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Bandwidth-Conserving Independent Amplitude and Phase Modulation

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Given two baseband signals $f(t)$ and $g(t)$, suitably restricted in amplitude and bandlimited to $[\lambda, \mu]$ and $[-\mu, -\lambda]$, $0 < \lambda < \mu < \infty$, it is shown how to generate a carrier signal, $s(t) = A(t) \cos\{ct + \phi(t)\}$, bandlimited to $[c - \beta, c + \beta]$ and $[-(c + \beta), -(c - \beta)]$, where β need be only sightly larger than μ, and such that $f(t)$ and $g(t)$ may be recovered by bandlimiting $\log A(t)$ and $\phi(t)$, respectively. The restriction $\lambda > 0$, i.e., that the baseband signals be bandpass, is not essential but it is a practical constraint in approximating the required operations. Also a modification is given for conserving bandwidth in case the signals $f(t)$ and $g(t)$ are of disparate bandwidths.

I. INTRODUCTION

Double-sideband amplitude modulation is wasteful of bandwidth, but it offers the advantage of envelope detection (with full carrier). A simple way to utilize the same bandwidth in transmitting two independent signals, $f(t)$ and $g(t)$, is the so-called in-phase and quadrature modulation

$$S_1(t) = f(t)\cos ct - g(t)\sin ct,$$

where synchronous demodulation is required to recover f and g. A modification that allows f to be recovered (approximately) by envelope detection is

$$S_2(t) = \{1 + f(t)\}\cos ct - g(t)\sin ct.$$

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The envelope of $S_2(t)$ is

$$A_2(t) = \sqrt{\{1 + f(t)\}^2 + g^2(t)}.$$

Then if g is made small (compared to $\min\{1 + f(t)\}$), we have

$$A_2(t) \cong 1 + f(t).$$

The phase of S_2 (i.e., the part due to signals) is

$$\phi_2(t) = \tan^{-1} \frac{g(t)}{1 + f(t)} \cong \frac{g(t)}{1 + f(t)}.$$

So making g small allows envelope and phase detection to be used so as to approximately recover f and g (multiplying the phase output by the envelope output).

A still further modification is

$$S_3(t) = \{1 + x_1(t)\}\cos ct - \{1 + x_2(t)\}\sin ct,$$

where x_1 and x_2 are both small. The envelope of S_3 is

$$A_3(t) = \sqrt{(1 + x_1)^2 + (1 + x_2)^2} = \sqrt{2 + 2x_1 + 2x_2 + x_1^2 + x_2^2}$$
$$\cong \sqrt{2}\left(1 + \frac{x_1 + x_2}{2}\right).$$

The phase of S_3 is

$$\phi_3(t) = \tan^{-1} \frac{1 + x_2}{1 + x_1} \cong \frac{\pi}{4} + \frac{x_2 - x_1}{2}.$$

So if

$$\frac{x_1 + x_2}{2} = f \quad \text{and} \quad \frac{x_1 - x_2}{2} = g,$$

i.e.,

$$x_1 = f + g$$
$$x_2 = f - g,$$

then envelope and phase detection of S_3 will give (approx.) the desired independent signals f and g.

An exact result of this type may be obtained using log of the envelope, rather than the envelope, and then *bandlimiting* the phase and log of the envelope to obtain the desired independent signals f and g. A slight increase in bandwidth is required to allow a guard band in the filtering operations. Also $|f|$ and $|g|$ cannot be too large if the increase in bandwidth is to be small. The basic theory is that of

Exponential Single-Sideband Modulation (ESSB) developed in Ref. 1.

II. THE EXACT METHOD

We assume that the desired signals, f and g, are bounded band-pass signals whose Fourier transforms vanish (in the sense detailed in Ref. 1) outside $[\lambda, \mu]$ and $[-\mu, -\lambda]$, $0 < \lambda < \mu < \infty$, which then (automatically) have bounded Hilbert transforms, $\hat{f}$ and $\hat{g}$. The band-pass assumption is not essential to the theory, but affords important practical simplifications in approximating the Hilbert transform operations as well as in effecting the subsequent analytic exponential modulation.

