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An Algebraic Theory of Relational Databases

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In this paper we present a theory of relational database systems based on the partition lattice, which represents a new mathematical approach to the structure of relational database systems. A partition lattice can be defined for any given relation. This partition lattice is shown to be a meet-morphic image of the Boolean algebra of subsets of the attribute set. The partial ordering in the lattice is proved to be equivalent to the concept of functional dependency, and thus Armstrong's axioms for functional dependencies are proved. We solve the problem of finding the list of all keys by seeking the prime implicants of the Boolean function associated with the principal ideals generated by the attributes. We demonstrate the properties of the Boyce-Codd Normal Form (BCNF), and give a modified algorithm for synthesizing an information-lossless BCNF based on the principal filter. The necessary and sufficient conditions for multivalued dependency (MVD) are given in terms of a lattice equation, and the inference rules of MVD are proved. The necessary and sufficient conditions for join dependency (JD) are given; consequently, we can prove the known result that acyclic join dependency (AJD) is equivalent to a set of MVDs. The concept of data independence is introduced, and is extended to conditional independence and mutual independence. We established this algebraic theory of relational databases in the same spirit that the theory of probability was constructed. We present a comparison that demonstrates the similarities.

I. INTRODUCTION

The existing theory of relational databases is based on Codd's relational model of data.[1,2] This relational database theory can be considered to be the study of data dependencies (or independencies).

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The theory was initiated by Codd with the introduction of the concept of functional dependency; Codd observed that this concept can be used to design better, normalized, database schemes. The advantage of normalized database schemes is that they remove the possibility of updating anomalies caused by undesirable data dependencies.[2–5]

In the existing theory of logical database design, functional dependencies are input constraints that must always hold in the relation.[6] In the present paper, however, we take a different approach. We assume that for a particular database designer, there exists a (finite) universal relation $R[\Omega]$ for a given set of attributes Ω, such that any relation T on Ω is a subset of $R[\Omega]$. Furthermore, each subset X of Ω corresponds to an equivalence relation (partition) on the set of tuples of $R[\Omega]$. That is, if two tuples in $R[\Omega]$ have the same X value, then they are in the same equivalence class. With this approach, the concept of functional dependency becomes equivalent to the refinement partial ordering of the partition lattice. The partitions on the (finite) set of tuples of the universal relation $R[\Omega]$ can then be considered as the fundamental constraints, from which the functional dependencies (partial ordering) can be derived. Consequently, with our approach, the functional dependencies are inherent properties of the universal relation $R[\Omega]$. The input constraints of course must be consistent with the inherent properties within the database.

Another kind of data dependency, proposed by Fagin[7] and Zaniolo,[8] is the multivalued dependency, which includes functional dependency as a special case. Multivalued dependency is the necessary and sufficient condition for the lossless-join decomposition of a relation into two subrelations, such that the original relation can be regenerated by the (natural) join operation.[7–11] Using the partition lattice we propose, we can formulate multivalued dependency as a lattice equation (see Section VI). We show that the axioms for functional dependencies[12] and the inference rules of multivalued dependencies[13] can all be proved as theorems within the framework of partition lattice theory. We show how the concept of join dependency[10,11,14] is connected to multivalued dependency. We give the necessary and sufficient condition for join dependency and, consequently, we can prove the known result that the acyclic join dependency is equivalent to a set of MVDs.[15,16] We also introduce the concept of data independence, and the extension to conditional independence and mutual independence of sets of attributes.

The problem of listing all the keys of a relation is solved by using the concept of principal ideals in lattice theory. One form of a relation having desirable properties is the Boyce-Codd Normal Form (BCNF); we show that the concept of the principal filter (dual ideal) can be used to produce information-lossless Boyce-Codd Normal Forms.

Both the theoretical foundation and the practical application of the existing theory of relational databases appear to be fragmented. This paper shows that all the diverse kinds of data dependencies can be formulated within the lattice theory, which has the important advantage of unifying the theory of relational databases into a coordinated whole. Because of this, it would appear that future work in relational databases should be conducted using lattice theory as the basic framework.

The establishment of this algebraic theory of relational databases is done in the same spirit as the construction of the theory of probability, although probability theory is of course unrelated to database theory. We are convinced that the lattice theory could play a role in the theory of relational databases similar to the role that measure theory plays in the theory of probability.[17]

The basic notion of relational databases is defined in Section II, and the partition lattice of the relation is introduced in Section III. The problem of listing all keys is solved in Section IV, where the Boolean functions associated with the principal ideals are defined. The properties of the Boyce-Codd Normal Form are studied in Section V, where we present a modified algorithm for synthesizing information-lossless BCNFs based on the principal filters. Section VI is devoted to the proof of equivalence between multivalued dependency and a lattice equation. Section VII discusses join dependency and acyclic join dependency. Finally, in Section VIII we outline a possible direction for future research, as well as a comparison that shows the similarities between probability theory and the algebraic theory of relational databases. In Appendix A we list the laws of lattice theory for reference. The proofs of the axioms for functional and multivalued dependencies are listed in Appendix B.

Unless otherwise stated, we refer to the universal relation as simply "the relation" in the remainder of this paper.

II. RELATIONS

An *attribute* is a symbol taken from a finite set $\Omega = \{A_1, A_2, \cdots, A_n\}$. For each attribute A there is a set of possible values called its *domain*, denoted DOM(A). We will use capital letters from the beginning of the alphabet ($A, B, \cdots$) for single attributes, and capital letters from the end of the alphabet ($X, Y, \cdots$) for sets of attributes. For a set of attributes $X \subseteq \Omega$, an X-value x is an assignment of values to the attributes of $X = \{A_{i_1}, A_{2_2}, \cdots, a_{i_k}\}$ from their respective domains. The notation XY will be used to represent the union of two arbitrary sets of attributes $X, Y \subseteq \Omega$.

A *relation* R on the set of attributes $\Omega = \{A_1, \cdots, A_n\}$ is a subset of the Cartesian product $\mathrm{DOM}(A_1) \times \cdots \times \mathrm{DOM}(A_n)$. The elements (rows) of R are called *tuples*. A relation R on $\{A_1, \cdots, A_n\}$ will be denoted by $R[A_1 \cdots A_n]$. Similarly, if R is defined on the union of sets $(X_1, X_2, \cdots, X_m)$, then the notation $R[X_1 \cdots X_m]$ will be used. A relation can be visualized as a table whose columns are labeled with attributes and whose rows depict tuples. The ordering of the rows and columns is immaterial. The *cardinal* of R is the total number of tuples in R and is denoted by $|R|$.

Let t be a tuple in $R[\Omega]$. For $X \subseteq \Omega$, $t[X]$ denotes the tuple that contains the components of t corresponding to the attributes of X. The *projection* of R on X, denoted by $R[X]$, is defined as follows:

$$R[X] = \{t[X] \mid t \in R\}.$$

Similarly, the *conditional projection* of R on X by a Y-value y, where $Y \subseteq \Omega$, is defined as follows:

$$R_y[X] = \{t[X] \mid t \in R \quad \text{and} \quad t[Y] = y\}.$$

Let $R[XZ]$ and $S[XZ]$ be relations where X, Y, and Z, are disjoint sets of attributes. The *join* (natural join) of R and S, denoted by $R \,|\!\times\!|\, S$, is the relation $T[XYZ]$ whose attributes are XYZ, and is defined as follows:

$$\begin{aligned} T[XYZ] &= R[XZ] \,|\!\times\!|\, S[YZ] \\ &= \{(x, y, z) \mid (x, z) \in R \quad \text{and} \quad (y, z) \in S\}. \end{aligned}$$

The join can also be defined as the union of a collection of Cartesian products:

$$\begin{aligned} T[XYZ] &= R[XZ] \,|\!\times\!|\, S[YZ] \\ &= \{R_z[X] \times S_z[Y] \times (z) \mid (z) \in R[Z] \cap S[Z]\}. \end{aligned}$$

Let R be a relation on the set of attributes Ω. We may have two sets of attributes $X, Y, \subseteq \Omega$, such that for any two tuples $t_1, t_2 \in R$, $t_1[X] = t_2[X]$ implies $t_1[Y] = t_2[Y]$. We say then that X *functionally determines* Y in R, and denote this fact by $X \rightarrow Y$. A functional dependency (FD) $X \rightarrow Y$ is *trivial*, meaning it holds in all relations, if $Y \subseteq X$. Note that FDs enjoy the projectivity and inverse projectivity properties.[3,4] For sets $X, Y \subseteq \Omega' \subseteq \Omega$, the FD: $X \rightarrow Y$ is valid in $R[\Omega]$ iff it is valid in $R[\Omega']$.

We say that a set of relations $\{R[\Omega_1], \cdots, R[\Omega_n]\}$ has the *information-lossless join* property if $\Omega = \Omega_1 \cdots \Omega_n$ and

$$R[\Omega] = R[\Omega_1] \,|\!\times\!|\, \cdots \,|\!\times\!|\, R[\Omega_n].$$

If the set $\{R[\Omega_1], \cdots, R[\Omega_n]\}$ does not have this property, we say that it has a *lossy join*.[14] An important property of functional dependency[10] is that if FD: $X \rightarrow Y$ is valid in $R[\Omega]$ then

$$R[\Omega] = R[(\Omega - Y)X] \,|\times|\, R[XY].$$

This property will be discussed in more detail in Section VI.

III. THE RELATION LATTICE

If S is a nonempty set, then a subset ρ of $S \times S$ is called a *binary relation* on S. The *product* of two binary relations ρ, $\rho' \subseteq S \times S$ is defined as:

$$\rho \circ \rho' = \{(a, b\} \in S \times S \mid \exists c \in S \text{ such that } (a, c) \in \rho, (c, b) \in \rho'\}.$$

We say that a relation ρ on S is *reflexive* if $(a, a) \in \rho$ for every a in S; that ρ is *symmetric* if $\rho^{-1} = \rho$, i.e., if

$$(\forall a, b \in S), \qquad (a, b) \in \rho \quad \text{implies} \quad (b, a) \in \rho;$$

and that ρ is *transitive* if $\rho \circ \rho \subseteq \rho$, *i.e.*, if

$$(\forall a, b, c \in S), \qquad (a, b) \in \rho \quad \text{and} \quad (b, c) \in \rho \quad \text{imply} \quad (a, c) \in \rho.$$

A binary relation is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

A family $\pi = \{B_i | \, i \in I\}$ of subsets, called *blocks* of S, is said to form a *partition* of S if the following conditions hold:

1. Each B_i is nonempty
2. For all $i \neq j$ in I, $B_i \cap B_j = \varnothing$
3. $\cup\{B_i | \, i \in I\} = S$.

The two apparently different notions of "equivalence relation" and "partition" are interchangeable: Let ρ be an equivalence relation on a set S. Then the family $a_\rho = \{b \mid (a, b) \in \rho\}$ of subsets of S is a partition of S. Conversely, if $\pi = \{B_i | \, i \in I\}$ is a partition of S, then the relation $\{(a, b) \mid (\exists i \in I), (a, b) \in B_i\}$ is an equivalence relation on S.

If ρ is an equivalence relation (partition) on S, we shall sometimes write $a\rho b$ as an alternative to $(a, b) \in \rho$. The sets a_ρ that form the associated partition of the equivalence relation are called *ρ-classes*. The set of ρ-classes is called the *quotient set* of S by ρ and is denoted by S/ρ.

A binary relation $\leqq$ on the set S is a *partial ordering* of S if and only if $\leqq$ is reflexive; antisymmetric, i.e., if

$$(\forall a, b \in S), \qquad a \leqq b \quad \text{and} \quad b \leqq a \quad \text{imply} \quad a = b;$$

and transitive. A set S with a partial ordering $\leqq$ is called a *partially ordered set* (poset) and it is denoted by the pair $(S, \leqq)$.

Let $(S, \leqq)$ be a poset and let T be a subset of S. Then, $a \in S$ is the *greatest lower bound* (g.l.b.) of T iff

1. $(\forall \mathrm{t} \in \mathrm{T}), \mathrm{a} \leqq \mathrm{t}$.

2. $(\forall t \in T)$, $a' \leqq t$ implies $a' \leqq a$.

Similarly, $a \in S$ is the *least upper bound* (l.u.b.) of T iff

1. $(\forall t \in T)$, $t \leqq a$.
2. $(\forall t \in T)$, $t \leqq a'$ implies $a \leqq a'$.

A *lattice* is a poset in which any two elements a and b have a g.l.b., called a *meet* and denoted by $a \cdot b$, and a l.u.b., called a *join* and denoted by $a + b$. We sometimes write the meet $a \cdot b$ as ab if no confusion is created. The properties of the meet and join operations of a lattice[18] are listed in Appendix A.

Let the set of all partitions π_i on S be denoted by $\prod(S)$, and define the partial ordering on $\prod(S)$ as follows:

$$\text{If } (\forall a, b \in S), \quad a\pi_1 b \text{ implies } a\pi_2 b, \quad \text{then} \quad \pi_1 \leqq \pi_2.$$

The poset $(\prod(S), \leqq)$ is seen to be a lattice $(\prod(S), \cdot, +)$ with a universal lower bound $\mathbf{0} = \{B_i | i \in I\}$ such that every block B_i is a singleton, and an universal upper bound $\mathbf{1} = \{S\}$. To specify a particular partition, we list the elements, and distinguish blocks with bars and semicolons. For example, if $S = \{1, 2, 3, 4, 5\}$ and partition π on S has blocks $\{1, 3, 4\}$, $\{2, 5\}$, then we write $\pi = \{\overline{1, 3, 4}; \overline{2, 5}\}$. The meet and join of any two partitions $\pi_1, \pi_2 \in \prod(S)$ can be determined as follows:

1. $(\forall a, b \in S)$, $a\pi_1 \cdot \pi_2 b$ iff $a\pi_1 b$ and $a\pi_2 b$.
2. $(\forall a, b \in S)$, $a\pi_1 + \pi_2 b$ iff $\exists n \in N$ and $c_0, \cdots, c_n \in S$ such that $a = c_0$, $b = c_n$ and $c_i\pi_1 c_{i+1}$ or $c_i\pi_2 c_{i+1}$ for each i, $0 \leqslant i \leqslant n - 1$.

A complemented distributive lattice is called a *Boolean algebra* (see Appendix A). The set of all subsets of S, called the *power set* of S, and denoted by $\mathbf{2}^S$, with the partial ordering $(\forall S_1, S_2 \in \mathbf{2}^S)$, $S_1 \leqq S_2$ iff $S_1 \supseteq S_2$, is a Boolean algebra $(\mathbf{2}^S, \cdot, +, ^-)$ with the universal bounds $\mathbf{0} = S$ and $\mathbf{1} = \varnothing$. The *dual* of a poset is the poset with the converse partial ordering on the same elements. The Boolean algebra defined above is the dual of the conventional Boolean algebra of the power set. The operations of meet and join are defined by

1. Meet (g.l.b.) $S_1 \cdot S_2 = S_1 \cup S_2$,
2. Join (l.u.b.) $S_1 + S_2 = S_1 \cap S_2$,

and the complement of $S_1 \in \mathbf{2}^S$ is $\overline{S_1} = S - S_1$.

