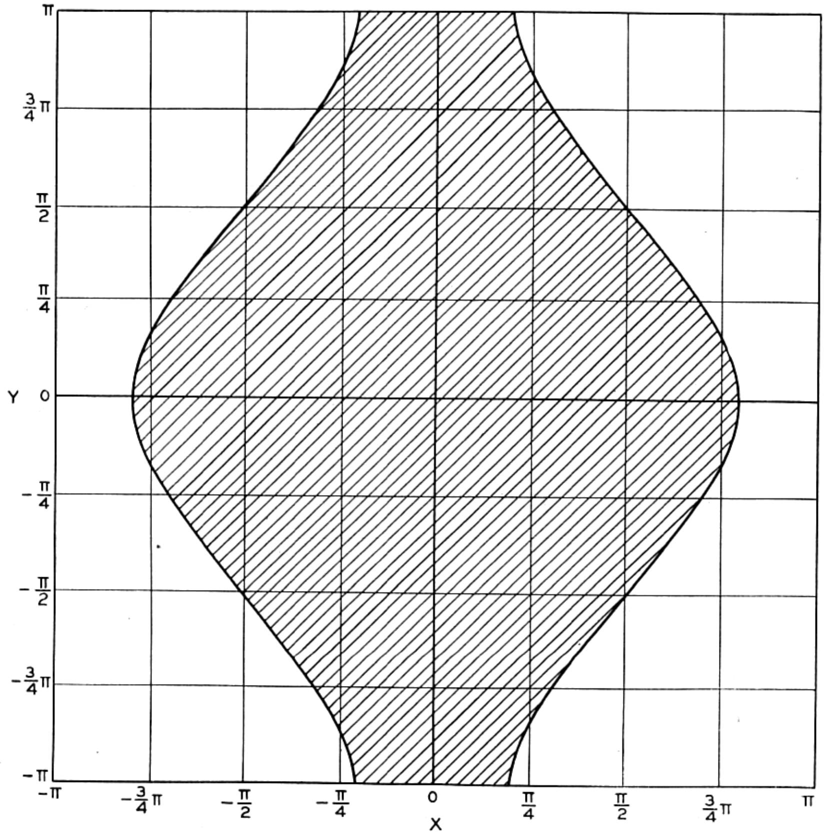

Fig. 1—Region of integration for the determination of the coefficients in the double Fourier series expansion of $f(x, y)$.

plying by four. We therefore obtain, on substitution of the proper limits,

$$A_{mn} = A_{\pm mn} = \frac{2P}{\pi^2} \int_0^{\pi} \cos nydy \times \int_0^{\text{arc cos } (-k \cos y)} (\cos x + k \cos y) \cos mxdx \quad (9)$$

This expression gives the amplitude of the typical component of frequency $(mp \pm nq)/2\pi$. The remaining steps are concerned merely with the calculation of the integral (9) for particular values of m and n.

It will suffice to work through one example in detail and give the results in tabular form for the other products up to the fourth order. The second order side frequencies, $(p \pm q)/2\pi$, will be taken as a typical case.

By direct substitution

$$A_{11} = \frac{2P}{\pi^2} \int_0^{\pi} \cos y dy \int_0^{\arccos(-k\cos y)} (\cos x + k \cos y) \cos x dx. \quad (10)$$

Performing the inner integration and substituting the limits for x, we obtain:

$$A_{11} = \frac{P}{\pi^2} \int_0^{\pi} \cos y [\arccos(-k\cos y) + k\cos y \sqrt{1 - k^2\cos^2 y}] dy. \quad (11)$$

Considering separately the integral,

$$\int_0^{\pi} \cos y \arccos(-k\cos y) dy,$$

integrate once by parts, letting

$$u = \arccos(-k\cos y),$$
$$dv = \cos y dy.$$

The result is, after combining with the remainder of the integral for A_{11},

$$A_{11} = \frac{kP}{\pi^2} \int_0^{\pi} \frac{\sin^2 y + \cos^2 y(1 - k^2\cos^2 y)}{\sqrt{1 - k^2\cos^2 y}} dy. \quad (12)$$

