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Series Solutions of Companding Problems

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A formal power series solution (i) $x(t) = \sum_1^\infty m^k x_k(t)$ is given for the companding problem (ii) $Bf\{x(t)\} = my(t)$, $B\{x(t)\} = x(t)$, where B is the bandlimiting operator defined by $Bg = (Bg)(t) = \int_{-\infty}^{\infty} g(s)[\sin \lambda(t-s)]/[\pi(t-s)]ds$ and $f(t)$ has a Taylor series with $f(0) = 0$, $f'(0) \neq 0$. Expressions for the x_k are given in terms of the coefficients of f, and operations on y, and in a different form in terms of the coefficients of the inverse function ϕ, $\phi\{f(x)\} = x$. A series development is given for a bandlimited $z(t)$, $Bz = z$, such that the solution of (ii) is given by $x = B\phi(z)$. Also a series development is given for the "approximate identity", $x \doteq B\phi\{Bf(x)\}$, where $x = x(t)$, $Bx = x$, which is shown to be a good approximation to x for fairly linear $f(x)$, not necessarily having a Taylor series expansion. As an example of one application of the results, a few terms are given for correction of the "inband" distortion arising in envelope detection of "full-carrier" single-sideband signals. The results should prove useful in correcting small distortions in other transmission systems. Finally, it is shown that the formal series solution (*i*) actually converges for sufficiently small $|m|$. This involves proving that the companding problem (*ii*) has a unique solution for arbitrary complex-valued $y(t)$ and complex m of sufficiently small magnitude, the solution $x(t; m)$ being, for each t, an analytic function of the complex variable m in a neighborhood of the origin. It is a curious fact, as shown by an interesting example, that the series (*i*) may converge for values of m for which it is not a solution of (*ii*).

I. INTRODUCTION

Suppose $x(t)$ is a bandlimited signal whose Fourier transform vanishes outside the interval $[-\lambda, \lambda]$. If such a signal is instantaneously

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distorted by a nonlinear (companding) function $f(x)$, the distorted signal $f\{x(t)\}$ will, in general, have frequency components outside the interval $[-\lambda, \lambda]$. If the out-of-band components of the distorted signal are removed by ideal low-pass filtering, the result is a bandlimited signal $y(t)$ whose Fourier transform agrees with that of $f\{x(t)\}$ over $(-\lambda, \lambda)$. How, and under what conditions, may $x(t)$ be recovered from $y(t)$? When the signals are real-valued, this is known as the *companding problem* of Landau and Miranker (Refs. 1 and 2), hereafter referred to as the *r.v. companding problem.* Before stating their result, and our purpose, we introduce some notation.

The symbol $\mathscr{B}_2(\lambda)$ will denote the subspace of $L_2 = L_2(-\infty, \infty)$ whose elements are those (square-integrable) functions whose Fourier transforms vanish outside $[-\lambda, \lambda]$. Associated with this subspace is the *bandlimiting operator* B_λ, defined for g in L_2 by

$$B_\lambda g = (B_\lambda g)(t) = \int_{-\infty}^{\infty} g(s)\, \frac{\sin \lambda(t-s)}{\pi(t-s)}\, ds.$$

The Fourier transforms of g and $B_\lambda g$ agree over $(-\lambda, \lambda)$, the transform of the latter vanishing outside $[-\lambda, \lambda]$. In the language of Hilbert space, $B_\lambda g$ is the projection of g on $\mathscr{B}_2(\lambda)$, being the best approximation to g in the subspace $\mathscr{B}_2(\lambda)$. In case g belongs to $\mathscr{B}_2(\lambda)$, we have

$$B_\lambda g = g.$$

It follows that

$$B_\lambda^n g = B_\lambda g, \qquad g \text{ in } L_2, \qquad n = 1, 2, \cdots .$$

The operator B_λ may be applied also to functions belonging to L_p, $1 \leqslant p < \infty$; i.e., to functions g satisfying

$$\|g\|_p = \left\{\int_{-\infty}^{\infty} |g(t)|^p dt\right\}^{1/p} < \infty \; (1 \leqslant p < \infty).$$

Here the notation $\|g\|_p$ designates the norm of g in L_p, or simply the L_p-norm of g. The space L_∞ consists of those functions g whose magnitude is bounded on the real line, their norm $\|g\|_\infty$ being the "essential supremum" of $|g(t)|$, which for functions we will be dealing with here, is simply the maximum value of $|g(t)|$. The operator B_λ may not be applied to an arbitrary bounded function, since the associated integral may not converge. However, the integral may converge conditionally for a large class of functions; in particular, $B_\lambda g = g$, for any constant function g.