Let ψ: $L \rightarrow M$ be a function from a lattice L into a lattice M. We say ψ is a *meet-morphism* if

$$(\forall a, b \in L), \qquad \psi(a \cdot b) = \psi(a) \cdot \psi(b),$$

and ψ is a *join-morphism* if

$$(\forall a, b \in L), \qquad \psi(a + b) = \psi(a) + \psi(b).$$

Meet-morphisms and join-morphisms are both *isotone* (order-preserving); i.e.,

$$(\forall a, b \in L), \qquad a \leqq b \quad \text{implies} \quad \psi(a) \leqq \psi(b),$$

and any order-preserving one-to-one mapping with an inverse is an *isomorphism.*[18]

Let R be a relation on the set of attributes Ω. The set of all subsets of Ω, denoted $\mathbf{2}^{\Omega}$, with the partial ordering defined by set-containment, is a Boolean algebra $(\mathbf{2}^{\Omega}, \cdot, +, ^{-})$,[18] where the meet, join, and complement operations are defined as above. For every $X \in \mathbf{2}^{\Omega}$, there is an equivalence relation (partition) on the set of tuples in $R[\Omega]$ defined as follows:

Definition 1: Let R be a relation on the set of attributes Ω. Each subset of Ω is associated with a partition of the set of tuples of R. We define the function θ: $\mathbf{2}^{\Omega} \rightarrow \prod(R[\Omega])$, which we call the *partition function* (associated with $R[\Omega]$), by

$$\theta: X \rightarrow \theta(X) = \{(t_1, t_2) \in R[\Omega] \times R[\Omega] \mid t_1[X] = t_2[X]\}. \quad \blacksquare$$

In general, the image set $Im(\theta)$ of θ is not a sublattice of $\prod(R[\Omega])$. Since $\pi_1, \pi_2 \in Im(\theta)$ implies $\pi_1 \cdot \pi_2 \in Im(\theta)$, $Im(\theta)$ is a complete lattice in its own right,[20] and it will be called the *relation lattice* of $R[\Omega]$, and denoted by $L(R[\Omega])$. Note that there are no duplicated tuples in $R[\Omega]$, so that $\theta(\Omega) = \mathbf{0}$. Since the tuples cannot be "differentiated" by the empty set of attributes, we define $\theta(\varnothing) = \mathbf{1}$. The universal bounds of $L(R[\Omega])$ are the same as those in $\prod[R)\Omega])$. We immediately recognize the concept of functional dependency to be equivalent to the refinement partial ordering of the partitions.

Lemma 1: Let $R[\Omega]$ be a relation on the set of attributes Ω, and let θ: $\mathbf{2}^{\Omega} \rightarrow \prod(R[\Omega])$ be the partition function associated with $R[\Omega]$, defined above. Then

$$X \rightarrow Y \quad \text{iff} \quad \theta(X) \leqq \theta(Y). \quad \blacksquare$$

An immediate consequence of the above lemma is that the projection $R[X]$ of $R[\Omega]$ on X is simply the quotient of $R[\Omega]$ by $\theta(X)$, i.e., $R[X] = R[\Omega]/\theta(X)$. Thus each tuple in $R[X]$ corresponds to a $\theta(x)$-class in $R[\Omega]/\theta(X)$ and it takes the X-value only. Note that $\theta(X) = \theta(Y)$ does not imply $R[X] = R[Y]$ because the attributes X and Y may have different sets of values.

Theorem 1: Let R be a relation on the set of attributes Ω, and let $L(R[\Omega])$ be the relation lattice of R. Then the partition function $\theta: \mathbf{2}^{\Omega} \rightarrow L(R[\Omega])$ is a meet-morphism.

Proof: We want to show that

$$\theta(XY) = \theta(X)\theta(Y), \qquad \forall X, Y \in \mathbf{2}^{\Omega}.$$

Suppose $t_1\theta(XY)t_2$. Then, $t_1[XY] = t_2[XY]$, which implies

$$t_1[X] = t_2[X] \quad \text{and} \quad t_1[Y] = t_2[Y].$$

Hence,

$$t_1\theta(X)t_2 \quad \text{and} \quad t_1\theta(Y)t_2.$$

By the definition of the meet operation, we have

$$t_1\theta(X)\theta(Y)t_2,$$

so that

$$\theta(XY) \leqq \theta(X)\theta(Y).$$

Suppose $t_1\theta(X)\theta(Y)t_2$. Then,

$$t_1\theta(X)t_2 \quad \text{and} \quad t_1\theta(Y)t_2,$$

so that

$$t_1[X] = t_2[X] \quad \text{and} \quad t_1[Y] = t_2[Y],$$

and thus

$$t_1[XY] = t_2[XY].$$

Consequently,

$$t_1\theta(XY)t_2,$$

so that

$$\theta(X)\theta(Y) \leqq \theta(XY).$$

Hence

$$\theta(XY) = \theta(X)\theta(Y). \quad \blacksquare$$

Note that the partition function θ is order-preserving, but it is in general not a join-morphism.* However, if $\theta(X + Y) = \theta(X) + \theta(Y)$ holds in $L[R]$, the pair (X, Y) has a special property in the relation. This is discussed further in Section VI.

It is clear now that Armstrong's axioms for functional dependencies become theorems within the framework of lattice theory. The proofs of the axioms for functional dependencies are given in Appendix B.

Let R be a relation on the set of attributes Ω, and let θ: $\mathbf{2}^\Omega \to L[R(\Omega)]$ be the partition function associated with $R[\Omega]$. Then the relation $\theta \circ \theta^{-1}$ on $\mathbf{2}^\Omega$ defined by

$$\theta \circ \theta^{-1} = \{(X, Y) \in \mathbf{2}^\Omega \times \mathbf{2}^\Omega \mid \theta(X) = \theta(Y)\}$$

is obviously an equivalence relation. Sets in the quotient set $\mathbf{2}^\Omega/\theta \circ \theta^{-1}$ will be called θ *classes.*

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* The join of π_1 and π_2 in $L(R[\Omega])$ may be different from their join in $\prod(R[\Omega])$. We will use the notation $\pi_1 \oplus \pi_2$ to denote the join of π_1 and π_2 in $\prod(R[\Omega])$, while $\pi_1 + \pi_2$ will denote the join of π_1 and π_2 in $L(R[\Omega])$; e.g., in Example 1 below, $\theta(E) \oplus \theta(S) = \{1, 2, 3, 4, 5, 6; 7, 8\}$, and $\theta(E) + \theta(S) = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Table I—Relation $R[ECSY]$

	Employee	Child	Salary	Year
1	Hilbert	Hubert	$35K	1975
2	Hilbert	Hubert	$40K	1976
3	Gauss	Gwendolyn	$40K	1975
4	Gauss	Gwendolyn	$50K	1976
5	Gauss	Greta	$40K	1975
6	Gauss	Greta	$50K	1976
7	Pythagoras	Peter	$15K	1975
8	Pythagoras	Peter	$20K	1976

Example 1: Consider the relation R in Table I (see Ref. 7). Let $\Omega = \{E, C, S, Y\}$ be the set of attributes, where E = employee, C = child, S = salary, Y = year. Then

$$\mathbf{2}^{\Omega} = \{\varnothing, E, C, S, Y, EC, ES, EY, CS, CY, SY, ECS, ECY, ESY, CYS, CESY\},$$

and

$$\theta(\varnothing = \{\overline{1, 2, 3, 4, 5, 6, 7, 8}\} = \mathbf{1},$$

$$\theta(E) = \{\overline{1, 2}; \overline{3, 4, 5, 6}; \overline{7, 8}\} = \pi_1,$$

$$\theta(C) = \theta(EC) = \{\overline{1, 2}; \overline{3, 4}; \overline{5, 6}; \overline{7, 8}\} = \pi_2,$$

$$\theta(S) = \{\overline{1}; \overline{2, 3, 5}; \overline{4, 6}; \overline{7}; \overline{8}\} = \pi_3,$$

$$\theta(Y) = \{\overline{1, 3, 5, 7}; \overline{2, 4, 6, 8}\} = \pi_4,$$

$$\theta(ES) = \theta(EY) = \theta(SY) = \theta(ESY) = \{\overline{1}; \overline{2}; \overline{3, 5}; \overline{4, 6}; \overline{7}; \overline{8}\} = \pi_5,$$

$$\theta(CS) = \theta(CY) = \theta(ECY) = \theta(ECS) = \theta(CSY) = \theta(ECSY) = \{\overline{1}; \overline{2}; \overline{3}; \overline{4}; \overline{5}; \overline{6}; \overline{7}; \overline{8}\} = \mathbf{0}.$$

The Hasse diagram[18,21] of the relation lattice is illustrated in Fig. 1. ■

IV. LIST OF KEYS

Let R be a relation on the set of attributes Ω. We say that $X \subseteq \Omega$ is a *superkey* of R if $X \rightarrow A$, $\forall A \in \Omega$. If X is a superkey and no proper subset of X is a superkey, X is said to be a *key* of R.[1,2,5]

Lemma 2: $X \subseteq \Omega$ is a superkey of R iff $\theta(X) = \mathbf{0}$.

Proof: (Necessity) Let $\Omega = \{A_1, \cdots, A_n\}$, and $X \rightarrow A_i$, $\forall A_i \in \Omega$. Then,

$$\theta(X) \leqq \theta(A_i), \qquad \forall A_i \in \Omega.$$

By the definition of the meet operation, we have

$$\theta(X) \leqq \theta(A_1)\theta(A_2) \cdots \theta(A_n).$$

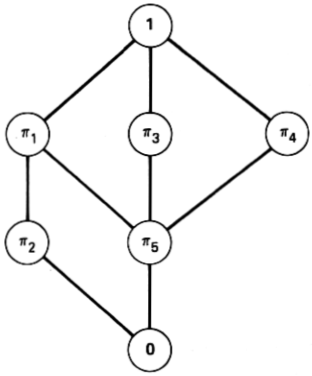

Fig. 1—Relation lattice $L(R[\Omega])$.

It follows from Theorem 1 that

$$\theta(X) \leqq \theta(A_1A_2 \cdots A_n) = \theta(\Omega) = \mathbf{0}.$$

Hence,

$$\theta(X) = \mathbf{0}.$$

(Sufficiency) Suppose $\theta(X) = \mathbf{0}$. Then

$$\theta(X) \leqq \theta(A_i), \qquad \forall A_i \in \Omega.$$

Hence,

$$X \rightarrow A_i, \qquad \forall A_i \in \Omega. \quad \blacksquare$$

An *ideal* is a subset J of a lattice L with the properties[18]
1. $a \in J$, $x \in L$, and $x \leqq a$, imply $x \in J$,
2. $a, b \in J$ implies $a + b \in J$.

For every $a \in L$, the subset of all elements "less than or equal to" a is evidently an ideal; it is called the *principal ideal* of L generated by a, and is denoted by $(a]$, i.e.,

$$(a] = \{x \in L \mid x \leqq a\}.$$

Definition 2: Let R be a relation on the set of attributes $\Omega = \{A_1, \cdots, A_n\}$. For each $A_i \in \Omega$, $J_i = (\theta(A_i)]$ is the principal ideal of the relation lattice $L(R[\Omega])$ generated by $\theta(A_i)$. A Boolean function

$$f_i(A_1, \cdots, A_n) = \sum_{\theta(X) \in J_i} X$$

defined on $\mathbf{2}^\Omega$ is the Boolean sum of all $X \in \mathbf{2}^\Omega$ such that $\theta(X) \leqq \theta(A_i)$. We will call f_i the *principal ideal function* (generated by A_i). $\blacksquare$

This function plays a role similar to the Boolean function used in Ref. 16.

Theorem 2: Let R be a relation on $\Omega = \{A_1, \cdots, A_n\}$. $X \subseteq \Omega$ is a superkey of R iff X is a product term in the expansion of the Boolean function

$$F(A_1, \cdots, A_n) = \prod_{i=1}^{n} f_i(A_1, \cdots, A_n),$$

where f_i is the principal ideal function generated by A_i.

Proof: The Boolean function $F(A_1, \cdots, A_n)$ has the expansion

$$F(A_1, \cdots, A_n) = \prod_{i=1}^{n} f_i = \sum_{\theta(X_i) \in J_i} X_1 \cdots X_n.$$

We want to show that every term $K = X_1 \cdots X_n$ is a superkey. Since $\theta(X_i) \in J_i = (\theta(A_i)]$, it follows that

$$\theta(X_i) \leqq \theta(A_i), \qquad 1 \leqslant i \leqslant n.$$

From L6 in Appendix A, we have

$$\theta(X_1)\theta(X_2) \cdots \theta(X_n) \leqq \theta(A_1)\theta(A_2) \cdots \theta(A_n).$$

It follows from Theorem 1 that

$$\theta(X_1X_2 \cdots X_n) \leqq \theta(A_1 \cdots A_n) = \theta(\Omega) = \mathbf{0},$$

and thus

$$\theta(X_1X_2 \cdots X_n) = \mathbf{0}.$$

Hence, $K = X_1X_2 \cdots X_n$ is a superkey of R.

Conversely, suppose X is a superkey of R. Then

$$X \rightarrow A_i, \qquad \forall A_i \in \Omega.$$

Thus,

$$\theta(X) \leqq \theta(A_i), \qquad 1 \leqslant i \leqslant n.$$

By the definition of the principal ideal J_i, we must have

$$\theta(X) \in J_i = (\theta(A_i)], \qquad 1 \leqslant i \leqslant n.$$

It follows that $X = X \cdots X$ (n times) is a product term in the expansion of $F(A_1, \cdots, A_n)$. ■

It is natural to call $F(A_1, \cdots, A_n)$ the *key Boolean function* of the relation $R[A_1 \cdots A_n]$. Since any key X is a superkey of R, X must be a product term of the key Boolean function $F(A_1, \cdots, A_n)$. Since no proper subset of X is a superkey, then by the definition of the *prime implicant* of a Boolean function,[22] we have

Corollary 1: Let R be a relation on the set of attributes $\Omega = \{A_1, \cdots, A_n\}$. $X \subseteq \Omega$ is a key of R iff X is a prime implicant of the key Boolean function $F(A_1, \cdots, A_n)$. ■

An attribute $A \in \Omega$ is *prime* in $R[\Omega]$ if A is in any key of R; otherwise A is *nonprime*. $A \subseteq \Omega$ is a nonprime attribute if and only if the key Boolean function is independent of A.

Theorem 3: Let R be a relation on Ω. $A \in \Omega$ is a nonprime attribute iff there exists $X \subseteq \Omega$ such that

1. $A \notin X$, $X \to A$,
2. $AZ \to X$ implies $Z \to X$.