Now substituting

$$\cos y = z,$$

we obtain

$$A_{11} = \frac{2kP}{\pi^2} \int_0^1 \frac{1 - k^2 z^4}{\sqrt{(1 - z^2)(1 - k^2 z^2)}} dz. \quad (13)$$

This is a standard elliptic form.[2] It is convenient here to let

[2] It may be remarked that a large number of the integrals required in the evaluation of the coefficients are listed by D. Bierens de Haan, "Nouvelles Tables d'Integrales Definies." See in particular Tables 8 and 12, pages 34 and 39.

$$Z_m = \int_0^1 \frac{z^m}{\sqrt{(1 - z^2)(1 - k^2z^2)}}\, dz. \tag{14}$$

By differentiating the expression $z^{m-3}\sqrt{(1 - z^2)(1 - k^2z^2)}$, we may easily derive the useful recurrence formula:

$$Z_m = \frac{(m - 2)(1 + k^2)Z_{m-2} - (m - 3)Z_{m-4}}{(m - 1)k^2}. \tag{15}$$

We may now calculate the value of Z_m for even values of m in terms of Z_0 and Z_2. Z_0 is a complete elliptic integral of the first kind which we shall designate as usual by K; i.e.,

$$Z_0 = K = \int_0^1 \frac{dz}{\sqrt{(1 - z^2)(1 - k^2z^2)}} = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta. \tag{16}$$

Furthermore from the identity:

$$\frac{z^2}{\sqrt{(1 - z^2)(1 - k^2z^2)}} = \frac{1}{k^2}\left[\frac{1}{\sqrt{(1 - z^2)(1 - k^2z^2)}} - \sqrt{\frac{1 - k^2z^2}{1 - z^2}}\right], \tag{17}$$

we have

$$Z_2 = \frac{1}{k^2}(K - E), \tag{18}$$

where E is a complete elliptic integral of the second kind defined by

$$E = \int_0^1 \sqrt{\frac{1 - k^2z^2}{1 - z^2}}\, dz = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta. \tag{19}$$

Now making use of (15), we calculate Z_4 in terms of Z_2 and Z_0 and get finally:

$$Z_4 = \frac{(2 + k^2)K - 2(1 + k^2)E}{3k^4}. \tag{20}$$

We can then evaluate (13) in terms of K and E. The result is

$$A_{11} = \frac{4P}{3\pi^2 k}\left[(1 + k^2)E - (1 - k^2)K\right]. \tag{21}$$

The process of evaluating the other coefficients is quite similar. Results are listed in Table I.

Convenient tables of K and E may be found in Peirce's Short Table of Integrals (page 121), Byerly's Integral Calculus, and the Jahnke und Emde tables. For a very extensive set of tables, see Legendre's