Now suppose $z_1(t)$ and $z_2(t)$ are bandlimited "analytic signals" whose Fourier transforms vanish outside $[0, \beta]$ and which signals are zero-free in the upper half-plane with

$$|z_{1,2}(t + iu)| \geqslant \epsilon \quad \text{for} \quad u \geqslant 0, \qquad -\infty < t < \infty. \tag{1}$$

Then $\log z_1$ and $\log z_2$ are analytic and bounded in the upper half-plane, and hence their Fourier transforms vanish over $(-\infty, 0)$.

Writing

$$z_1(t) = A_1(t)e^{i\phi_1(t)}, \qquad A_1 = |z_1| \tag{2}$$

$$z_2(t) = A_2(t)e^{i\phi_2(t)}, \qquad A_2 = |z_2|, \tag{3}$$

we have

$$\log z_1(t) = \log A_1(t) + i\phi_1(t) \tag{4}$$

$$\log z_2(t) = \log A_2(t) + i\phi_2(t). \tag{5}$$

Under further assumptions on z, e.g.,

$$z_{1,2}(t + iu) = 1 + 0(e^{-\lambda u}), \qquad u \to \infty, \tag{6}$$

$\log A$ and ϕ will be Hilbert transform pairs:

$$\phi_1(t) = \log\hat{\ }\ A_1(t), \qquad \log A_1(t) = -\hat{\phi}_1(t) \tag{7}$$

$$\phi_2(t) = \log\hat{\ }\ A_2(t) \qquad \log A_2(t) = -\hat{\phi}_2(t). \tag{8}$$

Now we consider the product

$$z_1(t)\overline{z_2(t)} = A_1(t)A_2(t)e^{i[\phi_1(t)-\phi_2(t)]},$$

where the bar denotes the complex conjugate. The F.T. (Fourier transform) of $z_1(t)$ vanishes outside $[0, \beta]$ and the F.T. of $\overline{z_2(t)}$ vanishes outside $[-\beta, 0]$. Therefore, the F.T. of $z_1(t)\overline{z_2(t)}$ vanishes outside $[-\beta, \beta]$. Then we form the signal

$$s(t) = \text{Re } e^{ict} z_1(t)\overline{z_2(t)}$$
$$= A(t)\cos\{ct + \phi(t)\}, \tag{9}$$

where $c > \beta$,

$$A(t) = A_1(t)A_2(t) \tag{9a}$$

$$\phi(t) = \phi_1(t) - \phi_2(t), \tag{9b}$$

and the spectrum of $s(t)$ is confined to $[c - \beta, c + \beta]$ and $[-c - \beta, -c + \beta]$.

We require

$$\mathbf{B}_{\mu,\alpha}\{\log A(t)\} = f(t) \tag{10}$$

$$\mathbf{B}_{\mu,\alpha}\{\phi(t)\} = g(t), \tag{11}$$

where $\mu < \alpha < \beta$, and, in general, $\mathbf{B}_{p,q}$ is any bandlimiting operator [with passband $(-p, p)$ and cut-off frequency $\pm q$] defined by

$$\mathbf{B}_{p,q}\{x(t)\} = \int_{-\infty}^{\infty} x(s)K_{p,q}(t - s)ds \tag{12}$$

and

$$\overset{>}{K}_{p,q}(\omega) = \int_{-\infty}^{\infty} K_{p,q}(t)e^{-i\omega t}dt = 1, \qquad -p < \omega < p$$
$$= 0, \qquad |\omega| > q. \tag{12a}$$

$$0 < p < q < \infty. \tag{12b}$$

The definition of $\overset{>}{K}_{p,q}(\omega)$ in the cut-off region (p, q) and $(-q, -p)$ is not important, but $\overset{>}{K}_{p,q}(\omega)$ must be sufficiently smooth to give

$$\int_{-\infty}^{\infty} |K_{p,q}(t)|\, dt < \infty \tag{12c}$$

so that $\mathbf{B}_{p,q}\{x(t)\}$ is defined for any bounded $x(t)$.