Proof: (Necessity) Let $A \in \Omega$ be a nonprime attribute, and let X be any key of R. Then

$$A \notin X, \quad \text{and} \quad X \to A.$$

Suppose $AZ \to X$. Then $\theta(AZ) \leqq \theta(X) = \mathbf{0}$. It follows that $\theta(AZ) = \mathbf{0}$ and thus AZ is a superkey; it contains a key $K \subseteq AZ$ and $A \notin K$. We have $K \subseteq Z$, so that

$$\theta(Z) \leqq \theta(K) = \mathbf{0} = \theta(X).$$

Hence,

$$Z \to X.$$

(Sufficiency) Let $\Omega = \{A_1, \cdots, A_n\}$, $n \geqslant 2$. Assume there is an $X = A_2, \cdots, A_m$, such that (1) $X \to A_1$, and (2) $A_1Z \to X$, implies $Z \to X$. We want to show that A_1 must be a nonprime attribute. The key Boolean function $F(A_1, \cdots, A_n)$ of $R[\Omega]$ can be written in the form

$$F(A_1, \cdots, A_n) = \prod_{i=1}^{n} f_i = f_1 f_2 \cdots f_m f_{m+1} \cdots f_n$$

$$= (f_1 f_X f_{m+1}) \cdots (f_1 f_X f_n),$$

where $f_X = f_2 \cdots f_m$. For any product term Y in f_X we have

$$\theta(Y) \leqq \theta(X) \leqq \theta(A_1).$$

Therefore, Y must be a term in f_1. It follows that f_1 has the form

$$f_1 = f_X + g$$

for some Boolean function g. Since $\theta(A_1Z) \leqq \theta(X)$ implies $\theta(Z) \leqq \theta(X)$, f_X can be written in the form

$$f_X = A_1 h + h + p = h + p$$

for some Boolean functions h and p which are independent of A_1. Also, every f_j, $j = m + 1, \cdots, n$, can be written in the form

$$f_j = f_1 e + f_X e + q$$

for some Boolean functions e and q, which are independent of A_1. It follows that

$$f_1 f_X f_j = f_1 f_X (f_1 e + f_X e + q)$$
$$= f_X (f_1 e + f_1 f_X e + f_1 q)$$
$$= f_X (f_1 e + f_1 q)$$
$$= f_1 f_X (e + q) = (f_X + g) f_X (e + q)$$
$$= f_X (e + q) = (h + p)(e + q).$$

Since h, p, e, and q are all independent of A_1, we know that $f_1 f_X f_j$ is independent of A_1 for all $j = m + 1, \cdots, n$. Clearly, no prime implicant of $F(A_1, \cdots, A_n)$ contains A_1, and therefore A_1 is a nonprime attribute. ■

Example 2: Consider the relation R in Example 1. To obtain the prime implicants of the key Boolean function F, we can first simplify each principal ideal function. The principal ideal functions of the relation $R[ECSY]$ are

$$f_E = E + C + SY,$$
$$f_C = C,$$
$$f_S = S + EY + CY,$$
$$f_Y = Y + ES + CE,$$

and the key Boolean function is

$$F(E, C, S, Y) = (E + C + SY) \cdot C \cdot (S + EY + CY) \cdot (Y + ES + CE)$$
$$= CS + CY.$$

The sets CS and CY are the keys, and E is the only nonprime attribute. ■

V. BOYCE-CODD NORMAL FORM

Normalization is a logical database design process that can be viewed as the decomposition of a relation into a set of subrelations, such that the original relation can be regenerated by the joins of the subrelations. The purpose of decomposition is to separate the independent components into distinct relations, to avoid updating anomalies.[2] It is claimed in Ref. 4 that the Boyce-Codd Normal Form is one that is free of insertion and deletion anomalies. This section is devoted to the BCNF and its relation lattice. A modified algorithm for synthesizing an information-lossless BCNF[6] is included, based on the concept of the principal filter of the relation lattice.

Recall that a functional dependency $X \rightarrow Y$ is trivial if $Y \subseteq X$. A relation $R[\Omega]$ is said to be in Boyce-Codd Normal Form if, for all nontrivial FDs $X \rightarrow Y$, X is a superkey.[2,4]

Definition 3: Relation $R[\Omega]$ is in BCNF if $X \rightarrow Y$ implies either

1. X is a superkey, i.e., $\theta(X) = \mathbf{0}$,

or

2. $Y \subseteq X$. ■

If a relation is in BCNF, we will show that its relation lattice has some special properties. To analyze these properties we need the concept of the principal filter.[18]

An ideal of the dual of the lattice L is called a *filter* of L. A subset M of L is a filter of L if

1. $a \in M$, $x \in L$, and $x \geqq a$, imply $x \in M$,
2. $a, b, \in M$ implies $a \cdot b \in M$.

For every $a \in L$, the subset of all elements "greater than or equal to" a is a filter; it is called the *principal filter* of L generated by a, and is denoted by $[a)$, i.e.,

$$[a) = \{x \in L \mid x \geqq a\}.$$

If a and b are elements of a lattice L, where $a < b$, and there is no $c \in L$ such that $a < c < b$, then we say that a is *covered* by b (or b covers a).[18] An element that covers the universal lower bound $\mathbf{0}$ of L is referred to as an *atom* of L.[18]

Definition 4: Let R be a relation on the set of attributes Ω, and let π be an atom of the relation lattice $L(R[\Omega])$. Let $\Omega_\pi = \{A \mid A \in \Omega, \theta(A) \geqq \pi\} \subseteq \Omega$. Then the projection $R[\Omega_\pi]$ of R and Ω_π is called an *atomic projection*, and $[\pi)$ is called an *atomic filter.* ■

It is easy to verify that the relation lattice of the atomic projection $R[\Omega_\pi]$ is isomorphic to the principal filter $[\pi)$ of $L(R[\Omega])$ generated by π.

Definition 5: Let R be a relation on the set of attributes Ω, and let $\pi \in L(R[\Omega])$ be an atom. The principal filter $[\pi)$ of $L(R[\Omega])$ is called *normal* iff whenever $X \rightarrow Y$ is valid in the atomic projection $R[\Omega_\pi]$ then $Y \subseteq X$; otherwise, it is called *abnormal.* ■

Lemma 3: A relation $R[\Omega]$ is in BCNF iff every atomic filter of $L(R[\Omega])$ is normal.

Proof: (Necessity) Trivial.

(Sufficiency) Suppose $X \rightarrow Y$ and X is not a superkey, i.e., $\theta(X) \neq \mathbf{0}$. Then there must exist an atom π, such that

$$\mathbf{0} < \pi \leqq \theta(X) \leqq \theta(Y).$$

It follows that $X, Y \subseteq \Omega_\pi$ and that $X \rightarrow Y$ is valid in the atomic projection $R[\Omega_\pi]$, which is assumed normal. Therefore $Y \subseteq X$. ■

The join operation in the Boolean algebra $(\mathbf{2}^{\Omega}, \cdot, +, ^{-})$ is not always preserved by the θ mapping. But for a relation $R[\Omega]$ in BCNF, if $X, Y \subseteq \Omega$ and neither X nor Y is a superkey of $R[\Omega]$, then the join $X + Y$ is preserved by θ. We have

Corollary 2: If $R[\Omega]$ is in BCNF, $X, Y \subseteq \Omega$, $\theta(X) \neq \mathbf{0}$, and $\theta(Y) \neq \mathbf{0}$, then

$$\theta(X + Y) = \theta(X) + \theta(Y).$$

Proof: Since $X + Y \subseteq X$ and $X + Y \subseteq Y$, we have

$$\theta(X) \leqq \theta(X + Y) \quad \text{and} \quad \theta(Y) \leqq \theta(X + Y).$$

By definition of the join operation, we have

$$\theta(X) + \theta(Y) \leqq \theta(X + Y).$$

Suppose there is a $Z \subseteq \Omega$ such that

$$\theta(X) \leqq \theta(Z) \quad \text{and} \quad \theta(Y) \leqq \theta(Z).$$

Given $\theta(X) \neq \mathbf{0}$ and $\theta(Y) \neq \mathbf{0}$, we have

$$Z \subseteq X \quad \text{and} \quad Z \subseteq Y.$$

Thus,

$$Z \subseteq X + Y,$$

so that

$$\theta(X + Y) \leqq \theta(Z).$$

By the definition of least upper bound, we have

$$\theta(X + Y) = \theta(X) + \theta(Y). \quad \blacksquare$$

The most important characteristic of the BCNF is given in the following theorem.

Theorem 4: The relation $R[\Omega]$ is in BCNF iff every atomic filter $[\pi)$ of $L(R[\Omega])$ is isomorphic to the Boolean algebra $(\mathbf{2}^{\Omega_\pi}, \cdot, +, ^{-})$.

Proof: (Necessity) Since $[\pi)$ is a meet-morphic image of θ restricted to $\mathbf{2}^{\Omega_\pi}$, it is sufficient to show that θ is a one-to-one mapping on $\mathbf{2}^{\Omega_\pi}$. Let $X, Y \in \mathbf{2}^{\Omega_\pi}$, and $\theta(X) = \theta(Y)$. It follows that

$$\theta(X) = \theta(Y - X)\theta(Y + X) \leqq \theta(Y - X),$$

which implies

$$X \rightarrow Y - X.$$

Since $\mathbf{0} < \pi \leqq \theta(X)$ and $[\pi)$ is normal, we have

$$Y - X \subseteq X.$$

Hence,

$$Y \subseteq X.$$

Similarly,

$$X \subseteq Y.$$

Therefore, $X = Y$, and θ is a one-to-one mapping on $\mathbf{2}^{\Omega_\pi}$.

(Sufficiency) Suppose $X \to Y$ is valid in $R[\Omega_\pi]$. Then $\theta(X) \leqq \theta(Y)$. Since the inverse of an isomorphism is also order-preserving, it follows that $X \supseteq Y$. Therefore, $[\pi)$ is normal and $R[\Omega]$ is in BCNF. ■

The above theorem implies that if $[\pi)$ is normal, the only key of $R[\Omega_\pi]$ is $\theta^{-1}(\pi) = \Omega_\pi$.

It is known that any relation has a lossless-join decomposition into Boyce-Codd Normal Form, and an algorithm for determining the decomposition is given in Ref. 6. We will show how the concept of the principal filter can be used to modify this algorithm. In the algorithm for synthesizing the Third Normal Form,[5] a concept similar to the principal filter is used implicitly by Bernstein when he partitions the functional dependencies (Step 2). Before describing the improved algorithm, we need the following:

Lemma 4: Let R be a relation on Ω. *Let* $\pi \in L(R[\Omega])$ *be an atom of the relation lattice, and let K be a key of the atomic projection* $R[\Omega_\pi]$. *Then,*

$$R[\Omega] = R[(\Omega - \Omega_\pi)K] | \times | R[\Omega_\pi].$$

Proof: $K \subseteq \Omega_\pi$ and $K \to \Omega_\pi$. ■

The algorithm for determining the lossless-join decomposition into BCNF is simply to construct a sequence of decompositions $D_i = (R_1, \cdots, R_m)$ of R, each with lossless join: Initially, let D_0 consist of R alone. If $T[\Omega]$ is a relation in D_i, and $T[\Omega]$ is not in BCNF, let π be an atom of $L(T[\Omega])$ for which the principal filter $[\pi)$ is abnormal. Let K

Table II—Relation $R[MSPCNY]$

	Model Number	Serial Number	Price	Color	Name	Year
1	1234	342	13.25	blue	pot	1974
2	1234	347	13.25	red	pot	1974
3	1234	410	14.23	red	pot	1975
4	1465	347	9.45	black	pan	1974
5	1465	390	9.82	black	pan	1976
6	1465	392	9.82	red	pan	1976
7	1465	401	9.82	red	pan	1976
8	1465	409	9.82	blue	pan	1976
9	1623	311	22.34	blue	kettle	1973
10	1623	390	30.21	blue	kettle	1976
11	1623	410	28.55	black	kettle	1975
12	1623	423	28.55	black	kettle	1975
13	1623	428	28.55	blue	kettle	1975
14	1654	435	28.55	red	kettle	1975

be a key of the atomic projection $T[\Omega_\pi]$. Now replace $T[\Omega]$ in D_i by $T[\Omega - \Omega_\pi)K]$ and $T[\Omega_\pi]$ to obtain D_{i+1}. Continue the process until all the relations in the decomposition D_k are in BCNF.

Example 3: Let us consider the relation $R[MSPCNY]$ from Ref. 23, where M = model number, S = serial number, P = price, C = color, N = name, and Y = year. The tuples of the relation $R[MSPCNY]$ are shown in Table II.

The Hasse diagram of the relation lattice $L(R[\Omega])$ is illustrated in Fig. 2, where

$$\pi_1 = \{\overline{1, 8, 9, 10, 13};\ \overline{2, 3, 6, 7, 14};\ \overline{4, 5, 11, 12}\},$$

$$\pi_2 = \{\overline{1, 2, 3};\ \overline{4, 5, 6, 7, 8};\ \overline{9, 10, 11, 12, 13, 14}\},$$

$$\pi_3 = \{\overline{1, 2, 4};\ \overline{3, 11, 12, 13, 14};\ \overline{5, 6, 7, 8, 10};\ \overline{9}\},$$

$$\pi_4 = \{\overline{1, 2, 3};\ \overline{4, 5, 6, 7, 8};\ \overline{9, 10, 11, 12, 13};\ \overline{14}\},$$

$$\pi_5 = \{\overline{1, 2};\ \overline{3};\ \overline{4};\ \overline{5, 6, 7, 8};\ \overline{9};\ \overline{10};\ \overline{11, 12, 13, 14}\},$$

$$\pi_6 = \{\overline{1};\ \overline{2, 3};\ \overline{4, 5};\ \overline{6, 7};\ \overline{8};\ \overline{9, 10, 13};\ \overline{11, 12};\ \overline{14}\},$$

$$\pi_7 = \{\overline{1, 2};\ \overline{3};\ \overline{4};\ \overline{5, 6, 7, 8};\ \overline{9};\ \overline{10};\ \overline{11, 12, 13};\ \overline{14}\},$$

$$\pi_8 = \{\overline{1};\ \overline{2, 4};\ \overline{3, 11};\ \overline{5, 10};\ \overline{6};\ \overline{7};\ \overline{8};\ \overline{9};\ \overline{12};\ \overline{13};\ \overline{14}\},$$

$$\pi_9 = \{\overline{1};\ \overline{2};\ \overline{3};\ \overline{4};\ \overline{5};\ \overline{6, 7};\ \overline{8};\ \overline{9};\ \overline{10};\ \overline{11, 12};\ \overline{13};\ \overline{14}\},$$

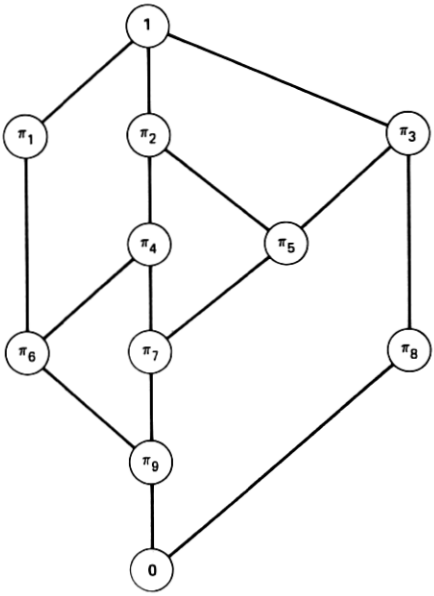

Fig. 2—Relation lattice $L(R[MSPCNY])$.

and $\theta(C) = \pi_1$, $\theta(N) = \pi_2$, $\theta(Y) = \pi_3$, $\theta(M) = \pi_4$, $\theta(P) = \pi_5$, $\theta(S) = \pi_8$. For $X \subseteq \Omega$, $\theta(X)$ can be obtained easily by carrying out the meet operations on the attributes in X.