TABLE I

TWO FREQUENCY MODULATION PRODUCTS

Applied Wave = $P[\cos (pt + \theta p) + k \cos (qt + \theta q)]$

Order of Product	Symbol for Coefficient	$2\pi \times$ Frequency of Product	Amplitude of Product: Half Wave Linear Rectifier	Amplitude of Product: Half Wave Square Law Rectifier
0	$\frac{1}{2} A_{00}$	0	$\frac{2P}{\pi^2}[2E - (1 - k^2)K]$	$\frac{1 + k^2}{4} P^2$
1	A_{10}	p	$\frac{P}{2}$	$\frac{8P^2}{9\pi^2}[(7 + k^2)E - 4(1 - k^2)K]$
	A_{01}	q	$\frac{kP}{2}$	$\frac{8P^2}{9\pi^2 k}[(1 + 7k^2)E - (1 + 3k^2)(1 - k^2)K]$
2	A_{20}	$2p$	$\frac{4P}{\pi^2}[2(2 - k^2)E - (1 - k^2)K]$	$\frac{P^2}{4}$
	A_{11}	$p \pm q$	$\frac{4P}{3\pi^2 k}[(1 + k^2)E - (1 - k^2)K]$	$\frac{k}{2} P^2$
	A_{02}	$2q$	$\frac{4P}{9\pi^2 k^2}[2(2k^2 - 1)E + (2 - 3k^2)(1 - k^2)K]$	$\frac{k^2}{4} P^2$
3	A_{30}	$3p$	0	$\frac{8P^2}{225\pi^2}[(23 - 23k^2 + 8k^4)E - 4(2 - k^2)(1 - k^2)K]$
	A_{21}	$2p \pm q$	0	$\frac{8P^2}{45\pi^2 k}[(3 + 7k^2 - 2k^4)E - (3 + k^2)(1 - k^2)K]$
	A_{12}	$p \pm 2q$	0	$\frac{8P^2}{45\pi^2 k^2}[(3k^4 + 7k^2 - 2)E + 2(1 - 3k^2)(1 - k^2)K]$
	A_{03}	$3q$	0	$\frac{8P^2}{225\pi^2 k^3}[(8 - 23k^2 + 23k^4)E - (8 - 19k^2 + 15k^4)(1 - k^2)K]$

Phase Angle of Product $(mp \pm nq) = m\theta p \pm n\theta q$

$$K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^1 \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}; \qquad E = \int_0^{\frac{2}{\pi}} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta = \int_0^1 \sqrt{\frac{1 - k^2 z^2}{1 - z^2}}\, dz$$

Traité des Fonctions Elliptiques. Numerical calculation of the coefficients making use of these tables and the formulae listed in Table I is a quite simple process. Curves of the coefficients as functions of k have been calculated in this way and are plotted in Fig. 2.

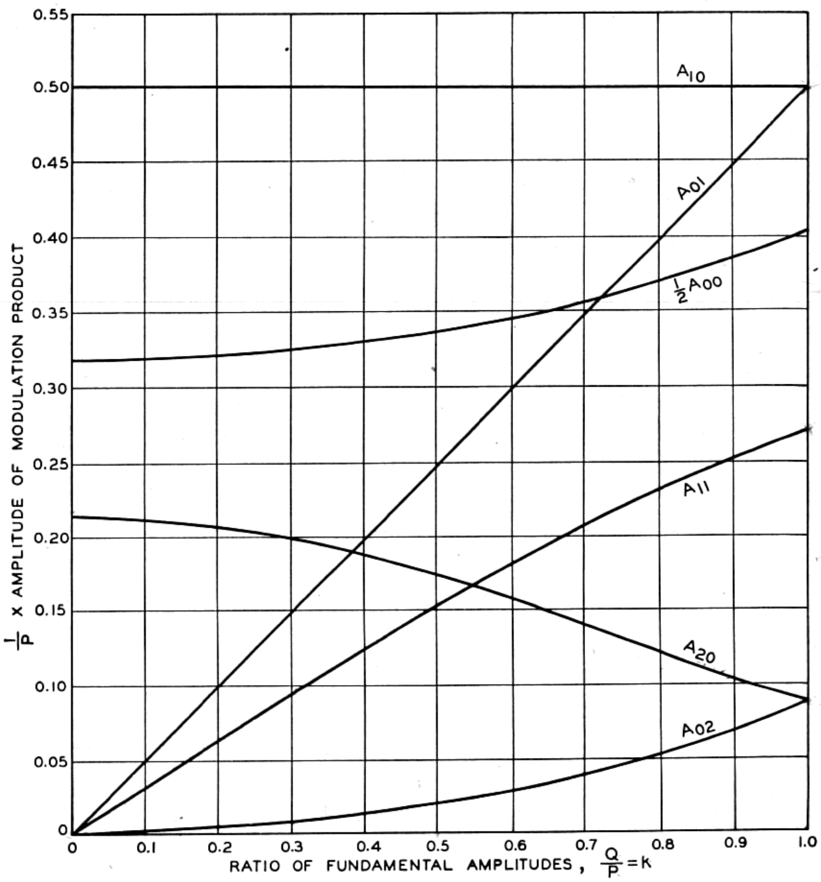

Fig. 2—Curves showing amplitudes of modulation products in output of half wave linear rectifier when input wave consists of two frequencies.