Writing (10) as

$$\mathbf{B}_{\mu,\alpha}\{\log A_1(t) + \log A_2(t)\} = f(t)$$

and taking Hilbert transforms of both sides of (10) and (11), using (7) and (8), we have

$$\mathbf{B}_{\mu,\alpha}\{\phi_1(t) + \phi_2(t)\} = \hat{f}(t) \tag{13}$$

$$\mathbf{B}_{\mu,\alpha}\{\phi_1(t) - \phi_2(t)\} = g(t) \tag{14}$$

or

$$\mathbf{B}_{\mu,\alpha}\{\phi_1(t)\} = \tfrac{1}{2}\{\hat{f}(t) + g(t)\} \tag{15}$$

$$\mathbf{B}_{\mu,\alpha}\{\phi_2(t)\} = \tfrac{1}{2}\{\hat{f}(t) - g(t)\}, \tag{16}$$

which according to (7) and (8) is equivalent to

$$\mathbf{B}_{\mu,\alpha}\{\log A_1(t)\} = \tfrac{1}{2}\{f(t) - \hat{g}(t)\} \tag{17}$$

$$\mathbf{B}_{\mu,\alpha}\{\log A_2(t)\} = \tfrac{1}{2}\{f(t) + \hat{g}(t)\}. \tag{18}$$

Setting

$$h_1(t) = \tfrac{1}{2}\{f(t) - \hat{g}(t)\}; \qquad \hat{h}_1(t) = \tfrac{1}{2}\{\hat{f}(t) + g(t)\}, \tag{19}$$

$$H_1(t) = h_1(t) + i\hat{h}_1(t), \tag{19a}$$

$$h_2(t) = \tfrac{1}{2}\{f(t) + \hat{g}(t)\}; \qquad \hat{h}_2(t) = \tfrac{1}{2}\{\hat{f}(t) - g(t)\}, \tag{20}$$

$$H_2(t) = h_2(t) + i\hat{h}_2(t), \tag{20a}$$

the four equations (15), (16), (17), and (18) are equivalent to the two equations, implying (6),

$$\mathbf{B}_{\mu,\alpha}\{\log z_1(t)\} = H_1(t) \tag{21}$$

$$\mathbf{B}_{\mu,\alpha}\{\log z_2(t)\} = H_2(t), \tag{22}$$

where H_1 and H_2 are given "analytic" band-pass signals whose Fourier transforms vanish outside the single interval $[\lambda, \mu]$ and z_1 and z_2 are bandlimited "analytic" signals whose Fourier transforms vanish outside the single interval $[0, \beta]$. The problem of finding z_1 and z_2 has been solved (see Ref. 1):

$$z_1(t) = \mathbf{B}_{\alpha,\beta}\{\exp H_1(t)\} \tag{23}$$

$$z_2(t) = \mathbf{B}_{\alpha,\beta}\{\exp H_2(t)\}, \tag{24}$$

where $\mathbf{B}_{\alpha,\beta}$ is any bandlimiting operator with passband $(-\alpha, \alpha)$ and cut-off frequency $\pm\beta$.

Now z_1 and z_2 given by (23) and (24) satisfy (21) and (22), *provided* $z_1(\tau)$ and $z_2(\tau)$, $\tau = t + iu$, are zero free in the upper half-plane $u > 0$. The filter characteristic $\overset{>}{K}_{\alpha,\beta}(\omega)$ in the cut-off region (α, β) becomes important, but not critical, in this respect. From theoretical considerations the linear cut-off characteristic is desirable (see Ref. 1):

$$\overset{>}{K}_{\alpha,\beta}(\omega) = \frac{\beta - \omega}{\beta - \alpha}, \qquad \alpha < \omega < \beta. \tag{25}$$

For a given α and β, and a smooth cut-off characteristic, z_1 and z_2 will be zero free in the upper half-plane provided $|H_1|$ and $|H_2|$ are not too large. In practice this means that the levels of f and g must not be too large if α and β are not much larger than μ, the top signal frequency, i.e., in the bandwidth conserving case. The results in Ref. 1 may be used as a rough guide. For example, if α is only slightly larger than μ and $\beta/\alpha = 1.1$ (relatively sharp cut-off), then z_1 and z_2 will be zero free in the upper half-plane if $|H_1|$ and $|H_2|$ are less than 0.6. (See the appendix for a modification of signals f and g of disparate bandwidths.)