The operator B_λ is a "contraction" operator on L_2; i.e.,

$$\|B_\lambda g\|_2 \leqslant \|g\|_2,$$

with equality attaining only for g in $\mathscr{B}_2(\lambda)$. This follows from Parseval's theorem and the definition of B_λ.

Applying Schwarz's inequality to the integral equation denoted by $B_\lambda g = g$, we obtain the useful inequality

$$\|g\|_\infty \leqslant \sqrt{\lambda/\pi}\ \|g\|_2, \qquad g \text{ in } \mathscr{B}_2(\lambda).$$

Also, it is easy to show from the integral equation that

$$\lim_{t\to\pm\infty} g(t) = 0, \qquad g \text{ in } \mathscr{B}_2(\lambda).$$

We shall also make use of the *high-pass operator* H_λ, defined by

$$H_\lambda = I - B_\lambda,$$

where I is the identity operator. H_λ is an identity operator for functions h of L_2 whose Fourier transforms vanish over $(-\lambda, \lambda)$, and it is also a contraction operator on L_2,

$$\|H_\lambda g\|_2 \leqslant \|g\|_2,$$

with equality attaining only for $H_\lambda g = g$, i.e., for $B_\lambda g = 0$. (In these operational equations, 0 is interpreted as the null function.)

It is clear from the operator definitions and the associated Fourier transform relations that any function f in L_2 has the decomposition

$$f = g + h,$$

where

$$g = B_\lambda f, \qquad h = H_\lambda f.$$

Since λ will be fixed throughout the paper, we will, except where emphasis is desired, simply write B, H, and $\mathscr{B}_2$ for B_λ, H_λ, $\mathscr{B}_2(\lambda)$, respectively.

Now, using our notation, we may state the important result of Landau and Miranker as follows:

Theorem (Landau and Miranker): *Let $f(x)$ be a real-valued function of the real variable x, satisfying*

(*i*) $f(0) = 0$

(*ii*) $0 < m_1 \leqslant f'(x) \leqslant m_2 < \infty, \qquad (-\infty < x < \infty).$

Then to each real-valued y in $\mathscr{B}_2$ there corresponds a unique x in $\mathscr{B}_2$, also real-valued, satisfying

(*iii*) $Bf(x) = y.$

The solution x of (iii) may be obtained as the limit of the sequence of approximants $\{x_n\}$ defined iteratively by

(*iv*) $x_{n+1} = x_n - cB\{f(x_n) - y\},$

provided only that x_1 is a real-valued function in $\mathscr{B}_2$ and the real constant c is so chosen that

(v) $\max_x |1 - cf'(x)| \leq r < 1.$

The beauty of this result is that, under the hypotheses on f, every (r.v.) y in $\mathscr{B}_2$ has the representation (*iii*) where x is a unique (r.v.) function in $\mathscr{B}_2$. In some r.v. companding problems of interest, $f(x)$ may not be defined outside some interval and/or the condition on $f'(x)$ may not be satisfied over the whole real axis, but rather over some interval including the origin. Then the conclusion will apply only to y of sufficiently small norm. In such cases, the companding problem has two essentially different interpretations. The first is the *recovery problem*: y is known to be of the form (*iii*); recover x. The second is the *design problem*: y is a prescribed (desired) signal; find x, if possible, so that y is given by (*iii*). In this case, one is faced with the problem of determining for what y the problem has a solution.