It is perhaps worth noting that the special case of equal fundamental amplitudes ($P = Q$ or $k = 1$) yields the simple result,

$$A_{mn} = \frac{8(-)^m P}{[(m+n)^2 - 1][(m-n)^2 - 1]\pi^2}, \tag{22}$$

where $m + n$ is even. When $m + n$ is odd and greater than one, A_{mn} is zero.

II. Fourier Series Method

The second method is of interest because it obtains the same results as the only previously known solution,[3] which is in terms of infinite series involving Bessel functions. The fact that the results agree is a check on the validity of certain doubtful rearrangements of multiple series necessary in the process by which these results were originally obtained. Furthermore by comparison with the corresponding results of the first method we can sum the infinite series in terms of complete elliptic integrals; a number of interesting mathematical theorems are thus proved, which have been made the basis of a paper by the author in the December, 1932 issue of the *Bulletin of the American Mathematical Society.*

By expanding the function:

$$\left.\begin{aligned}\phi(u) &= -\frac{u}{2}, \qquad & -c \leq u \leq 0 \\ &= \frac{u}{2}, & 0 \leq u \leq c\end{aligned}\right\} \tag{23}$$

in a Fourier series in u, we may verify that:

$$\left.\begin{aligned}\frac{c}{4}+\frac{u}{2}-\frac{2c}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}\cos\frac{(2r-1)\pi u}{c} &= 0, \qquad & -c \leq u \leq 0 \\ &= u, & 0 \leq u \leq c.\end{aligned}\right\} \tag{24}$$

If we let $u = P\cos x + Q\cos y$, the left hand member of (24) is equal to $f(x, y)$ provided $|P| + |Q| < c$. With this restriction on c, we may substitute the resulting expression for $f(x, y)$ in the integrand of (6), and no change in the limits of integration are required. Term by term integration of the series may be justified without difficulty, and making use of well known definite integrals, we obtain finally:

$$A_{mn} = \frac{4c}{\pi^2}(-)^{\frac{m+n+2}{2}}\sum_{r=1}^{\infty}\frac{J_m\left(\frac{2r-1}{c}\pi P\right)J_n\left(\frac{2r-1}{c}\pi Q\right)}{(2r-1)^2}, \tag{25}$$

where $m + n$ is an even integer. When $m + n = 0$, the extra term $c/4$ must be added. When $m + n$ is odd and greater than one, the value of A_{mn} is zero; when $m + n = 1$, the values are $A_{10} = P/2$, $A_{01} = Q/2$.

Peterson and Keith obtained the above result [3] by substituting

[3] Peterson and Keith, "Grid Current Modulation," *Bell System Technical Journal,* Vol. 7, pp. 138–9, January, 1928.

$u = P\cos x + Q\cos y$ in the left hand member of (24), applying Jacobi's expansions in series of Bessel coefficients, and rearranging the resulting triple series. It appears that it is much more difficult to justify the series rearrangement than term by term integration. From the results obtained by the first method it follows that the series in (25), which might be termed a generalized Schlömilch series,[4] is summable in terms of elliptic integrals.

III. Trigonometric Integral Method

Following a suggestion of Mr. S. O. Rice, we may make use of the following relation:

$$\left.\begin{aligned} \frac{u}{2} + \frac{u}{\pi}\int_0^\infty \frac{\sin u\lambda}{\lambda}\,d\lambda &= u, \qquad u \geq 0 \\ &= 0, \qquad u \leq 0. \end{aligned}\right\} \tag{26}$$