III. IMPLEMENTATION

The block diagram of an implementation is shown in Fig. 1. The transmitter is shown in Fig. 1a. The inputs are labeled $f(t + T)$ and $g(t + T)$ to account for a delay T incurred in the Hilbert transform filters. The delay T need not be more than one or two periods of the lower signal frequency λ to obtain a good approximation to the Hilbert transforms, $\hat{f}(t)$ and $\hat{g}(t)$. (The inputs $f(t + T)$ and $g(t + T)$ must be delayed accordingly to obtain $f(t)$ and $g(t)$.) The signals $f(t)$, $\hat{f}(t)$, $g(t)$, and $\hat{g}(t)$ are summed to obtain

$$h_1 = \tfrac{1}{2}(f - \hat{g})$$

$$\hat{h}_1 = \tfrac{1}{2}(\hat{f} + g)$$

$$h_2 = \tfrac{1}{2}(f + \hat{g})$$

$$\hat{h}_2 = \tfrac{1}{2}(\hat{f} - g)$$

in accord with (19) and (20). (The gain factor of the summing networks, shown as 1/2, may be any constant, which may be simply reflected as a gain factor on the inputs.) Then these outputs are fed to two analytic exponential modulators that furnish outputs

$$X_1 = e^{h_1}\cos \hat{h}_1 = \mathrm{Re}\{\exp H_1\}$$

$$Y_1 = e^{h_1}\sin \hat{h}_1 = \mathrm{Im}\{\exp H_1\}$$

$$X_2 = e^{h_2}\cos \hat{h}_2 = \mathrm{Re}\{\exp H_2\}$$

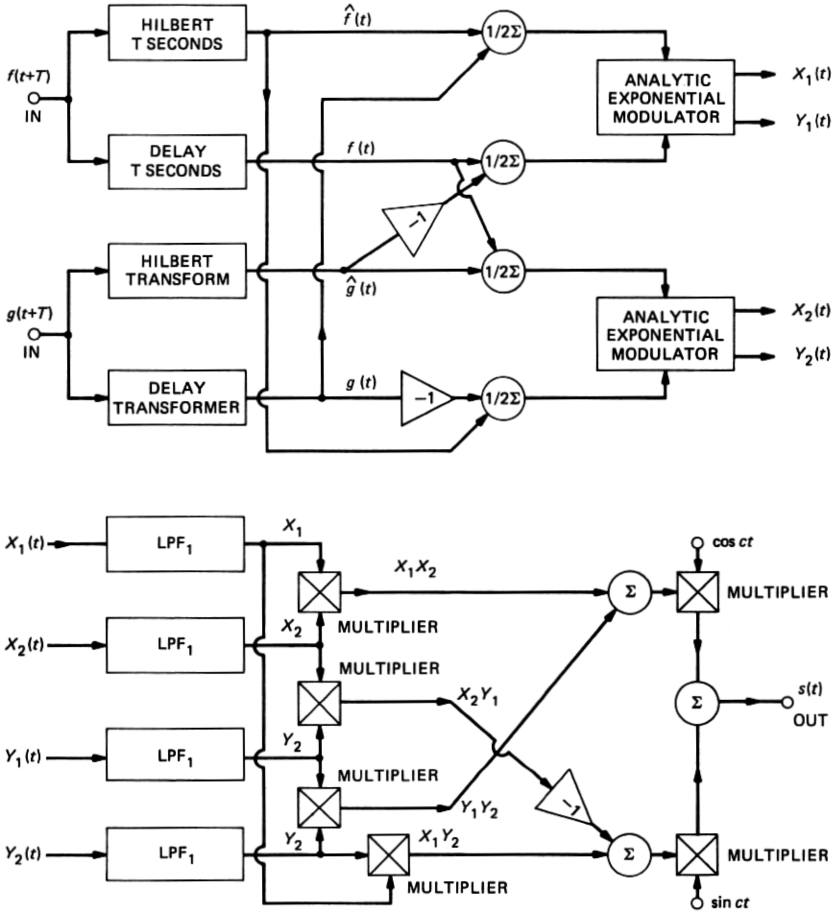
$$Y_2 = e^{h_2}\sin \hat{h}_2 = \mathrm{Im}\{\exp H_2\}.$$

A feedback circuit for accomplishing the analytic exponential modulation is described in Ref. 2. The outputs of the modulators are then bandlimited with identical low-pass filters LPF_1 having the characteristic shown in Fig. 2a to obtain