The principal ideal functions of $R[MSPCNY]$ are

$$f_C(M, S, P, C, N, Y) = C + MS + NS + PS,$$

$$f_N(M, S, P, C, N, Y) = N + M + P + CY + CS,$$

$$f_P(M, S, P, C, N, Y) = P + MY + CS + MS + NS,$$

$$f_M(M, S, P, C, N, Y) = M + CN + CP + CY + NS + PS + CS,$$

$$f_Y(M, S, P, C, N, Y) = Y + P + S,$$

$$f_S(M, S, P, C, N, Y) = S,$$

and the key Boolean function is

$$\begin{aligned} F(M, S, P, C, N, Y) &= (C + MS + NS + PS)\cdot(N + M + P + CY \\ &\qquad + CS) \\ &\quad \cdot(P + MY + CS + MS + NS) \\ &\quad \cdot(M + CN + CD + CY + NS + PS + CS) \\ &\quad \cdot(Y + P + S)\cdot S \\ &= CS + MS + NS + PS. \end{aligned}$$

The keys of $R[MSPCNY]$ are $\{CS, MS, NS, PS\}$, and Y is the only nonprime attribute.

Initially, let $D_0 = \{R(MSPCNY)\}$. Since both atomic filters $[\pi_8)$ and $[\pi_9)$ are abnormal, we arbitrarily choose π_9, and let $\Sigma = \Omega_{\pi_9} = MPCNY$. The relation lattice of $R[\Sigma]$ is isomorphic to $[\pi_9)$. The principal ideal functions of $R[\Sigma]$ are

$$g_C(M, P, C, N, Y) = f_C(M, \mathbf{0}, P, C, N, Y) = C,$$

$$g_N(M, P, C, N, Y) = f_N(M, \mathbf{0}, P, C, N, Y) = N + M + P + CY,$$

$$g_P(M, P, C, N, Y) = f_P(M, \mathbf{0}, P, C, N, Y) = P + MY,$$

$$g_M(M, P, C, N, Y) = f_M(M, \mathbf{0}, P, C, N, Y) = M + CN + CP + CY,$$

$$g_Y(M, P, C, N, Y) = f_Y(M, \mathbf{0}, P, C, N, Y) = Y + P,$$

and the key Boolean function is

$$\begin{aligned} G(M, P, C, N, Y) &= C\cdot(N + M + P + CY)\cdot(P + MY) \\ &\quad \cdot(M + CN + CP + CY)\cdot(Y + P) \\ &= CP + CMY. \end{aligned}$$

We choose the key CP and replace $R[MSPCNY]$ in D_0 by $R[(\Omega - \Sigma)K] = R[SPC]$ and $R[\Sigma] = R[MPCNY]$ to obtain $D_1 = \{R[SPC], R[MPCNY]\}$. The relation $R[SPC]$ and its lattice are shown in Table III and Fig. 3, respectively.

The relation $R[SPC]$ is in BCNF, but the relation $R[MPCNY]$ is not. The relation lattice of $R[MPCNY]$ is isomorphic to the filter $[\pi_9)$. We will not duplicate the figure. Both "atoms" π_6 and π_7 of $[\pi_9)$ are abnormal. We choose the filter $[\pi_7)$. The principal ideal functions of $R[\Sigma_{\pi_7}] = R[MPNY]$ are

$$h_M(M, P, N, Y) = g_M(M, P, \mathbf{0}, N, Y) = M,$$

$$h_N(M, P, N, Y) = g_N(M, P, \mathbf{0}, N, Y) = N + M + P,$$

$$h_P(M, P, N, Y) = g_P(M, P, \mathbf{0}, N, Y) = P + NY,$$

$$h_Y(M, P, N, Y) = g_Y(M, P, \mathbf{0}, N, Y) = Y + P,$$

and the key Boolean function of $R[MPNY]$ is given by

$$H(M, P, N, Y) = M \cdot (N + M + P) \cdot (P + MY) \cdot (Y + P)$$

$$= MP + MY.$$

We choose the key $K' = MP$ and replace $R[MPCNY]$ in D_1 by $R[(\Sigma - \Sigma_{\pi_7})K'] = R[MPC]$ and $R[MPNY]$ to obtain $D_2 = \{R[SPC], R[MPC], R[MPNY]\}$. The relation $R[MPC]$ and its relation lattice are illustrated in Table IV and Fig. 4, respectively.

Now we have to decompose the relation $R[MPNY]$ in D_2. The relation lattice of $R[MPNY]$ is isomorphic to $[\pi_7)$ of $L(R[MSPCNY])$. We choose the abnormal filter that is isomorphic to $[\pi_5)$. Since $\Sigma_{\pi_5} = PNY$ and the only key is P, we can replace $R[MPNY]$ in D_2 by $R[MP]$ and $R[PNY]$ to obtain $D_3 = \{R[SPC], R[MPC], R[MP], R[PNY]\}$. All the relations in D_3 are in BCNF. The relations $R[MP]$, $R[PNY]$ and

Table III—Relation $R[SPC]$

	Serial Number	Price	Color
1	342	13.25	blue
2	347	13.25	red
3	410	14.23	red
4	347	9.45	black
5	390	9.82	black
6	392	9.82	red
7	401	9.82	red
8	409	9.82	blue
9	311	22.34	blue
10	390	30.21	blue
11	410	28.55	black
12	423	28.55	black
13	428	28.55	blue
14	435	28.55	red

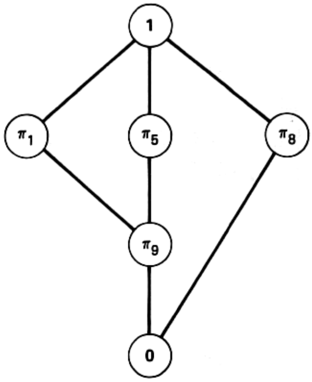

Fig. 3—Relation lattice $L(R[SPC])$.

Table IV—Relation $R[MPC]$

	Model Number	Price	Color
1	1234	13.25	blue
2	1234	13.25	red
3	1234	14.23	red
4	1465	9.45	black
5	1465	9.82	black
(6, 7)	1465	9.82	red
8	1465	9.82	blue
9	1623	22.34	blue
10	1623	30.21	blue
(11, 12)	1623	28.55	black
13	1623	28.55	blue
14	1654	28.55	red

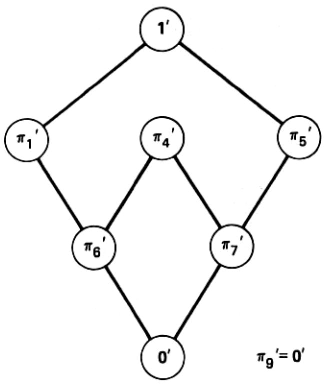

Fig. 4—Relation lattice $L(R[MPC])$.

their respective lattices are shown in Tables V and VI, and Figs. 5 and 6. ■

VI. MULTIVALUED DEPENDENCIES

Multivalued dependency (MVD) proposed by Fagin[7] and Zaniolo[8] is the necessary and sufficient condition for a (binary) lossless-join decomposition. A similar concept, called hierarchical dependency, was defined by Delobel.[24] A bit later, the concept of multivalued dependency was generalized to join dependency by Rissanen.[10,11] A set of "axioms" or inference rules for multivalued dependencies was given by Beeri, Fagin, and Howard.[25] We know from our previous discussion that functional dependency is equivalent to partial ordering in the partition lattice. In this section we show that multivalued dependency

Table V—Relation $R[MP]$

	Model Number	Price
(1, 2)	1234	13.25
3	1234	14.23
4	1465	9.45
(5, 6, 7, 8)	1465	9.82
9	1623	22.34
10	1623	30.21
(11, 12, 13)	1623	28.55
14	1654	28.55

Table VI—Relation $R[PNY]$

	Price	Name	Year
(1, 2)	13.25	pot	1974
3	14.23	pot	1975
4	9.45	pan	1974
(5, 6, 7, 8)	9.82	pan	1976
9	22.34	kettle	1973
10	30.21	kettle	1976
(11, 12, 13, 14)	28.55	kettle	1975

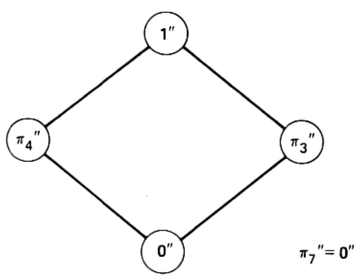

Fig. 5—Relation lattice $L(R[MP])$.

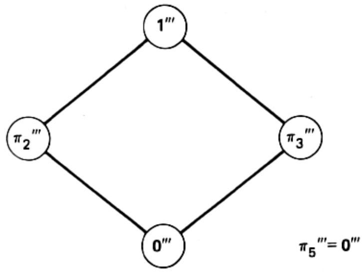

Fig. 6—Relation lattice $L(R[PNY])$.

is equivalent to a lattice equation. First, however, we state the definition of MVD and show that MVD guarantees information-lossless join decomposition.

Definition 5: Let R be a relation on the set of attributes $\Omega = XYZ$, where X, Y, and Z are disjoint subsets of Ω. We say there is a multivalued dependency $X \twoheadrightarrow Y$ if

$$R_{xz}[Y] = R_x[Y], \qquad \forall (x) \in R[X], \qquad (z) \in R[Z]. \quad \blacksquare$$

Lemma 5: Let R be a relation on $\Omega = XYZ$, where X, Y, and Z are disjoint subsets. Then,

$$R[XYZ] = R[XY] \mid \times \mid R[XZ]$$

iff

$$R_x[YZ] = |R_x[Y]| \cdot |R_x[Z]|, \qquad \forall (x) \in R[X].$$

Proof: (Necessity) $R[XYZ] = R[YX] \mid \times \mid R[XZ]$ implies

$$R_x[YZ] = R_x[Y] \times R_x[Z], \qquad \forall (x) \in R[X].$$

Hence,

$$|R_x[YZ]| = |R_x[Y]| \cdot |R_x[Z]|.$$

(Sufficiency) It is easy to verify that

$$R_x[YZ] \subseteq R_x[Y] \times R_x[Z], \qquad \forall (x) \in R[X].$$

The given cardinal identity assures that

$$R_x[YZ] = R_x[Y] \times R_x[Z], \qquad \forall (x) \in R[X]. \quad \blacksquare$$

Theorem 5: Let R be a relation on the set of attributes $\Omega = XYZ$, where X, Y, and Z are disjoint subsets. Then,*

$$R[XYZ] = R[XY] \mid \times \mid R[XZ] \text{ iff } X \twoheadrightarrow Y.$$

* For convenience, we assume X, Y, and Z to be disjoint. It will later become clear that this assumption is not necessary.

Proof: (Necessity) From Lemma 5, it is sufficient to show that

$$|R_x[YZ]| = |R_x[Y]| \cdot |R_x[Z]|, \qquad \forall (x) \in R(X)$$

iff

$$R_{xz}[Y] = R_x[Y], \qquad \forall (x) \in R(X), \qquad (z) \in R(Z).$$

Since

$$R[XYZ] = R[XY] \mid \times \mid R[XZ]$$

implies

$$R_{xz}[Y] \times (x, z) = R_x[Y] \times (x, z), \qquad \forall (x) \in R[X], \qquad (z) \in R[Z].$$

Hence,

$$R_{xz}[Y] = R_x[Y].$$

(Sufficiency) For every $(x) \in R[X]$, we have

$$\begin{aligned}(x) \times R_x[ZY] &= \{(x, z_i) \times R_{xz_i}[Y] \mid (x, z_i) \in R[XZ]\} \\ &= \{(x, z_i) \times R_x[Y] \mid (x, z_i) \in R[XZ]\} \\ &= (x) \times R_x[Z] \times R_x[Y].\end{aligned}$$

Since $|x| = 1$, it follows that

$$|R_x[YZ]| = |R_x[Y]| \cdot |R_x[Z]|, \qquad \forall (x) \in R[X]. \quad \blacksquare$$

We need the commutative property of the product of two equivalence relations (partitions) to establish the lattice equation of multivalued dependency. The product of two equivalence relations may not be an equivalence relation; if it is an equivalence relation then the product must be commutative and vice versa.

Definition 6: Two binary relations ρ and ρ' and S are *permutable* (commute) if and only if $\rho \circ \rho' = \rho' \circ \rho$. This means that if $a\ \rho\ x\ \rho'\ b$ for some $x \in S$, then $a\ \rho'\ y\ \rho\ b$ for some $y \in S$, and conversely.[18] ■

Lemma 6: *Let ρ and ρ' be equivalence relations (partitions) on S. Then the following are equivalent*:

1. $\rho \circ \rho' = \rho \circ \rho$
2. $\rho \circ \rho' = \rho \oplus \rho'$
3. $\rho \circ \rho'$ is an equivalence relation
4. $\rho \circ \rho'$ is symmetric.

Proof: The proof of Lemma 6 is given in Ref. 21. ■

Lemma 7: *Let R be a relation on the set of attributes Ω and let X, Y, Z $\subseteq \Omega$. Then,*

$$\theta(X) = \theta(XY) + \theta(XZ) = \theta(XY) \circ \theta(XZ) = \theta(XZ) \circ \theta(XY)$$

iff

$$\theta(X) \subseteq \theta(XY) \circ \theta(XZ).$$

Proof: (Necessity) Trivial.

(Sufficiency) Suppose $t_1\theta(XY) \circ \theta(XZ)t_2$. Then there exists $t_3 \in R[\Omega]$ such that

$$t_1\theta(XT)t_3\theta(XZ)t_2,$$

which implies

$$t_1\theta(X)t_3\theta(X)t_2.$$

Therefore,

$$t_1\theta(X)t_2.$$

Hence,

$$\theta(XY) \circ \theta(XZ) \subseteq \theta(X).$$

It follows that

$$\theta(X) = \theta(XY) \circ \theta(XZ).$$

From Lemma 6, we have

$$\theta(X) = \theta(XY) \oplus \theta(XZ) = \theta(XY) \circ \theta(XZ) = \theta(XZ) \circ \theta(XY).$$

Since

$$\theta(XY) \leqq \theta(XY) + \theta(XZ) \quad \text{and} \quad \theta(XZ) \leqq \theta(XY) + \theta(XZ),$$

by the definition of the join operation $\oplus$ in $\prod(R[\Omega])$, we must have

$$\theta(X) = \theta(XY) \oplus \theta(XZ) \leqq \theta(XY) + \theta(XZ).$$

But,

$$\theta(XY) + \theta(XZ) \leqq \theta(X) + \theta(X) = \theta(X),$$

so it follows that

$$\theta(X) = \theta(XY) + \theta(XZ) = \theta(XY) \circ \theta(XZ) = \theta(XZ) \circ \theta(XY). \quad \blacksquare$$

The following theorem shows that the multivalued dependency can be formulated as a lattice equation.