Evidently if we substitute $u = P\cos x + Q\cos y$, the left hand member of (26) represents the function $f(x, y)$ and may be substituted in the integrand of (6) without change in the limits. Interchange of the order of integration may then be justified without difficulty and the following result is obtained in terms of a special case of the integral of Weber and Schafheitlin:

$$A_{mn} = \frac{2}{\pi}(-)^{\frac{m+n+2}{2}}\int_0^\infty \frac{J_m(P\lambda)J_n(Q\lambda)}{\lambda^2}\,d\lambda, \tag{27}$$

where $m + n$ is even and greater than zero. When $m + n = 0$, the above integral should be replaced by an infinite contour integral taken along the real axis except for an indentation to avoid the origin and with all other quantities remaining the same except for a division by two. For all even order modulation products it may now be deduced[5] that:

$$A_{mn} = \frac{(-)^{\frac{m+n}{2}+1}\Gamma\left(\frac{m+n-1}{2}\right)k^n P}{2\pi\Gamma(n+1)\Gamma\left(\frac{m-n+3}{2}\right)} \times F\left(\frac{m+n-1}{2}, \quad \frac{n-m-1}{2}; \quad n+1; \quad k^2\right). \tag{28}$$

The case of $m + n = 0$ requires a special investigation, which shows that (28) holds for this case also.

[4] Cf. Watson, "Theory of Bessel Functions," Chapter XIX.
[5] Watson, "Theory of Bessel Functions," p. 401.

The hypergeometric function in (28) may always be expressed in terms of K and E by successive applications of recurrence formulae and use of the known relations:

$$\left.\begin{aligned} K &= \frac{\pi}{2} F\left(\frac{1}{2}, \quad \frac{1}{2}; \quad 1; \quad k^2\right), \\ E &= \frac{\pi}{2} F\left(-\frac{1}{2}, \quad \frac{1}{2}; \quad 1; \quad k^2\right). \end{aligned}\right\} \tag{29}$$

By means of the hypergeometric recurrence formulæ we may also show that

$$A_{mn} = -\frac{2[(m-1)k^2 + n - 1]A_{m-1,\,n-1} + (m+n-5)kA_{m-2,\,n-2}}{(m+n+1)k} \tag{30}$$

when $m + n$ is even. A discussion of the hypergeometric function and a derivation of (30) are given in the appendix.

From (30), we can compute successively all even order modulation products starting with say A_{00} and A_{11} known. If negative subscripts occur in applying the formula, they may be replaced by positive subscripts without changing the validity of the results; this is proved in the appendix.

Half Wave Square Law Rectifier—Two Applied Frequencies

The solution for two frequencies applied to a square law rectifier, or in fact to any rectifier operating on an integer power law, can be obtained in a manner quite similar to that used in solving the linear rectifier. In the case of a square law rectifier, we have to represent the function

$$\left.\begin{aligned} f(x, y) &= P^2 (\cos x + k \cos y)^2, &\quad \cos x + k \cos y \geq 0 \\ &= 0, &\quad \cos x + k \cos y < 0. \end{aligned}\right\} \tag{31}$$

Going through the same steps with this function that we did with that of (3), we find that the amplitudes of the modulation products can be expressed in terms of K and E as in the case of the linear rectifier; the results are listed in Table I. A set of curves is plotted in Fig. 3.

We may also show that

$$A_{mn} = (-)^{\frac{m+n+1}{2}} \frac{8c^2}{\pi^3} \sum_{r=1}^{\infty} \frac{J_m\left(\frac{2r-1}{c}\pi P\right) J_n\left(\frac{2r-1}{c}\pi Q\right)}{(2r-1)^3} \tag{32}$$