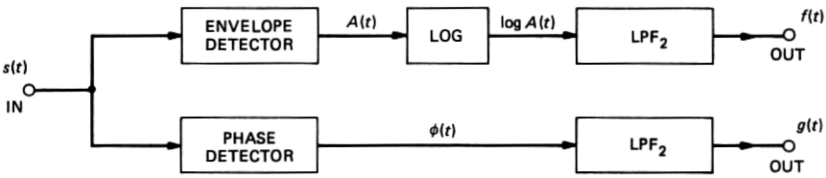
$$x_1 = \mathrm{Re}\{z_1\}, \qquad y_1 = \mathrm{Im}\{z_1\}$$

$$x_2 = \mathrm{Re}\{z_2\}, \qquad y_2 = \mathrm{Im}\{z_2\}.$$

These outputs are then combined to form



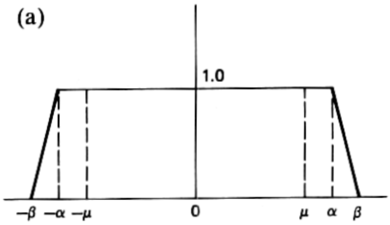
(a)



(b)

Fig. 1a—Transmitter.

Fig. 1b—Receiver.



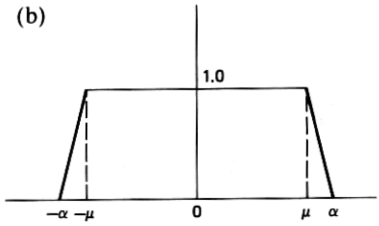


Fig. 2a—Characteristic of LPF_1.

Fig. 2b—Characteristic of LPF_2.

$$
\begin{aligned}
s(t) &= \mathrm{Re}\{[x_1(t) + iy_1(t)][x_2(t) - iy_2(t)]e^{ict}\} \\
&= \mathrm{Re}\{[x_1x_2 + y_1y_2 + i(y_1x_2 - y_2x_1)]e^{ict}\} \\
&= (x_1x_2 + y_1y_2)\cos ct - (y_1x_2 - y_2x_1)\sin ct \\
&= A(t)\cos\{ct + \phi(t)\}.
\end{aligned}
$$

The signal $s(t)$ is then transmitted to the receiver, Fig. 1b, where an envelope detector is used to obtain the envelope $A(t)$, which is then fed to a device having a logarithmic characteristic to furnish the output $\log A(t)$. This output is then filtered with LPF_2 to obtain $f(t)$. A phase detector, e.g., a phase-locked loop, is used to detect the phase $\phi(t)$, which is subsequently filtered with another LPF_2 to obtain $g(t)$. The characteristic of the filters LPF_2 is shown in Fig. 2b.

Note that $\phi(t)$ is high pass with lower frequency λ; so $\phi(t)$ may be recovered from $\{\phi'(t) + c\}$, if desired.

REFERENCES

1. B. F. Logan, "Theory of Analytic Modulation Systems," B.S.T.J., *57*, No. 3 (March 1978), pp. 491–576.
2. B. F. Logan, "Click Modulation," unpublished work.

APPENDIX

Modification for Signals of Disparate Bandwidths

Note that the bandwidth of the transmitted signal is the sum (or twice the sum, counting positive and negative frequencies) of the bandwidths of the analytic signals $z_1(t)$ and $z_2(t)$, which need be only slightly larger than the sum of the bandwidths of the analytic signals $H_1(t)$ and $H_2(t)$. Owing to the linear combinations in (19) and (20), the bandwidths of $H_1(t)$ and $H_2(t)$ will be the same, equal to the larger of the bandwidths of $f(t)$ and $g(t)$. In case the bandwidth of, say, $g(t)$ is (considerably) larger than that of $f(t)$, the bandwidth of the transmitted signal may be reduced by setting

$$H_1(t) = f(t) + i\hat{f}(t) \tag{26}$$

$$H_2(t) = g(t) + i\hat{g}(t). \tag{27}$$

Here we assume that the Fourier transforms of $H_1(t)$ and $H_2(t)$ vanish outside the intervals $[\lambda, \mu_1]$ and $[\lambda_2, \mu_2]$, respectively. Now we set

$$z_1(t) = \mathbf{B}_{\alpha_1,\beta_1}\{\exp H_1(t)\}, \qquad \mu_1 < \alpha_1 < \beta_1 \tag{28}$$