Theorem 6: Let R be a relation on the set of attributes $\Omega = XYZ$, where X, Y, and Z are disjoint subsets. Then, $R[XYZ] = R[XY] \mid \times \mid R[XZ]$ iff

$$\theta(X) = \theta(XY) + \theta(XZ) = \theta(XY) \circ \theta(XZ) = \theta(XZ) \circ \theta(XY).$$

Proof: (Necessity) Since $R[XYZ] = R[XY] \mid \times \mid R[XZ]$ implies

$$R_x[YZ] = R_x[Y] \times R_x[Z], \qquad \forall(x) \in R[X],$$

there is a one-to-one and onto mapping ϕ_x: $R_x[YZ] \to R_x[Y] \times R_x[Z]$, which takes every tuple $(y, z) \in R_x[YZ]$ into $\phi_x((y, z)) = ((y), (z)) \in R_x[Y] \times R_z[Z]$, $\forall (x) \in R[X]$. Suppose $t_1, t_2 \in R[XYZ]$ and $t_1\theta[X]t_2$, and assume $t_1 = (x, y_1, z_1)$ and $t_2 = (x, y_2, z_2)$. As

$$(y_1, z_1), (y_2, z_2) \in R_z[YZ] = R_x[Y] \times R_x[Z],$$

we have

$$(y_1), (y_2) \in R_x[Y] \quad \text{and} \quad (z_1), (z_2) \in R_x[Z].$$

Since ϕ_x is an onto mapping, there must exist two tuples $t_3 = (x, y_1, z_2)$, and $t_4 = (x, y_2, z_1) \in R[XYZ]$. Hence,

$$t_1\theta(XY)t_3\theta(XZ)t_2,$$

which means

$$t_1\theta(XY) \circ \theta(XZ)t_2.$$

It follows that

$$\theta(X) \subseteq \theta(XY) \circ (XZ).$$

From Lemma 7, we have

$$\theta(X) = \theta(XY) + \theta(XZ) = \theta(XY) \circ \theta(XZ) = \theta(XZ) \circ \theta(XY).$$

(Sufficiency) We know $R[XYZ] \subseteq R[XY] \mid \times \mid R[XZ]$. Suppose $t = (x, y, z) \in R\,[XY] \mid \times \mid R[XZ]$. Then there exist $t_1 = (x, y, z')$, $t_2 = (x, y', z) \in R[XYZ]$. Thus,

$$t_1\theta(X)t_2,$$

which implies

$$t_1\theta(XY) \circ \theta(XZ)t_2.$$

There must exist $t_3 \in R[XYZ]$ such that

$$t_1\theta(XY)t_3\theta(XZ)t_2.$$

Therefore,

$$t_3 = (x, y, z) = t \in R[XYZ],$$

and thus

$$R[XY] \mid \times \mid R[XZ] \subseteq R[XYZ].$$

Hence,

$$R[XYZ] = R[XY] \mid \times \mid R[XZ]. \quad \blacksquare$$

It should be noted that in the above proof we use the fact that $\Omega = XYZ$ and $\theta(\Omega) = \theta(XYZ) = \mathbf{0}$, i.e., there are no duplicated tuples in

$R[\Omega]$. The inference rules of MVD are given and proved in Appendix B.

Example 4: Consider the relation $R[ECSY]$ of Example 1. We have the MVD: $E \twoheadrightarrow SY$, where $\theta(E) = \pi_1 = \{\overline{1, 2; 3, 4, 5, 6; 7, 8}\}$, $\theta(EC) = \pi_2 = \{\overline{1, 2}; \overline{3, 4}; \overline{5, 6}; \overline{7, 8}\}$, $\theta(ESY) = \pi_5 = \{\overline{1}; \overline{2}; \overline{3, 5}; \overline{4, 6}; \overline{7}; \overline{8}\}$. It is easy to verify that

$$\pi_1 = \pi_2 + \pi_5 = \pi_2 \circ \pi_5 = \pi_5 \circ \pi_2. \quad \blacksquare$$

It is known that if R is a relation on $\Omega = XYZ$, and $X \twoheadrightarrow Y$ then $X \twoheadrightarrow Z$. The symmetricity of the MVD can easily be seen in the lattice equation of Lemma 7.

If $XYZ \subset \Omega$ and $\theta(X) = \theta(XY) + \theta(XZ) = \theta(XY) \circ \theta(XZ) = \theta(XZ) \circ \theta(XY)$ holds, then $X \twoheadrightarrow Y|Z$ is called an embedded multivalued dependency (EMVD)[7]; this is simply a multivalued dependency in the projection $R[XYZ]$ of $R[\Omega]$.

Theorem 6 clearly indicates that the MVD is actually a condition pertaining to data *independency* rather than data dependency. For this reason, we introduce the notion of decomposition of two sets of attributes in a relation as follows.

Definition 7: Let R be a relation on the set of attributes Ω. The two sets of attributes $\Omega_1, \Omega_2 \subseteq \Omega$ are decomposable in R if

$$\theta(\Omega_1 + \Omega_2) = \theta(\Omega_1) + \theta(\Omega_2) = \theta(\Omega_1) \circ \theta(\Omega_2) = \theta(\Omega_2) \circ \theta(\Omega_1). \quad \blacksquare$$

It is easy to see that Ω_1 and Ω_2 are decomposable in Ω iff $\Omega_1 + \Omega_2 \twoheadrightarrow \Omega_1 - \Omega_2 | \Omega_2 - \Omega_1$ is an EMVD in R. Furthermore, if $\Omega_1\Omega_2 = \Omega$ then $\Omega_1 + \Omega_2 \twoheadrightarrow \Omega_1 - \Omega_2$ (or $\Omega_1 + \Omega_2 \twoheadrightarrow \Omega_2 - \Omega_1$) is an MVD in R. In the latter case, (Ω_1, Ω_2) is called a *decomposition pair* by Armstrong and Delobel.[26]

We feel that decomposition is a basic concept in the study of the structure of databases. It can be naturally generalized to the concepts of projective decomposition and mutual decomposition. Projective composition concerns the data independence of two sets of attributes on the projection of a relation. Mutual decomposition extends the concept of decomposition to more than two sets of attributes.

Let ρ be a partition on the set S, the function $\rho^* = S \rightarrow S/\rho$ maps $a \in S$ into $(a)\rho^* = a_\rho$ is called the *canonical function* of ρ. For $S = \{a, b, \ldots, e\}$, we will use the notation

$$\rho^* = \begin{pmatrix} a\, b & \cdots & e \\ a_\rho b_\rho & \cdots & e_\rho \end{pmatrix}$$

to illustrate the canonical function ρ^*. The equivalence relation

$$\ker \rho^* = \rho^* \circ \rho^{*-1} = \{(a, b) \in S \times S \,|\, \rho^*(a) = \rho^*(b)\}.$$

is called the *kernel* of ρ^*. Notice that $\ker \rho^* = \rho$.

Let ρ and σ be partitions on $S = \rho \leqq \sigma$; then there is a unique function f from S/ρ onto S/σ such that $(a_\rho)f = a_\sigma$. The kernal of f,

$$\ker f = f \circ f^{-1} = \{(a_\rho, b_\rho) \in S/\rho \times S/\rho \,|\, a \;\sigma\; b\},$$

is an equivalence on S/ρ. It is usual to write $\ker f$ as σ/ρ, the quotient of σ and ρ. Note that $a_\rho(\sigma/\rho)b_\rho$ if and only if $a \;\sigma\; b$ and the mapping g: $(S/\rho)/(\sigma/\rho) \to S/\sigma$ defined by $((a_\rho)_{\sigma/\rho})g = a_\sigma$ is one-to-one and onto. Thus the function f defined above is in fact the canonical function of σ/ρ, i.e., $f = (\sigma/\rho)^*$. It is easy to see the diagram in Fig. 7 commutes, that is $\rho^* \circ (\sigma/\rho)^* = \sigma^*$.

Example 5: Let ρ, σ be partitions on the set $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$, such that $\rho = \{\overline{1, 2}; \overline{3, 4}; \overline{5, 6}; \overline{7, 8}\}$ and $\sigma = \{\overline{1, 2, 3, 4}; \overline{5, 6, 7, 8}\}$ with $\rho \leqq \sigma$. Then $S/\rho = \{I, II, III, IV\}$, where $I = \overline{1, 2}$, $II = \overline{3, 4}$, $III = \overline{5, 6}$, $IV = \overline{7, 8}$, and $S/\sigma = \{\alpha, \beta\}$, where $\alpha = \overline{1, 2, 3, 4}$, $\beta = \overline{5, 6, 7, 8}$. The canonical functions of ρ and σ are

$$\rho^* = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ I & I & II & II & III & III & IV & IV \end{pmatrix}$$

and

$$\sigma^* = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \alpha & \alpha & \alpha & \alpha & \beta & \beta & \beta & \beta \end{pmatrix}.$$

It follows that

$$\sigma/\rho = \{\overline{I, II}; \overline{III, IV}\}$$

and

$$(\sigma/\rho)^* = \begin{pmatrix} I & II & III & IV \\ \alpha & \alpha & \beta & \beta \end{pmatrix}. \quad \blacksquare$$

Lemma 8: Let ρ, σ_1, σ_2 be partitions on S such that $\rho \leqq \sigma_1$, $\rho \leqq \sigma_2$, and $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1$. Then

$$(\sigma_1 \circ \sigma_2)/\rho = (\sigma_1/\rho) \circ (\sigma_2/\rho).$$

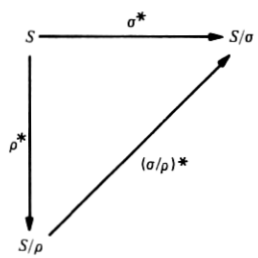

Fig. 7—Canonical function of quotient partition.

Proof: It is clear that $\sigma_1 \circ \sigma_2$ is a partition on S and $\rho \leqq \sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1$. It follows from the definition of quotient partition that the lemma is true. ■

Lemma 9: Let ρ, σ_1, σ_2 be partitions on S, such that $\rho \leqq \sigma_1$, $\rho \leqq \sigma_2$. Then

$$\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1$$

iff

$$(\sigma_1/\rho) \circ (_2/\rho) = (\sigma_2/\rho) \circ (\sigma_1/\rho).$$

Proof: (Necessity)

$$(\sigma_1/\rho) \circ (\sigma_2/\rho) = (\sigma_1 \circ \sigma_2)/\rho = (\sigma_2 \circ \sigma_1)/\rho = (\sigma_2/\rho) \circ (\sigma_1/\rho).$$

(Sufficiency) Suppose $a\sigma_1 \circ \sigma_2$b. Then there is a $c \in S$ such that $a\sigma_1 c \sigma_2 b$. It follows that

$$a_\rho(\sigma_1/\rho)c_\rho(\sigma_2/\rho)b_\rho.$$

There must be a $d \in S$ such that

$$a_\rho(\sigma_2/\rho)d_\rho(\sigma_1/\rho)b_\rho.$$

Thus

$$a\sigma_2 d\sigma_1 b,$$

and

$$a\sigma_2 \circ \sigma_1 b.$$

Hence

$$\sigma_1 \circ \sigma_2 \subseteq \sigma_2 \circ \sigma_1.$$

Similarly, we have $\sigma_2 \circ \sigma_1 \subseteq \sigma_1 \circ \sigma_2$. Then $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1$. ■

Definition 8: Let R be a relation on the set of attributes Ω. For Ω_1, $\Omega_2 \subseteq \Omega$, the *projective partition* defined by

$$\theta(\Omega_1 | \Omega_2) = \theta(\Omega_1 + \Omega_2)/\theta(\Omega_2)$$

is a partition on the set of tuples of $R[\Omega]/\theta(\Omega_2) = R[\Omega_2]$. The canonical function of $\theta(\Omega_1|\Omega_2)$ is denoted by $\theta^*(\Omega_1|\Omega_2) = (\theta(\Omega_1 + \Omega_2)/\theta(\Omega_2))^*$, which satisfies $\theta^*(\Omega_2) \circ \theta^*(\Omega_1|\Omega_2) = \theta^*(\Omega_1 + \Omega_2)$. ■

Certain properties of the projective partition are demonstrated in the following theorems and their proof directly follows from the definition of projective partition.

Theorem 7: Let R be a relation on the set of attributes Ω and $\Omega_1, \cdots, \Omega_n \subseteq \Omega$. Then

$$\theta^*\left(\bigcap_{k=1}^{n} \Omega_k\right) = \theta^*(\Omega_1) \circ \theta^*(\Omega_2|\Omega_1) \circ \cdots \circ \theta^*(\Omega_n| \bigcap_{k=1}^{n-1} \Omega_k).$$ ■

Theorem 8: Let R be a relation on the set of attributes Ω *and* $X, \Omega_1, \cdots, \Omega_n \subseteq \Omega$*, such that*

$$\bigcup_{k=1}^{n} \Omega_k = \Omega.$$

Then

1. $\theta(X) = \prod_{k=1}^{n} \ker(\theta^*(\Omega_k) \circ \theta^*(X \mid \Omega_k))$
2. $\theta(\Omega_m \mid X) = \ker(\theta^*(\Omega_m) \circ \theta^*(X \mid \Omega_m)) / \prod_{k=1}^{n} \ker(\theta^*(\Omega_k) \circ \theta^*(X \mid \Omega_k))$. ■

Definition 9: Let R be a relation on the set of attributes Ω and $\Omega_1, \Omega_2, \Sigma \subseteq \Omega$. We say Ω_1 and Ω_2 are projectively decomposable on Σ if

$$\begin{aligned}\theta(\Omega_1 + \Omega_2 \mid \Sigma) &= \theta(\Omega_1 \mid \Sigma) + \theta(\Omega_2 \mid \Sigma) \\ &= \theta(\Omega_1 \mid \Sigma) \circ \theta(\Omega_2 \mid \Sigma) \\ &= \theta(\Omega_2 \mid \Sigma) \circ \theta(\Omega_1 \mid \Sigma). \quad \blacksquare\end{aligned}$$

The EMVD is a special case of projective decomposition, which can be seen from the following theorem.