The speed of covergence of the iterative solution of Landau and Miranker is a matter of practical concern. They show that

$$\| x_{n+1} - x_n \|_2 \leq r \| x_n - x_{n-1} \|_2.$$

Then the constant c in (v) should be chosen to make r as small as possible. Assuming that equality may attain on both sides in (*ii*), one should choose

$$c = \frac{2}{m_1 + m_2}, \qquad \text{giving} \quad r = \frac{m_2 - m_1}{m_2 + m_1}.$$

Thus rapid convergence is assured if (m_2/m_1) is not much larger than 1. If this is not the case, a large number of iterations are, in general, required to obtain a close approximation to the solution of the problem. In a practical implementation of the iterative scheme of solution (Ref. 1), the ideal bandlimiting operator is replaced by an approximate operator, incurring a certain delay, in addition to (eventually) significant spectral distortions, with the result that the sequence $\{x_n\}$ will not converge to the solution x. Thus, in practice, the number of iterations to be performed is limited both by practical and theoretical considerations. The conclusion is that good approximate solutions to companding problems may be conveniently obtained in practice only in those cases where the companding function $f(x)$ is fairly linear over the range of $x(t)$.

We should remark at this point that there is only one known (nonlinear) r.v. companding problem (see Ref. 3) admitting of an explicit noniterative solution; viz.,

$$B\{\log(1 + x)\} = y, \qquad x > -1, \qquad x \text{ in } \mathscr{B}_2,$$

which has a solution if, and only if, the function†

† Here we are applying the B operator to a function not in L_2, the proper interpretation being $w = 1 + B\{-1 + \exp(\cdot)\}$.

$$w(t) = B\{\exp 1/2[y(t) + \hat{y}(t)]\},$$

where $\hat{y}$ is the Hilbert transform of y, extends as a function zero-free in the upper half-plane, which will be the case if $\| y \|_2$ is sufficiently small. Then the solution is given by

$$x(t) = | w(t) |^2 - 1.$$

Motivated by the above considerations, pure curiosity, and the fact that in many cases of practical interest the companding function and/or its inverse can be well approximated by a polynomial of low degree over the range of interest, we are led to consider the case where the companding function has a Taylor series expansion, allowing the possibility of developing a corresponding series solution to the problem. To obtain the terms (1st order, 2nd order, etc.) in the series solution it is convenient to multiply y by a scalar parameter m, and consider the problem

$$Bf(x) = my \tag{1}$$

to be solved for x in $\mathscr{B}_2$, given y in $\mathscr{B}_2$, for companding functions

$$f(x) = \sum_1^\infty b_k x^k, \qquad |x| < R_0 \tag{2}$$

$$b_1 \neq 0.$$

For sufficiently small $|x|$, f will have an inverse ϕ,

$$x = \phi\{f(x)\}$$

$$\phi(y) = \sum_1^\infty a_k y^k, \qquad |y| \leqslant R_0^*. \tag{3}$$

We assume that the solution $x = x(t; m)$ of (1) has a series expansion in the parameter m,

$$x(t; m) = \sum_1^\infty m^k x_k(t), \tag{4}$$

where the $x_k(t)$, aptly described as kth order corrections, (not to be confused with the Landau-Miranker approximants) depend only on $y(t)$ and f. Presumably, in cases of small distortion, a few terms of the series would give a satisfactory approximation to the solution.

Explicit expressions for the first five of the $x_k(t)$ are given in the sequel, first in formulas involving the coefficients of ϕ, and next, the coefficients of f, together with certain operations on y. These formulas reveal how the Fourier transforms of the $x_k(t)$ may be calculated from the Fourier transform of $y(t)$, if this be given.

Next, we find a series development of z in $\mathscr{B}_2$,

$$z(t) = z(t;\, m) = \sum_1^\infty m^k z_k(t) \tag{5}$$

such that the solution to (1) is given (presumably for sufficiently small $|m|$) by

$$x = B\phi(z). \tag{6}$$

We find that $z_1 = y$, $z_2 = 0$, and in case a_2 in (3), or b_2 in (2), vanishes, we have, in addition, $z_3 = z_4 = 0$. That is, under certain conditions, $z \doteq my$, implying that $B\phi\{Bf(x)\}$ is an "approximate identity" ($\doteq x$) for x in $\mathscr{B}_2$, especially if f is odd and fairly linear, or if $x(t)$ is a predominantly low-frequency function.