when $m + n$ is odd and greater than one and $c \geq |P| + |Q|$. For

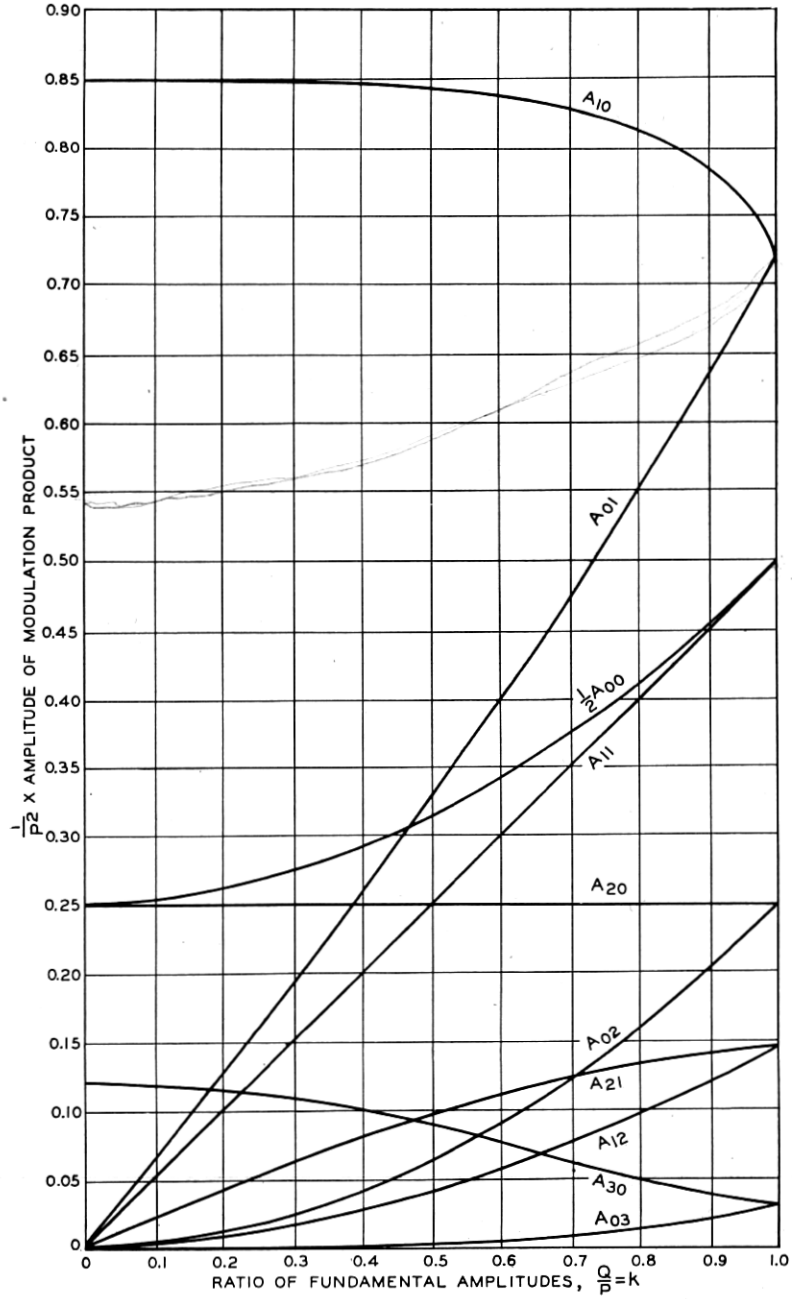

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Fig. 3—Curves showing amplitudes of modulation products in output of half wave square law rectifier when input wave consists of two frequencies.

A_{10} and A_{01} we must add $cP/2$ and $cQ/2$ respectively. The value of A_{mn} is zero for $m + n$ even and greater than two; the other even order products are listed in Table I. Another form of the result for odd order products is

$$A_{mn} = \frac{4}{\pi}(-)^{\frac{m+n+1}{2}} \int_0^\infty \frac{J_m(P\lambda)J_n(Q\lambda)}{\lambda^3}\, d\lambda \tag{33}$$

or

$$A_{mn} = (-)^{\frac{m+n+1}{2}} \frac{k^n P^2 \Gamma\left(\frac{m+n-2}{2}\right)}{2\pi\Gamma(n+1)\Gamma\left(\frac{m-n+4}{2}\right)} \times F\left(\frac{m+n-2}{2}, \quad \frac{n-m-2}{2}; \quad n+1; \quad k^2\right). \tag{34}$$

A three term recurrence formula for odd order products is:

$$A_{mn} = -\frac{2[(m-1)k+n-1]A_{m-1,\, n-1}+(m+n-6)kA_{m-2,\, n-2}}{(m+n+2)k}. \tag{35}$$