Theorem 9: Let R be a relation on Ω*, and let* $\Omega_1, \Omega_2, \Sigma \subseteq \Omega$*. Then* Ω_1 *and* Ω_2 *are projectively decomposable on* Σ *iff*

$$\begin{aligned}\theta(\Omega_1 + \Omega_2 + \Sigma) &= \theta(\Omega_1 + \Sigma) + \theta(\Omega_2 + \Sigma) \\ &= \theta(\Omega_1 + \Sigma) \circ \theta(\Omega_2 + \Sigma) = \theta(\Omega_2 + \Sigma) \circ \theta(\Omega_1 + \Sigma).\end{aligned}$$

Proof: The proof follows from Lemma 8 and 9. ■

Example 6: Consider the relation R on $\Omega = ABCDE$ in Table VII. The Hasse diagram of the relation lattice $L(R[\Omega])$ is shown in Fig. 8, where

$$\begin{aligned}
\pi_1 &= \{\overline{1, 2, 3, 5, 6, 7}; \overline{4}\} = \theta(A),\\
\pi_2 &= \{\overline{1, 3, 4}; \overline{2, 5, 6, 7}\} = \theta(B),\\
\pi_3 &= \{\overline{1, 6, 7}; \overline{2, 3, 4, 5}\} = \theta(C),\\
\pi_4 &= \{\overline{1, 3}; \overline{2, 5, 6, 7}; \overline{4}\} = \theta(AB),\\
\pi_5 &= \{\overline{1, 6, 7}; \overline{2, 3, 5}; \overline{4}\} = \theta(AC),\\
\pi_6 &= \{\overline{1}; \overline{2, 5}; \overline{3, 4}; \overline{6, 7}\} = \theta(BC),\\
\pi_7 &= \{\overline{1, 6, 7}; \overline{2, 5}; \overline{3}; \overline{4}\} = \theta(D),\\
\pi_8 &= \{\overline{1, 3}; \overline{2, 6}; \overline{4}; \overline{5, 7}\} = \theta(E),\\
\pi_9 &= \{\overline{1}; \overline{2, 5}; \overline{3}; \overline{4}; \overline{6, 7}\} = \theta(ABC) = \theta(ABD).
\end{aligned}$$

Let $\Omega_1 = ABD$, $\Omega_2 = ACE$, $\Sigma = ABC$. We find that

Table VII—Relation lattice $L(R[\Omega])$

	A	B	C	D	E
1	a_1	b_1	c_1	d_1	e_1
2	a_1	b_2	c_2	d_2	e_2
3	a_1	b_1	c_2	d_3	e_1
4	a_2	b_1	c_2	d_4	e_3
5	a_1	b_2	c_2	d_2	e_4
6	a_1	b_2	c_1	d_1	e_2
7	a_1	b_2	c_1	d_1	e_4

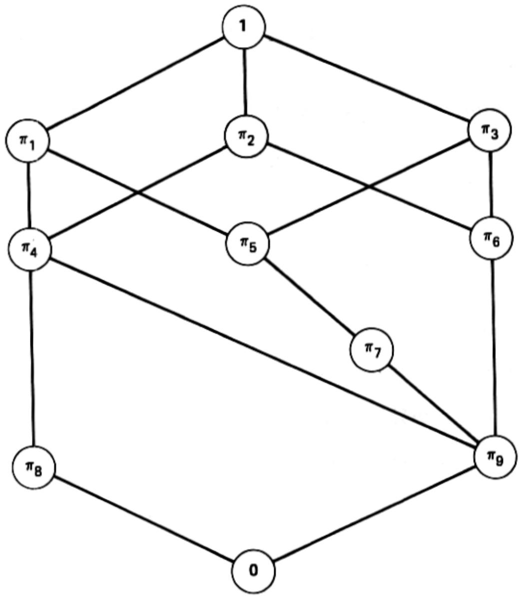

Fig. 8—Relation lattice $L(R[\Omega])$.

$$\theta(\Omega_1 + \Omega_2 | \Sigma) = \theta(A|ABC) = \theta(A)/\theta(ABC) = \{\overline{I, II, III, V}; \overline{IV}\},$$

$$\theta(\Omega_1 | \Sigma) = \theta(ABD|ABC) = \theta(AB)/\theta(ABC) = \{\overline{I, III}; \overline{II, V}; \overline{IV}\},$$

$$\theta(\Omega_2 | \Sigma) = \theta(ACE|ABC) = \theta(AC)/\theta(ABC) = \{\overline{I, V}; \overline{II, III}; \overline{IV}\},$$

and

$$\theta(ABC) = \{I, II, III, IV, V\},$$

where $I = \overline{1}$, $II = \overline{2, 5}$, $III = \overline{3}$, $IV = \overline{4}$, $V = \overline{6, 7}$.

It is easy to see that ABD and ACE are projectively decomposable on ABC, i.e.,

$$\begin{aligned} \theta(A|ABC) &= \theta(ABD|ABC) + \theta(ACE|ABC) \\ &= \theta(ABD|ABC) \circ \theta(ACE|ABC) \\ &= \theta(ACE|ABC) \circ \theta(ABD|ABC). \end{aligned}$$

But the MVD: $A \rightarrow\rightarrow BD$ (or $A \rightarrow\rightarrow CE$) does not hold in $R[\Omega]$. Nevertheless, the EMVD: $A \rightarrow B \mid C$ does hold in $R[\Omega]$. ■

So far we have discussed the properties of decomposition of two sets of attributes. The concept of decomposition certainly can be extended to any $n > 2$ sets of attributes. We define the notion of mutual decomposition as follows:

Definition 10: Let R be a relation on the set of attributes Ω. The sets of attributes $\Omega_1, \Omega_2, \cdots, \Omega_n \subseteq \Omega$ are mutually decomposable, if for any $I \subseteq N = \{1, \cdots, n\}$ and $J \subseteq N - I$, the two sets of attributes $\Omega_I = \bigcup_{i \in I} \Omega_i$ and $\Omega_J = \bigcup_{j \in J} \Omega_j$ are decomposable. ■

Theorem 10: Let R be a relation on the set of attributes Ω *and* $\Omega_1 \cdots \Omega_n = \Omega$. *Suppose* $\Omega_1, \cdots, \Omega_n$ *are mutually decomposable. Then*

$$R[\Omega] = R[\Omega_1 \cdots \Omega_n] = R[\Omega_1] \,|\times|\, \cdots \,|\times|\, R[\Omega_n].$$

Proof: It follows from the definition of mutual decomposition that

$$R[\Omega_1 \cdots \Omega_m] = R[\Omega_1 \cdots \Omega_{m-1}] \,|\times|\, R[\Omega_m], \qquad m = 2, \cdots n.$$

Therefore the assertion is true by induction. ■

The above theorem states that mutual decomposition implies an information-lossless join. The converse is not true in general. The necessary and sufficient condition of an information-lossless join is called join dependency, which will be discussed in the next section.

VII. JOIN DEPENDENCIES

Join dependency (JD)[10,11,14] is a generalization of MVD. It refers to a collection $\{\Omega_1, \cdots, \Omega_n\}$ of subsets of Ω such that

$$\Omega = \Omega_1 \cdots \Omega_n$$

and

$$R[\Omega] = R[\Omega_1] \,|\times|\, \cdots \,|\times|\, R[\Omega_n].$$

Join dependency can be considered as a "set of coordinates" of the relation. The connection between join dependencies and multivalued dependencies is given by the following lemma:

Lemma 10: Let $R[\Omega] = R[\Omega_1] \,|\times|\, \cdots \,|\times|\, R[\Omega_n]$, *let* N_0 *be a subset of* $\{1, \cdots, n\}$, *and let* $N_1 = \{1, \cdots, n\} - N_0$. *Then* $(\Omega_{N_0}, \Omega_{N_1})$ *is a decomposition pair, where*

$$\Omega = \Omega_1 \cdots \Omega_n, \qquad \Omega_{N_0} = \bigcup_{i \in N_0} \Omega_i, \qquad \text{and} \quad \Omega_{N_1} = \bigcup_{i \in N_1} \Omega_i.$$

Proof: Since

$$R\left[\bigcup_{i \in N_0} \Omega_i\right] \subseteq \underset{i \in N_0}{|\times|} R[\Omega_i],$$

and

$$R\left[\bigcup_{i\in N_1} \Omega_i\right] \subseteq \underset{i\in N_1}{|\times|} R[\Omega_i],$$

it follows that

$$R[\Omega_{N_0}]\,|\times|\,R[\Omega_{N_1}] \subseteq \left(\underset{i\in N_0}{|\times|} R[\Omega_i]\right) |\times| \left(\underset{i\in N_1}{|\times|} R[\Omega_i]\right).$$

Since the natural join operation is commutative and associative,[6] we have

$$R[\Omega_{N_0}]\,|\times|\,R[\Omega_{N_1}] \subseteq R[\Omega_1]\,|\times|\,\cdots\,|\times|\,R[\Omega_n] = R(\Omega).$$

But we know

$$R[\Omega] \subseteq R[\Omega_{N_0}]\,|\times|\,R[\Omega_{N_1}].$$

Hence,

$$R[\Omega] = R[\Omega_{N_0}]\,|\times|\,R[\Omega_{N_1}]. \quad \blacksquare$$

Let x be an X-value, and assume $Y \subseteq X$. We shall denote the Y-value in x as $x[Y]$. Let $t \in R[\Omega]$ be a tuple and let $\Omega = \Omega_1 \cdots \Omega_n$. The notation $t \triangleq (w_1, \cdots, w_n)$ will be used to indicate that $t[\Omega_i] = w_i$, $\forall i \in N$, where $N = \{1, \cdots, n\}$ denotes the index set.

Before we state the necessary and sufficient conditions for join dependency, we first introduce the concepts of *a set of consistent values* and an *indexed family of tuples.*

Definition 11: Let R be a relation on the set of attributes Ω, and let $\{X_i \,|\, i \in N\}$ be a collection of subsets of Ω. The set of values $\{x_i \,|\, x_i$ is an X_i-value, $i \in N\}$ is called *a set of consistent values* of $\{X_i \,|\, i \in N\}$ if the values of $X_i \cap X_j$ in x_i and x_j agree, i.e., if

$$x_i[X_i \cap X_j] = x_j[X_i \cap X_j], \qquad \forall i, j \in N.$$

The set of tuples $\{t_i \,|\, i \in N\}$ of $R[\Omega]$ is called an *indexed family of tuples* with respect to $\{X_i \,|\, i \in N\}$ if $\{x_i \,|\, t_i[X_i] = x_i,\ i \in N\}$ is a set of consistent values. $\blacksquare$

Theorem 11: Let R be a relation on the set of attributes Ω, and let $\Omega = \Omega_1 \cdots \Omega_n$. Then

$$R[\Omega] = R[\Omega_1]\,|\times|\,\cdots\,|\times|\,R[\Omega_n]$$

iff for every indexed family of tuples $\{t_i \,|\, i \in N\}$ with respect to $\{X_i \,|\, i \in N\}$ there is a tuple $t \in R[\Omega]$ such that $t[\Omega_i] = t_i[\Omega_i]$, $\forall i \in N$, where $X_i = \Omega_i \cap \hat{\Omega}_i$, and $\hat{\Omega}_i = \cup_{j\neq i}\Omega_j$.

Proof: (Necessity) Let $\{t_i \,|\, i \in N\}$ be an indexed family of tuples of $R[\Omega]$ with respect to $\{X_i \,|\, i \in N\}$. Thus, $\{x_i \,|\, t_i[X_i] = x_i,\ i \in N\}$ is a set of consistent values. Suppose $t_i[\Omega_i] = w_i$, $i \in N$. We want to show that there exists a tuple $t \triangleq (w_1, \cdots, w_n) \in R[\Omega]$. We will prove this by mathematical induction. We know that $s_1 = (w_1) \in R[\Omega_1]$ and $(w_2) \in R[\Omega_2]$. Thus,

$$w_1[X_1] = x_1 \quad \text{and} \quad w_2[X_2] = x_2.$$

Since $\{x_i \mid i \in N\}$ is consistent, it follows that

$$w_1[X_1 \cap X_2] = x_1[X_1 \cap X_2] = x_2[X_1 \cap X_2] = w_2[X_1 \cap X_2].$$

It is known that

$$X_i \cap X_j = (\Omega_i \cap \hat{\Omega}_i) \cap (\Omega_j \cap \hat{\Omega}_j) = \Omega_i \cap \Omega_j, \qquad i \neq j.$$

Therefore,

$$w_1[\Omega_1 \cap \Omega_2] = w_2[\Omega_1 \cap \Omega_2].$$

By the definition of natural join, we know that there exists a tuple $s_2 \triangleq (w_1, w_2) \in R[\Omega_1] |\times| R[\Omega_2]$.

Suppose there is a tuple $s_{n-1} \triangleq (w_1, \cdots, w_{n-1}) \in R[\Omega_1] |\times| \cdots |\times| R[\Omega_{n-1}]$. Then

$$\begin{aligned} s_{n-1}[X_i \cap X_n] &= w_i[X_i \cap X_n] = x_i[X_i \cap X_n] \\ &= x_n[X_i \cap X_n] = w_n[X_i \cap X_n], \qquad i = 1, \cdots, n-1. \end{aligned}$$

Hence,

$$\begin{aligned} s_{n-1}[\Omega_n \cap \hat{\Omega}_n] &= s_{n-1}[\Omega_n \cap (\Omega_1 \cup \cdots \cup \Omega_{n-1})] \\ &= s_{n-1}[(\Omega_1 \cap \Omega_n) \cup \cdots \cup (\Omega_{n-1} \cap \Omega_n)] \\ &= s_{n-1}[(X_1 \cap X_n) \cup \cdots \cup (X_{n-1} \cap X_n)] \\ &= w_n[(X_1 \cap X_n) \cup \cdots \cup (X_{n-1} \cap X_n)] \\ &= w_n[\Omega_n \cap \hat{\Omega}_n]. \end{aligned}$$

It follows that there exists a tuple t such that

$$t = s_n \triangleq (w_1, \cdots, w_n) \in R[\Omega_1] |\times| \cdots |\times| R[\Omega_n] = R[\Omega].$$

(Sufficiency) We know that

$$R[\Omega] \subseteq R[\Omega_1] |\times| \cdots |\times| R[\Omega_n].$$

For any $t \triangleq (w_1, \cdots, w_n) \in R[\Omega_1] |\times| \cdots |\times| R[\Omega_n]$, there exists an indexed family of tuples $\{t_i \mid t_i[\Omega_i] = w_i, i \in N\}$ of $R[\Omega]$ with respect to $\{X_i \mid i \in N\}$ that has a set of consistent values $\{x_i \mid w_i[X_i] = x_i, i \in N\}$. It follows that $t \triangleq (w_1, \cdots, w_n) \in R[\Omega]$. Hence,

$$R[\Omega] = R[\Omega_1] |\times| \cdots |\times| R[\Omega_n]. \quad \blacksquare$$

The necessary and sufficient conditions for JD given in the above theorem are similar to the notion of template dependency introduced by Sadri and Ullman.[27] The following condition can be considered as an extension of the binary natural join operation.