To further investigate the approximate identity, we introduce the parameter m again, and obtain expressions for u_k in

$$B\phi\{Bf(mx)\} = \sum_1^\infty m^k u_k, \qquad x \quad \text{in} \quad \mathscr{B}_2. \tag{7}$$

To see how interchanging f and ϕ affects the approximate identity, we compare u_k with v_k in

$$Bf\{B\phi(mx)\} = \sum_1^\infty m^k v_k, \qquad x \text{ in } \mathscr{B}_2. \tag{8}$$

As expected from the series development of z, we find $u_1 = v_1 = x$, and $u_2 = v_2 = 0$. Further comparisons [with the same m in (7) and (8)] should be made for the case $f'(0) = \phi'(0) = 1$. For this case, we find $u_3 = v_3 = 2b_2^2 B(x \cdot Hx^2)$, which may be small if b_2 is small or if Hx^2 is small. In case $b_2 = 0$, we find $u_k = v_k = 0$ for $k = 2, 3, 4$, and $u_5 = v_5 = 3b_3^2 B(x^2 \cdot Hx^3)$.

These series developments of the approximate identity suggest that it would be useful in obtaining an approximate solution to the r.v. companding problem for fairly linear companding functions, not necessarily having a Taylor series expansion, but merely satisfying $f(0) = 0$ and

$$0 < m_1 \leqslant f'(x)\; m_2 < \infty, \qquad (-\infty < x < \infty). \tag{9}$$

Compelled by this suggestion, we digress in the Appendix to show for such f that

$$\| x - B\phi\{Bf(x)\} \|_2 \leqslant \gamma \| x \|_2, \qquad x \quad \text{in} \quad \mathscr{B}_2, \tag{10}$$

where

$$\gamma = \frac{\epsilon^2}{4(1+\epsilon)}, \qquad \epsilon = \frac{m_2}{m_1} - 1.$$

(Note that $\gamma = 1/8$ for $m_2/m_1 = 2$.) Thus in many companding problems, $B\phi(y)$, involving only one filtering operation, would be an adequate approximation to the solution x. We go on to define an iterative procedure, involving both f and its inverse ϕ, obtaining approximants converging to x for $\gamma < 1$, offering an alternative to the solution of Landau and Miranker in cases where $(m_2/m_1) < 3 + 2\sqrt{2}$. In any case, $B\phi(y)$ is suggested as a good choice for x_1 in their iterative solution. We note, in leaving this topic, that the inequality (10) is invariant to the interchange of f and its inverse ϕ.

Returning to the series solution, we apply the results to the problem of compatible single-sideband transmission (Ref. 4), obtaining a few terms for correction of the "in-band" distortion arising in envelope detection of "full-carrier" single-sideband signals.

Although the original intent of the work here was to obtain expressions for the first few terms of the series solution (4), supposedly adequate for correcting small distortions, the mathematical question naturally arises in the end as to whether the series actually converges for sufficiently small $|m|$ (or equivalently, for $|m| = 1$ and $\|y\|_2$ sufficiently small), or whether it is merely an asymptotic series. It is indeed a pertinent mathematical question, since the expressions for the $x_k(t)$ were obtained by purely formal manipulations of power series and application of the operators B and H. The resulting expressions become progressively cumbersome and complicated, with no obvious general form, offering no possibility of establishing (from them) bounds on $|x_k(t)|$ which would ensure convergence of the series. The remainder of the paper is addressed to the problem of establishing the convergence of the series.

If we suppose in the r.v. companding problem that the series converges for real-valued m of sufficiently small magnitude, then it would also converge for similar complex m, suggesting that the companding problem [for fixed $y(t)$] would have a solution for all complex m of sufficiently small magnitude. This, in turn, suggests that the problem would have a solution for arbitrary *complex-valued* $y(t)$ in $\mathscr{B}_2$ and all complex m of sufficiently small magnitude, depending on f and the norm of y. That this is a fact has been established previously (Ref. 3) only for complex-valued $y(t)$ whose Fourier transforms vanish outside $[0, \lambda]$, (or $[-\lambda, 0]$), the Fourier transform of the solution $x(t)$ having the same property. In this case (with $m = 1$), the solution is given by $x = B\phi(y)$ for y of sufficiently small norm; i.e., in case the Fourier transform of $x(t)$ vanishes outside $[0, \lambda]$, (or $[-\lambda, 0]$) the "approximate identity" is an exact identity,

$$x = B\phi\{Bf(x)\} = \phi\{f(x)\}$$

for x of sufficiently small norm. This result can be explained, roughly, by the fact that nonlinear (analytic) distortion of such $x(t)$ does not produce both "sum and difference" frequency components, but only "sum" components.