When $P = Q$, and $m + n$ is odd,

$$A_{mn} = \frac{64\,(-)^{m+1}P^2}{(m^2-n^2)[(m+n)^2-4][(m-n)^2-4]\pi^2}. \tag{36}$$

Other Applications and Results

The solution for any full wave rectifier can be obtained from the solution for the corresponding half wave rectifier. Thus we may easily show that the output of a full wave linear rectifier contains neither of the fundamentals and that the amplitudes of all other modulation products are twice as large as the corresponding amplitudes in the output of a half wave linear rectifier. It is also evident that by superposing the solutions for the linear and square law rectifiers we can obtain the solution for a quadratic law rectifier having an output equal to $a_1e(t) + a_2[e(t)]^2$ when $e(t)$ is positive and no output when $e(t)$ is negative. Biased rectifiers, peak choppers, and saturating devices can be solved by the same methods used above, the solution of course becoming more complicated for the more complicated kinds of characteristics. Nor is the method restricted to "cut off" type modulation. Curvature type modulators can be treated in the same way and in many cases solution by the above method is simpler than by the usual power series expansion. The method also appears to have promise in the solution of magnetic modulation problems, where the effect of hysteresis must be considered.

When three frequencies are applied, a triple Fourier series is required, and in the general case of n frequencies, a Fourier series in n variables would be used. The work becomes more complicated as the number of frequencies increases, but there is no theoretical limitation.

In conclusion the writer wishes to express his appreciation of the valuable advice of Messrs. T. C. Fry and L. A. MacColl on the technical features of the paper.

Appendix

The hypergeometric function $F(\alpha, \beta; \gamma; z)$ may be defined by the power series:

$$F(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha\beta}{1!\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)} z^2 + \cdots.$$

When any one of the three quantities α, β, γ is increased or decreased by unity a new hypergeometric function is formed which is said to be contiguous to the first. Gauss listed fifteen linear relations which connect $F(\alpha, \beta; \gamma; z)$ with pairs of its contiguous functions. In deriving the recurrence formula for A_{mn} we require difference relations between functions which are not contiguous, but the required relations may be obtained from those listed by Gauss by a process of substitution and elimination.

We shall find it convenient to designate $F(\alpha, \beta; \gamma; z)$ by F, $F(\alpha + 1, \beta; \gamma; z)$ by $F_{\alpha+}$, $F(\alpha + 1, \beta; \gamma - 1; z)$ by $F_{\alpha+\gamma-}$, etc., and to let

$$\alpha = \frac{m+n-3}{2}; \qquad \beta = \frac{n-m-1}{2}; \qquad \gamma = n, \qquad z = k^2.$$

In this notation, Equation (27) becomes:

$$A_{mn} = \frac{(-)^{\frac{m+n}{2}+1}\Gamma\left(\frac{m+n-1}{2}\right)k^n P}{2\pi\Gamma(n+1)\Gamma\left(\frac{m-n+3}{2}\right)} F_{\alpha+\gamma+}.$$

The corresponding expressions for $A_{m-1,\, n-1}$ and $A_{m-2,\, n-2}$ are by direct substitution:

$$A_{m-1,\, n-1} = \frac{(-)^{\frac{m+n}{2}}\Gamma\left(\frac{m+n-3}{2}\right)k^{n-1} P}{2\pi\Gamma(n)\Gamma\left(\frac{m-n+3}{2}\right)} F,$$

$$A_{m-2,\, n-2} = \frac{(-)^{\frac{m+n}{2}-1}\Gamma\left(\frac{m+n-5}{2}\right)k^{n-2} P}{2\pi\Gamma(n-1)\Gamma\left(\frac{m-n+3}{2}\right)} F_{\alpha-\gamma-}.$$

Thus a recurrence relation expressing A_{mn} in terms of $A_{m-1,\, n-1}$ and $A_{m-2,\, n-2}$ evidently requires a relation between $F_{\alpha+\gamma+}$, F, and $F_{\alpha-\gamma-}$.