Corollary 3: Let R be a relation on the set of attributes Ω, *and* $\Omega = \Omega_1 \cdots \Omega_n$. *Then,*

$$R[\Omega] = R[\Omega_1] |\times| \cdots |\times| R[\Omega_n]$$

iff

$$R_{x_1,\cdots,x_n}[\Omega] = R_{x_1}[\Omega_1] |\times| \cdots |\times| R_{x_n}[X_n]$$

for every set of consistent values $\{x_i \mid i \in N\}$ of $\{X_i \mid i \in N\}$, *where* $X_i = \Omega_i \cup \hat{\Omega}_i$, $\forall i \in N$, *and* $R_{x_1,\cdots,x_n}[\Omega] = \{t \mid t \in R[\Omega],\ t[X_i] = x_i,\ \forall i \in N\}$.

Proof: The proof follows from Theorem 11. ■

Clearly, for any $t \in R[\Omega] = R[\Omega_1 \cdots \Omega_n]$, the set of values $\{x_i \mid t[X_i] = x_i,\ X_i = \Omega_i \cap \hat{\Omega}_i,\ i \in N\}$ is always consistent. The converse is not necessarily true. Suppose for any set of consistent values $\{x_i \mid x_i$ is an X_i-value, $i \in N\}$ there is a tuple $t \in R[\Omega]$ such that $t[X_i] = x_i$, $\forall i \in N$; in this case we say $\{\Omega_i \mid i \in N\}$ is *complete.*

Corollary 4: Let R be a relation on the set of attributes Ω, *and* $\Omega = \Omega_1 \cdots \Omega_n$. *Then* $\{\Omega_i \mid i \in N\}$ *is complete iff*

$$R[X_1 \cdots X_n] = R[X_1] |\times| \cdots |\times| R[X_n],$$

where $X_i = \Omega_i \cap \hat{\Omega}_i$, $i \in N$.

Proof: The proof follows directly from Theorem 11. ■

The necessary and sufficient conditions for JD may be stated in a different form, as follows:

Theorem 12: Let R be a relation on the set of attributes Ω, *and* $\Omega = \Omega_1 \cdots \Omega_n$. *Then*

$$R[\Omega] = R[\Omega_1] |\times| \cdots |\times| R[\Omega_n]$$

iff

1. $\{\Omega_i, \hat{\Omega}_i\}$ *is a decomposition pair,* $i \in N$,
2. $\{\Omega_i \mid i \in N\}$ *is complete, i.e.,*

$$R[X_1 \cdots X_n] = R[X_1] |\times| \cdots |\times| R[X_n], \quad X_i = \Omega_i \cap \hat{\Omega}_i, \quad i \in N.$$

Proof: (Necessity) Condition 1 follows from Lemma 10. Condition 2 is a consequence of Theorem 11.

(Sufficiency) We know that

$$R[\Omega] \subseteq R[\Omega_1] |\times| \cdots |\times| R[\Omega_n].$$

Suppose $t \triangleq (w_1, \cdots, w_n) \in R[\Omega_1] |\times| \cdots |\times| R[\Omega_n]$. Then there is an indexed family of tuples $\{t_i \mid t_i[\Omega_i] = w_i,\ i \in N\}$ of $R[\Omega]$ with respect to $\{X_i \mid i \in N\}$ and the set of consistent values $\{x_i \mid w_i[X_i] = x_i, X_i = \Omega_i \cap \hat{\Omega}_i, i \in N\}$. We will prove by mathematical induction that $t \triangleq (w_1, \cdots, w_n) \in R[\Omega]$.

Since $\{\Omega_i \mid i \in N\}$ is complete, there exists a tuple $s \triangleq (y_1, \cdots, y_n) \in R[\Omega]$ such that

$$s[X_i] = x_i, \qquad \forall i \in N.$$

We know that

$$t_1[\Omega_1 \cap \hat{\Omega}_1] = t_1[X_1] = w_1[X_1] = x_1 = s[X_1] = s[\Omega_1 \cap \hat{\Omega}_1],$$

which means

$$t_1\theta(\Omega_1 \cap \hat{\Omega}_1)s.$$

Since $(\Omega_1, \hat{\Omega}_1)$ is a decomposition pair, there exists a tuple $s_1 \in R[\Omega]$ such that

$$t_1\theta(\Omega_1)s_1\theta(\hat{\Omega}_1)s.$$

Hence,

$$s_1 \triangleq (w_1, y_2, \cdots, y_n) \in R[\Omega].$$

Suppose there is a tuple $s_{n-1} \triangleq (w_1, \cdots, w_{n-1}, y_n) \in R[\Omega]$. It follows that

$$t_n[\Omega_n \cap \hat{\Omega}_n] = t_n[X_n] = w_n[X_n] = x_n = s_{n-1}[X_n] = s_{n-1}[\Omega_n \cap \hat{\Omega}_n].$$

Thus there is a tuple $s_n \in R[\Omega]$ such that

$$t_n\theta(\Omega_n)s_n\theta(\hat{\Omega}_n)s_{n-1}.$$

Hence

$$t = s_n \triangleq (w_1, \cdots, w_n) \in R[\Omega]. \quad \blacksquare$$

It is known that a special class of JD, called *acyclic join dependency*, has many desirable properties; this class makes operations like updates and joins especially easy.[15, 28] A collection of subsets $\{\Omega_i \mid i \in N\}$ of the set of attributes Ω is called *acyclic* if all the attributes can be deleted by repeatedly applying the following two operations:[15,28]

1. Delete from some Ω_i an attribute A that appears in no other Ω_j
2. Delete one Ω_i if there is an Ω_j, $i \neq j$, such that $\Omega_i \subseteq \Omega_j$.

A reduction $\{Y_j \mid j \in J \subseteq N$, and $\forall i \in N - J \exists j \in J$ such that $Y_i \subseteq Y_j\}$ is obtained by removing from $\{Y_i \mid i \in N\}$ each Y_i that is contained in another Y_j.

Definition 12: Let $\mathbf{S} = \{\Omega_i \mid i \in N\}$ be a collection of subsets of Ω. The *core* of $\mathbf{S}$, denoted by $\dot{\mathbf{S}}$, is defined as follows:

1. $\dot{\mathbf{S}} = \varnothing$, for $|\mathbf{S}| = N = 1$
2. $\dot{\mathbf{S}}$ is the reduction of $\{\Omega_i \cap \hat{\Omega}_i \mid \mathrm{i} \in N\}$, for $|\mathbf{S}| = N > 1$. $\blacksquare$

There are many different but equivalent conditions that characterize a collection of subsets as acyclic.[15] We will use the following one:

Lemma 11: A collection $\mathbf{S} = \{\Omega_i \mid i \in N\}$ *of subsets of* Ω *is acyclic iff its core* $\dot{\mathbf{S}}$ *is acyclic.*

Proof: $\dot{\mathbf{S}}$ can be obtained from $\mathbf{S}$ by performing the operations 1 and 2 defined above. It follows that if $\mathbf{S}$ is acyclic then $\dot{\mathbf{S}}$ is acyclic and vice versa. ■

Corollary 5: Let $\mathbf{S} = \{\Omega_i \mid i \in N\}$ *be an acyclic collection of subsets of* Ω. *Then* $|\mathbf{S}| > |\dot{\mathbf{S}}|$.

Proof: For $|\mathbf{S}| = 1$, $|\dot{\mathbf{S}}| = |\emptyset| = 0$. For $|\mathbf{S}| \geqslant 2$, we know that $|\mathbf{S}| \geqslant |\dot{\mathbf{S}}|$. Suppose $|\mathbf{S}| = |\dot{\mathbf{S}}|$. Then any attribute A in $\dot{\mathbf{S}}$ must be contained in at least two distinct subsets of $\dot{\mathbf{S}}$. Let $A \in \Omega_i \cap \hat{\Omega}_i$. Then $A \in \Omega_i$ and $A \in \hat{\Omega}_i$. There is a $j \neq i$ such that $A \in \Omega_j$. Since $\Omega_i \subseteq \hat{\Omega}_j = \cup_{k \neq j} \Omega_k$ it follows that

$$A \in \Omega_j \cap \hat{\Omega}_j \in \dot{\mathbf{S}}.$$

Since $|\dot{\mathbf{S}}| = |\mathbf{S}| \geqslant 2$, $\dot{\mathbf{S}}$ is not empty. Now, neither operation 1 nor 2 can be applied to reduce $\dot{\mathbf{S}}$. From Lemma 11 we know this contradicts the assumption that $\mathbf{S}$ is acyclic. Thus, $|\mathbf{S}| > |\dot{\mathbf{S}}|$. ■

A JD $R[\Omega] = R[\Omega_1] |\times| \cdots |\times| R[\Omega_n]$, $\Omega = \Omega_1 \cdots \Omega_n$, is an acyclic join dependency if $\{\Omega_i \mid i \in N\}$ is acyclic. A recursive condition for acyclic join dependency is as follows:

Corollary 6: Let R be a relation on the set of attributes $\Omega = \Omega_1 \cdots \Omega_n$. *Then*

$$R[\Omega] = R[\Omega_1] |\times| \cdots |\times| R[\Omega_n]$$

is an acyclic join dependency iff

1. $(\Omega_i, \hat{\Omega}_i)$ *is a decomposition pair of* $R[\Omega]$, $i = 1, \cdots, n$,
2. $R[X_1 \cdots X_m] = R[X_1] |\times| \cdots |\times| R[X_m]$ *is an acyclic join dependency over the set* $X_1 \cdots X_m \subseteq \Omega$, *where* $\{X_i \mid i = 1, \cdots, m\}$ *is the core of* $\{\Omega_i \mid i \in N\}$.

Proof: The join dependency of a collection of sets and the join dependency of its reduction are equivalent.[25] The proof easily follows from Theorem 12 and Lemma 11. ■

The above corollary simply states that acyclic join dependency is equivalent to a set of MVDs and EMVDs, i.e., a set of simultaneous lattice equations that can be derived recursively. It has been shown by hypergraph theory that an acyclic join dependency is equivalent to a set of MVDs.[15,16] That is, the converse of Lemma 10 is true for acyclic join dependency; we will prove that the converse of Lemma 10 is a consequence of Corollary 6.

Theorem 10: Let R be a relation on the set of attributes $\Omega = \Omega_1 \cdots \Omega_n$ *such that* $\{\Omega_i \mid i \in N\}$ *is acyclic. Suppose for any* $N_0 \subseteq N = \{1, \cdots, n\}$, $N_1 = N - N_0$, *and*

$$R[\Omega] = R[\Omega_{N_0}] \,|\times|\, R[\Omega_{N_1}].$$

Then

$$R[\Omega] = R[\Omega_1] \,|\times|\, \cdots \,|\times|\, R[\Omega_n]$$

is an acyclic join dependency.

Proof: This theorem will be proved by mathematical induction on n. For the smallest nontrivial case $n = 3$, let the core set $\{X_i \mid i = 1, \cdots, m\}$ of $\{\Omega_i \mid i = 1, 2, 3\}$ be the reduction of $\{Y_i = \Omega_i \cap \hat{\Omega}_i \mid i = 1, 2, 3\}$. First we want to show that

$$R[X_1 \cdots X_m] = R[X_1] \,|\times|\, \cdots \,|\times|\, R[X_m].$$

We know $m < 3$ from Corollary 5. There is nothing to be proved if $m < 2$. For $m = 2$, without loss of generality, let $X_1 = Y_1$, $X_2 = Y_2$, and $Y_3 \subseteq Y_2 = X_2$. Then

$$X_1 \cap X_2 = Y_1 \cap Y_2 = Y_1 \cap Y_2Y_3 = (Y_1 \cap Y_2) \cup (Y_1 \cap Y_3)$$
$$= (\Omega_1 \cap \Omega_2) \cup (\Omega_1 \cap \Omega_3) = \Omega_1 \cap \Omega_2\Omega_3.$$

Since $(\Omega_1, \Omega_2\Omega_3)$ is a decomposition pair,

$$\theta(X_1 \cap X_2) = \theta(\Omega_1 \cap \Omega_2\Omega_3) \subseteq \theta(\Omega_1) \circ \theta(\Omega_2\Omega_3).$$

Also we have

$$\Omega_1 \supseteq Y_1 = X_1,$$

and

$$\Omega_2\Omega_3 \supseteq Y_2Y_3 = Y_2 = X_2.$$

Thus

$$\theta(\Omega_1) \leqq \theta(X_1),$$
$$\theta(\Omega_2\Omega_3) \leqq \theta(X_2),$$

and

$$\theta(\Omega_1) \circ \theta(\Omega_2\Omega_3) \subseteq \theta(X_1) \circ \theta(X_2).$$

It follows that

$$\theta(X_1 \cap X_2) \subseteq \theta(X_1) \circ \theta(X_2).$$

Hence

$$R[X_1X_2] = R[X_1] \,|\times|\, R[X_2].$$

It follows from Corollary 6 that

$$R[\Omega] = R[\Omega_1] \,|\times|\, R[\Omega_2] \,|\times|\, R[\Omega_3].$$

Suppose the theorem is true for all $k < n$. Let the core set $\{X_i \mid i = 1,$

$\cdots, m\}$ of $\{\Omega_i | i \in N\}$ be the reduction of $\{Y_i = \Omega_i \cap \hat{\Omega}_i | i \in N\}$. We know $m < n$ and for any $M_0 \subseteq M = \{1, \cdots, m\}$ and $M_1 = M - M_0$ there is an $N_0 \subseteq N$ and $N_1 = N - N_0$ such that

$$M_0 \subseteq N_0, \qquad M_1 \subseteq N_1,$$

and

$$X_{M_0} = Y_{N_0}, \qquad X_{M_1} = Y_{N_1}.$$

Then,

$$X_{M_0} \cap X_{M_1} = Y_{N_0} \cap Y_{N_1} = \bigcup_{i \in N_0, j \in N_1} (Y_i \cap Y_j)$$

$$= \bigcup_{i \in N_0, j \in N_1} (\Omega_i \cap \Omega_j) = \Omega_{N_0} \cap \Omega_{N_1}.$$

Since $(\Omega_{N_0}, \Omega_{N_1})$ is a decomposition pair, we have

$$\theta(X_{M_0} \cap X_{M_1}) = \theta(\Omega_{N_0} \cap \Omega_{N_1}) \subseteq \theta(\Omega_{N_0}) \circ \theta(\Omega_{N_1}).$$

Also, we know

$$\Omega_{N_0} \supseteq Y_{N_0} = X_{M_0},$$

and

$$\Omega_{N_1} \supseteq Y_{N_1} = X_{M_1}.$$

Thus

$$\theta(\Omega_{N_0}) \leqq \theta(X_{M_0}),$$

$$\theta(\Omega_{N_1}) \leqq \theta(X_{M_1}),$$

and

$$\theta(\Omega_{N_0}) \circ \theta(\Omega_{N_1}) \subseteq \theta(X_{M_0}) \circ \theta(X_{M_1}).$$

It follows that

$$\theta(X_{M_0} \cap X_{M_1}) \subseteq \theta(X_{M_0}) \circ \theta(X_{M_1}).$$

Hence

$$R[X_1 \cdots X_m] = R[X_{M_0}] |\times| R[X_{M_1}]$$

for any $M_0 \subseteq M$, $M_1 = M - M_0$.