In order to prove that the series solution actually converges for sufficiently small $|m|$, we show that the companding problem (1), where $y(t)$ is an arbitrary *complex-valued* function in $\mathscr{B}_2$, has a solution $x(t; m)$ for all complex m of sufficiently small magnitude, this solution being, for each fixed t, an anaytic function of the complex variable m, from which it follows that the solution has a Taylor series expansion in m; i.e.,

$$x(t;m) = \sum_{1}^{\infty} m^k x_k(t), \qquad |m| < m_0. \tag{11}$$

To obtain this result, we first have to establish for the complex-valued (c.v.) companding problem the analogue of A. Beurling's uniqueness theorem (see Ref. 1) for the r.v. companding problem.

We then examine in detail a specific problem illustrative of the theory and some of its nuances; viz., the problem (taking $\lambda = 2$ for convenience)

$$B\left\{\frac{x}{1-x}\right\} = m\,\frac{\sin 2t}{2t} \tag{12}$$

for which the solution is (at least for sufficiently small $|m|$)

$$x = x(t;m) = 2\beta\,\frac{\sin 2t}{2t} - \beta^2\left(\frac{\sin t}{t}\right)^2, \tag{13}$$

where

$$\beta = m/(2 + m).$$

The rather surprising revelation of this example is that, although the series expansion in m of $x(t;m)$ converges, uniformly in t, for $|m| < 2$, it is not a solution of (12) for all such m. Furthermore, one might reasonably assume that (13) is a solution of (12) for all m other than -2, but this is not true either. As an illuminating exercise, we determine precisely the set of m for which (13) is a solution of (12).

II. THE INVERSE SERIES METHOD

To obtain a series solution to (1), we first think of recovering from $y(t)$ the out-of-band components of $f\{x(t)\}$, so that we might apply the inverse function ϕ to the whole in order to recover $x(t) = x(t;m)$. We have

$$f\{x(t)\} = my(t) + h(t), \tag{14}$$

where $h(t) = h(t;m)$ is some unknown "high-pass" function satisfying

$$Bh(t) = 0 \tag{15}$$

and hence

$$x(t) = \phi\{my(t) + h(t)\}. \tag{16}$$

It is convenient at this point to introduce the high-pass operator defined by

$$H = I - B, \tag{17}$$

where I is the identity operator. Thus applying H to (16) we have

$$H\phi\{my(t) + h(t)\} = 0. \tag{18}$$

We would like to solve (18) for $h(t)$, which we think of as small in cases of interest.

Now we assume that

$$\phi(y) = \sum_1^\infty a_k y^k \quad \text{for sufficiently small} \quad |y|, \tag{19}$$

$$a_1 \neq 0$$

and that in (16)

$$h(t) = h(t;m) = \sum_2^\infty m^k h_k(t), \qquad Hh_k = h_k, \qquad k \geqslant 2, \tag{20}$$

$$x(t) = x(t;m) = \sum_1^\infty m^k x_k(t), \qquad Bx_k = x_k, \qquad k \geqslant 1, \tag{21}$$

where $h_k(t)$ and $x_k(t)$ do not depend on m.

We want to expand $\phi\{my(t) + h(t)\}$ as a power series in m. To do this it is convenient to write

$$my(t) + h(t) = \sum_1^\infty m^k h_k(t), \tag{22}$$

where we identify

$$y(t) = h_1(t) \quad \text{in} \quad \mathscr{B}_2. \tag{23}$$

Then we write

$$\phi\left\{\sum_1^\infty m^k h_k(t)\right\} = F(m;t) = \sum_1^\infty m^k F_k(t). \tag{24}$$