Referring to Gauss' tables,[6] we find

$$(\gamma - \alpha - 1)F + \alpha F_{\alpha+} - (\gamma - 1)F_{\gamma-} = 0,$$
$$\gamma(1 - z)F - \gamma F_{\alpha-} + (\gamma - \beta)zF_{\alpha+} = 0.$$

From the second of these two equations we form two more equations by substituting $\alpha + 1$ for α in one case and $\gamma - 1$ for γ in the other, giving

$$\gamma(1 - z)F_{\alpha+} - \gamma F + (\gamma - \beta)zF_{\alpha+\gamma+} = 0$$
$$(\gamma - 1)(1 - z)F_{\gamma-} - (\gamma - 1)F_{\alpha-\gamma-} + (\gamma - 1 - \beta)zF = 0.$$

Now eliminating $F_{\alpha+}$ and $F_{\gamma-}$ from the first, third, and fourth of the equations, we obtain

$$\alpha(\beta - \gamma)zF_{\alpha+\gamma+} + \gamma[\gamma - 1 + (\alpha - \beta)z]F - \gamma(\gamma - 1)F_{\alpha-\gamma-} = 0,$$

which is the relation desired. Substituting the value of $F_{\alpha+\gamma+}$ in terms of A_{mn}, F in terms of $A_{m-1,\, n-1}$, and $F_{\alpha-\gamma-}$ in terms of $A_{m-2,\, n-2}$ gives the recurrence formula of Equation (30).

In using (30) we may find, as for instance in calculating A_{m0}, A_{m1}, A_{0n}, A_{1n}, that the right hand member involves coefficients with negative subscripts. A simple rule for treating such cases may be demonstrated as follows. We first note that if we replace m by $-m$ in (28) the value of the right hand member is unchanged.[7] Hence since (30) is derivable directly from (28), we conclude that correct results are obtained from (30) if we adopt the convention,

$$A_{-m,\, n} = A_{mn}.$$

The case of n negative is a little more difficult because if n is a negative integer in (28), an indeterminate form results. However, making use of the result just obtained on the interchangeability of sign of the subscripts, m, $m - 1$, $m - 2$ in (30), we can demonstrate a

[6] Gauss, Werke, Bd. III, page 130. The equations used here are numbered (5) and (8) by Gauss.

[7] If we express $(-)^{m/2}\Gamma\left(\frac{m+n-1}{2}\right)\Big/\Gamma\left(\frac{m-n+3}{2}\right)$ in terms of $(-)^{-m/2}\Gamma\left(\frac{-m+n-1}{2}\right)\Big/\Gamma\left(\frac{-m-n+3}{2}\right)$ by successive applications of the recurrence formula for the gamma function, we find the two quantities are equivalent. Changing the sign of m in the hypergeometric function merely interchanges α and β, and hence does not change the value of the function.

similar rule for the subscripts $n, n - 1, n - 2$, valid when (30) is used. For example, by direct application of (30), we deduce that

$$A_{2-m,\, n+2} = -\frac{2[(1 - m)k^2 + n + 1]A_{1-m,\, n+1} + (n - m - 1)kA_{-m,\, n}}{(n - m + 5)k}.$$

Now since it is known that we may replace the subscripts $2 - m$, $1 - m$, and $-m$ by $m - 2$, $m - 1$, and m respectively, we may show that

$$A_{mn} = -\frac{2[(m - 1)k^2 - n - 1]A_{m-1,\, n+1} + (m - n + 5)kA_{m-2,\, n+2}}{(m - n + 1)k},$$

which is exactly equivalent to the relation we get if we replace n by $-n$ throughout in (30) and then substitute $A_{m,\, -n} = A_{mn}$, $A_{m,\, -n-1} = A_{m,\, n+1}$, $A_{m,\, -n-2} = A_{m,\, n+2}$.

It may be remarked that it would be incorrect to base a proof of interchangeability of sign of subscripts on (27) because the equivalence of (27) and (28) has not been demonstrated for a sufficient range of values of m and n.