Since the theorem is true for $m < n$, we have

$$R[X_1 \cdots X_m] = R[X_1] |\times| \cdots |\times| R[X_m].$$

It follows from Corollary 6 that

$$R[\Omega_1 \cdots \Omega_n] = R[\Omega_1] |\times| \cdots |\times| R[\Omega_n]. \quad \blacksquare$$

Further discussion of the properties of acyclic join dependencies can be found in Refs. 15 and 16. A linear-time algorithm for testing acyclicity is given in Ref. 28.

VIII. CONCLUSIONS

We have shown that lattice theory is a powerful tool in the analysis of the structure of relational database systems. Using this tool, we have established a unified theory of relations. As we have seen, almost every concept in the existing relational database theory has a counterpart in the lattice theory. This suggests that further study of relations should be carried out within the framework of lattice theory. The independency theory of lattices, which is a generalization of the familiar notion of independency in the geometries,[18,21] is especially important and relevant to the structure of relational database systems if its relation lattice is modular. This approach may lead to a geometric interpretation of data dependencies and independencies, which would make the theory more intuitive and also more useful for practical application.

The establishment of this algebraic theory of relational databases is done in the same spirit as the construction of probability theory. A probability space is a triple (Ω, Σ, P), where Ω is the sample space, Σ is a σ-algebra of the subsets of Ω, and P is a real-valued function, called a *probability measure,* defined on the σ-algebra Σ.[17,29] The notion

Table VIII—Comparison of probability theory and the theory of relational databases

Probability Theory	Theory of Relational Databases
Sample space Ω	Set of attributes Ω
Σ, the σ-Algebra of subsets of Ω	$\mathbf{2}^{\Omega}$, the Boolean algebra of subsets of Ω
Probability measure $P: \Sigma \rightarrow R[0, 1]$	Partition function $\theta: \mathbf{2}^{\Omega} \rightarrow \Pi[R(\Omega)]$
σ-additivity: $\{X_k\}$ is an denumerable union of disjoint events $P\left(\bigcup_{k=1}^{\infty} X_k\right) = \sum_{k=1}^{\infty} P(X_k)$	Meet-morphism: $\{X_k\}$ is a finite collection of sets of attributes $\theta\left(\bigcup_{k=1}^{n} X_k\right) = \theta(X_1) \cdots \theta(X_n)$
$P(\Omega) = 1$ $P(\varnothing) = 0$	$\theta(\Omega) = \mathbf{0}$ $\theta(\varnothing) = \mathbf{1}$
$0 \leqslant P(X) \leqslant 1, \forall X \in \Sigma$	$\mathbf{0} \leqq \theta(X) \leqq \mathbf{1}, \forall X \in \mathbf{2}^{\Omega}$
If $X \subseteq Y$, $P(X) \leqslant P(Y)$	If $X \supseteq Y$, $\theta(X) \leqq \theta(Y)$
If Ω_1 and Ω_2 are independent, $P(\Omega_1 \cap \Omega_2) = P(\Omega_1)P(\Omega_2)$	If Ω_1 and Ω_2 are decomposable, $\theta(\Omega_1 \cap \Omega_2) = \theta(\Omega_1) + \theta(\Omega_2)$ $= \theta(\Omega_1) \circ \theta(\Omega_2) = \theta(\Omega_2) \circ \theta(\Omega_1)$

of a σ-algebra of sets also has an abstract generalizaton, namely it is a particular case of a Boolean σ-algebra.[30] A comparison of the algebraic theory of relational databases and probability theory is shown in Table VIII.

We feel that this theory of relational databases can be used to analyze the nonquantitative aspects of data dependencies (or independencies), whereas probability theory is the basis of quantitative data analysis, namely statistics. This comparison is not meant to imply that there is a one-to-one correspondence between the theory of relational databases and the theory of probability. Nevertheless, we are convinced that the lattice theory could play a role in the theory of relational databases similar to the role measure theory plays in the theory of probability.[17]

The computational algorithms for meet and join operations of partitions are given in Ref. 31, which provides the basic tools for future development of algorithms for relations.

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APPENDIX A

Properties of Meet and Join Operations

In any lattice $(L, \cdot, +)$, the operations of meet and join satisfy the following laws:

L1—$a \cdot a = a$, $a + a = a$; (*Idempotent*)

L2—$a \cdot b = b \cdot a$, $a + b = b + a$; (*Commutative*)

L3—$a \cdot (b \cdot c) = (a \cdot b) \cdot c$,

$a + (b + c) = (a + b) + c$; (*Associative*)

L4—$a \cdot (a + b) = a + (a \cdot b) = a$; (*Absorption*)

L5—$a \leqq b$ iff $a \cdot b = a$,

$a \leqq b$ iff $a + b = b$; (*Consistency*)

L6—$b \leqq c$ implies $a \cdot b \leqq a \cdot c$

$b \leqq c$ implies $a + b \leqq a + c$; (*Isotone*)

L7—$a \cdot (b + c) \geqq (a \cdot b) + (a \cdot c)$

$a + (b \cdot c) \leqq (a + b) \cdot (a + c)$; (*Distributive Inequalities*)

L8—$a \leqq c$ implies $a + (b \cdot c) \leqq (a + b) \cdot c$. (*Modular Inequality*)

A lattice is called *distributive* if equality holds in L7 and is called *modular* if equality holds in L8. A Boolean algebra is a lattice $(L, \cdot, +, ^{-})$ with the following additional properties:[30]

L9—$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$,

$a + (b \cdot c) = (a + b) \cdot (a + c)$; (*Distributive Identities*)

L10—$a \leqq c$ implies $a + (b \cdot c) = (a + b) \cdot c$; (*Modular Identity*)

L11—L contains universal bounds $\mathbf{0}$, $\mathbf{1}$, which satisfy

$\mathbf{0} \cdot a = \mathbf{0}$, $\mathbf{0} + a = a$,

$\mathbf{1} \cdot a = a$, $\mathbf{1} + a = \mathbf{1}$;

L12—$\forall a \in L$, $\exists \bar{a} \in L$ such that

$a \cdot \bar{a} = \mathbf{0}$, $a + \bar{a} = \mathbf{1}$, $\bar{\bar{a}} = a$,

$\overline{(a \cdot b)} = \bar{a} + \bar{b}$, $\overline{(a + b)} = \bar{a} \cdot \bar{b}$.

APPENDIX B

The Proofs of Axioms for Functional and Multivalued Dependencies

The first three of the following are Armstrong's axioms for functional dependencies:[12]

B1. (Reflexivity for functional dependencies)

If $Y \subseteq X \subseteq \Omega$, then $X \rightarrow Y$.

Proof: $\theta(X) = \theta(Y(X - Y)) = \theta(Y)\theta(X - Y) \leqq \theta(Y)$. ■

B2. (Augmentation for functional dependencies)

If $X \rightarrow Y$ and $Z \subseteq \Omega$, then $XZ \rightarrow YZ$.

Proof: $\theta(XZ) = \theta(X)\theta(Z) \leqq \theta(Y)\theta(Z) = \theta(YZ)$. ■

B3. (Transitivity for functional dependencies)

If $X \rightarrow Y$ and $Y \rightarrow Z$, then $X \rightarrow Z$.

Proof: $\theta(X) \leqq \theta(Y)$ and $\theta(Y) \leqq \theta(Z)$ imply $\theta(X) \leqq \theta(Z)$. ■

The next three axioms apply to multivalued dependencies:[13]

B4. (Complementation for multivalued dependencies)

If $X \twoheadrightarrow Y$ then $X \twoheadrightarrow \Omega - X - Y$.

Proof: $\theta(X) = \theta(XY) + \theta(XZ) = \theta(XY) \circ \theta(XZ) = \theta(XZ) \circ \theta(XY)$, where $Z = \Omega - X - Y$. ■

B5. (Augmentation for multivalued dependencies)

If $X \twoheadrightarrow Y$, and $V \subseteq W$, then $WX \twoheadrightarrow VY$.

Proof: Without loss of generality,* we can let $\Omega = ABCDEFGHIJKL$, $X = ABCDEF$, $Y = BCGHFI$, $W = CDEFHIJK$, $V = EFIJ$ (see Fig. 9). Then $\Omega - X - Y = JKL$ and $\Omega - WX - VY = L$.

We want to show that

$$\theta(ABCDEF) \subseteq \theta(ABCDEFGHI) \circ \theta(ABCDEFJKL)$$

* This proof is carried out in terms of equivalence relations (partitions). It is irrelevant here whether an equivalence relation is the image of a single attribute or the image of a set of attributes.

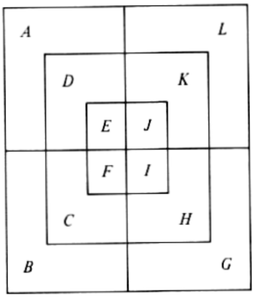

Fig. 9—Set of attributes Ω for B5.

implies

$$\theta(ABCDEFHIJK) \subseteq \theta(ABCDEFGHIJK) \circ \theta(ABCDEFHIJKL).$$

Suppose

$$t_1\theta(ABCDEFHIJK)t_2. \tag{1}$$

Then

$$t_1\theta(ABCDEF)t_2.$$

There exists t_3, such that

$$t_1\theta(ABCDEFGHI)t_3\theta(ABCDEFJKL)t_2. \tag{2}$$

From (1) and (2), we have

$$t_3\theta(JK)t_2\theta(JK)t_1.$$

It follows from (2) that

$$t_1\theta(JKG)t_3. \tag{3}$$

Combining (2) and (3), we have

$$t_1\theta(ABCDEFGHIJK)t_3. \tag{4}$$

Relation (2) also implies that

$$t_1\theta(HI)t_3.$$

From (1), we know

$$t_1\theta(HI)t_2,$$

and therefore

$$t_3\theta(HI)t_2. \tag{5}$$

It follows from (2) and (5) that

$$t_3\theta(ABCDEFHIJKL)t_2. \tag{6}$$

Combining (4) and (6), we have

$$t_1\theta(ABCDEFGHIJK)t_3\theta(ABCDEFHIJKL)t_2.$$

It follows that

$$\theta(ABCDEFHIJK) \subseteq \theta(ABCDEFGHIJK) \circ \theta(ABCDEFHIJKL). \quad \blacksquare$$

B6. (Transitivity for multivalued dependencies)

If $X \twoheadrightarrow Y$ and $Y \twoheadrightarrow Z$, then $X \twoheadrightarrow Z - Y$.

Proof: Again, without loss of generality, we can let $\Omega = ABCDEFGH$, $X = AFGH$, $Y = BCFG$, $Z = CDGH$ (see Fig. 10). Then $Z - Y = DH$, $\Omega - XY = DE$, $\Omega - YZ = AE$, $\Omega - X(Z - Y) = BCE$.
We want to show that

$$\theta(AFGH) \subseteq \theta(ADEFGH) \circ \theta(ABCFGH)$$

and

$$\theta(BCFG) \subseteq \theta(ABCEFG) \circ \theta(BCDFGH)$$

imply

$$\theta(AFGH) \subseteq \theta(ADFGH) \circ \theta(ABCEFGH).$$

Suppose $t_1\theta(AFGH)t_2$. Then there exists t_3 such that

$$t_1\theta(ADEFGH)t_3\theta(ABCFGH)t_2. \tag{7}$$

Since $t_2\theta(BCFG)t_3$, there exists t_4 such that

$$t_2\theta(ABCEFG)t_4\theta(BCDFGH)t_3. \tag{8}$$

It follows that

$$t_1\theta(AFG)t_3\theta(AFG)t_2\theta(AFG)t_4. \tag{9}$$

From (7) and (8), we have

$$t_1\theta(DH)t_3\theta(DH)t_4. \tag{10}$$

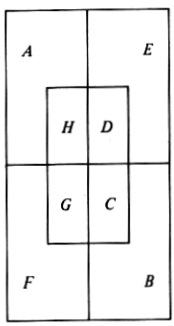

Fig. 10—Set of attributes Ω for B6.

Combining (9) and (10) yields

$$t_1\theta(ADFGH)t_4. \tag{11}$$

From (7) and (8), we have

$$t_4\theta(H)t_3\theta(H)t_2.$$

It follows from (8) that

$$t_4\theta(ABCEFGH)t_2. \tag{12}$$

Relations (11) and (12) yield

$$t_1\theta(ADFGH)t_4\theta(ABCEFGH)t_2.$$

Hence

$$\theta(AFGH) \subseteq \theta(ADFGH) \circ \theta(ABCEFGH). \quad \blacksquare$$

The last two axioms relate functional and multivalued dependencies.

B7. If $X \rightarrow Y$ then $X \rightarrow\rightarrow Y$.

Proof: Let $Z = \Omega - XY$. We want to show that

$$\theta(X) \leqq \theta(Y) \quad \text{implies} \quad \theta(X) \subseteq \theta(XY) \circ (XZ).$$

Suppose $t_1\theta(X)t_2$. Since $\theta(X) \leqq \theta(Y)$ implies $\theta(XY) = \theta(X)$, then $t_1\theta(XY)t_2$. It follows that

$$t_1\theta(XY)t_2\theta(XZ)t_2.$$

Hence

$$\theta(X) \subseteq \theta(XY) \circ \theta(XZ). \quad \blacksquare$$

B8. If $X \rightarrow\rightarrow Y$, $Z \subseteq Y$, and for some W disjoint from Y, we have $W \rightarrow Z$, then $X \rightarrow Z$.

Proof: Again, without loss of generality, we can let $\Omega = ABCDEFGH$,

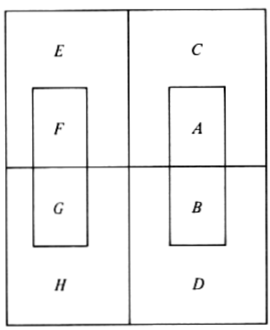

Fig. 11—Set of attributes Ω for B8.

$X = ACEF$, $Y = EFGH$, $Z = FG$, and $W = AB$ (see Fig. 11). Then $\Omega - XY = BD$.
We want to show that

$$\theta(ACEF) \subseteq \theta(ACEFGH) \circ \theta(ABCDEF)$$

and

$$\theta(AB) \leqq \theta(FG)$$

imply

$$\theta(ACEF) \leqq \theta(FG).$$

Suppose $t_1\theta(ACEF)T_2$. Then there exists t_3 such that

$$t_1\theta(ACEFGH)t_3\theta(ABCDEF)t_2.$$

Since

$$t_1\theta(FG)t_3 \quad \text{and} \quad t_3\theta(AB)t_2,$$

we have

$$t_1\theta(FG)t_3 \quad \text{and} \quad t_3\theta(FG)t_2,$$

and thus

$$t_1\theta(FG)t_2.$$

Hence

$$\theta(ACEF) \leqq \theta(FG). \quad \blacksquare$$

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