$$z_2(t) = \mathbf{B}_{\alpha_2,\beta_2}\{\exp H_2(t)\}, \qquad \mu_2 < \alpha_2 < \beta_2, \tag{29}$$

where β_1 and β_2 need be only slightly larger than μ_1 and μ_2 (respectively), the top frequencies of $f(t)$ and $g(t)$ (respectively). The Fourier transform of $z_1(t)\overline{z_2(t)}$ now vanishes outside the interval $[-\beta_2, \beta_1]$, which is smaller than would obtain in the previous scheme. Thus the Fourier transform of the transmitted signal,

$$s(t) = \text{Re } e^{ict}z_1(t)\overline{z_2(t)}, \qquad c > \beta_2 \tag{30}$$

vanishes outside the interval $[c - \beta_2, c + \beta_1]$ (and its reflection about the origin). The price paid for the saving in bandwidth is another Hilbert transform operation required in separating the signals at the receiver.

We have

$$s(t) = A(t)\cos[ct + \phi(t)], \tag{31}$$

where

$$A(t) = |z_1(t)z_2(t)|$$

$$\phi(t) = \phi_1(t) - \phi_2(t).$$

Then, assuming as before that z_1 and z_2 are zero free in the upper half-plane, we have

$$L(t) = \log A(t) = L_1(t) + L_2(t), \tag{32}$$

where $L_1(t) = \log|z_1(t)|$, $L_2(t) = \log|z_2(t)|$ and $L(t)$ is related to $\phi(t)$ by

$$\phi(t) = \phi_1(t) - \phi_2(t) = \hat{L}_1(t) - \hat{L}_2(t) \tag{33}$$

$$\hat{\phi}(t) = \hat{\phi}_1(t) - \hat{\phi}_2(t) = -L_1(t) + L_2(t). \tag{34}$$

In accord with (28) and (29) and the zero-free hypothesis, we have (as shown in Ref. 1)

$$\mathbf{B}_{\mu_1,\alpha_1}\{L_1(t)\} = f(t) \tag{35}$$

$$\mathbf{B}_{\mu_2,\alpha_2}\{L_2(t)\} = g(t) \tag{36}$$

To obtain $L_1(t)$ and $L_2(t)$ from $\log A(t)$ and $\phi(t)$, we need the Hilbert transform $\hat{\phi}(t)$, where according to (32) and (34),

$$L_1(t) = \tfrac{1}{2} \log A(t) - \tfrac{1}{2}\hat{\phi}(t) \tag{37}$$

$$L_2(t) = \tfrac{1}{2} \log A(t) + \tfrac{1}{2}\hat{\phi}(t). \tag{38}$$

However, to recover $f(t)$ and $g(t)$, we may use a modified version of $\hat{\phi}(t)$. We define

$$\hat{\phi}_{\lambda,\mu}(t) = \mathbf{H}_{\lambda,\mu}\{\phi(t)\}, \tag{39}$$

where $\mathbf{H}_{\lambda,\mu}$ is a modified (e.g., band-pass) Hilbert transform operator defined by

$$\mathbf{H}_{\lambda,\mu}\{\phi(t)\} = \int_{-\infty}^{\infty} h_{\lambda,\mu}(t - x)\phi(x)dx \tag{40a}$$

$$h_{\lambda,\mu}(\omega) = \int_{-\infty}^{\infty} h_{\lambda,\mu}(t)e^{i\omega t}dt = -i \operatorname{sgn} \omega, \tag{40b}$$

$$\text{for} \quad 0 < \lambda \leqslant |\omega| \leqslant \mu.$$

Now $\phi_1(t)$ and $\phi_2(t)$ are high-pass functions with lower frequencies λ_1 and λ_2 (respectively). Thus, if we require

$$0 < \lambda \leqslant \min(\lambda_1, \lambda_2), \qquad \mu \geqslant \max(\mu_1, \mu_2), \tag{41}$$

then we have

$$f(t) = \tfrac{1}{2}\mathbf{B}_{\mu_1,\alpha_1}\{\log A(t) - \hat{\phi}_{\lambda,\mu}(t)\} \tag{42}$$

$$g(t) = \tfrac{1}{2}\mathbf{B}_{\mu_2,\alpha_2}\{\log A(t) + \hat{\phi}_{\lambda,\mu}(t)\}. \tag{43}$$

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