For convenience we suppress the variable t and write simply x_k, h_k, F_k. In terms of the coefficients a_k in

$$\phi(y) = \sum_1^\infty a_k y^k,$$

we find, equating coefficients of m^k in (24),

$$F_1 = a_1 h_1 \tag{25.1}$$

$$F_2 = a_1 h_2 + a_2 h_1^2 \tag{25.2}$$

$$F_3 = a_1 h_3 + a_2(2h_1h_2) + a_3 h_1^3 \tag{25.3}$$

$$F_4 = a_1 h_4 + a_2(2h_1h_3 + h_2^2) + a_3(3h_1^2h_2) + a_4 h_1^4 \tag{25.4}$$

$$F_5 = a_1h_5 + a_2(2h_1h_4 + 2h_2h_3) + a_3(3h_1^2h_3 + 3h_1h_2^2) + a_4(4h_1^3h_2) + a_5h_1^5 \tag{25.5}$$

$$F_6 = a_1h_6 + a_2(2h_1h_5 + 2h_2h_4 + h_3^2) + a_3(3h_1^2h_4 + 6h_1h_2h_3 + h_2^3) + a_4(4h_1^3h_3 + 6h_1^2h_2^2) + a_5(5h_1^4h_2) + a_6h_1^6 \tag{25.6}$$

$$\begin{aligned} F_7 = a_1h_7 &+ a_2(2h_1h_6 + 2h_2h_5 + 2h_3h_4) \\ &+ a_3(3h_1^2h_5 + 6h_1h_2h_4 + 3h_1h_3^2 + 3h_2^2h_3) \\ &+ a_4(4h_1^3h_4 + 12h_1^2h_2h_3 + 4h_1h_2^3) \\ &+ a_5(5h_1^4h_3 + 10h_1^3h_2^2) + a_6(6h_1^5h_2) + a_7h_1^7 \end{aligned} \tag{25.7}$$

$$\begin{aligned} F_8 = a_1h_8 &+ a_2(2h_1h_7 + 2h_2h_6 + 2h_3h_5 + h_4^2) \\ &+ a_3(3h_1^2h_6 + 6h_1h_2h_5 + 6h_1h_3h_4 + 3h_2^2h_4 + 3h_2h_3^2) \\ &+ a_4(4h_1^3h_5 + 12h_1^2h_2h_4 + 6h_1^2h_3^2 + 12h_1h_2^2h_3 + h_2^4) \\ &+ a_5(5h_1^4h_4 + 20h_1^3h_2h_3 + 10h_1^2h_2^3) \\ &+ a_6(6h_1^5h_3 + 15h_1^4h_2^2) + a_7(7h_1^6h_2) + a_8h_1^8 \end{aligned} \tag{25.8}$$

$$\begin{aligned} F_9 = a_1h_9 &+ a_2(2h_1h_8 + 2h_2h_7 + 2h_3h_6 + 2h_4h_5) \\ &+ a_3(3h_1^2h_7 + 6h_1h_2h_6 + 6h_1h_3h_5 + 3h_1h_4^2 + 3h_2^2h_5 \\ &+ 6h_2h_3h_4 + h_3^3) + a_4(4h_1^3h_6 + 12h_1^2h_2h_5 + 12h_1^2h_3h_4 \\ &+ 12h_1h_2^2h_4 + 12h_1h_2h_3^2 + 4h_2^3h_3) + a_5(5h_1^4h_5 \\ &+ 20h_1^3h_2h_4 + 10h_1^3h_3^2 + 30h_1^2h_2^2h_3 + 5h_1h_2^4) + a_6(6h_1^5h_4 \\ &+ 30h_1^4h_2h_3 + 20h_1^3h_2^3) + a_7(7h_1^6h_3 + 21h_1^5h_2^2) \\ &+ a_8(8h_1^7h_2) + a_9h_1^9 \end{aligned} \tag{25.9}$$

$$\begin{aligned} F_{10} = a_1h_{10} &+ a_2(2h_1h_9 + 2h_2h_8 + 2h_3h_7 + 2h_4h_6 + h_5^2) \\ &+ a_3(3h_1^2h_8 + 6h_1h_2h_7 + 6h_1h_3h_6 + 6h_1h_4h_5 + 3h_2^2h_6 \\ &+ 6h_2h_3h_5 + 3h_2h_4^2 + 3h_3^2h_4) + a_4(4h_1^3h_7 + 12h_1^2h_2h_6 \end{aligned}$$

$$+ 12h_1^2h_3h_5 + 6h_1^2h_4^2 + 12h_1h_2^2h_5 + 24h_1h_2h_3h_4 + 4h_1h_3^3$$
$$+ 4h_2^3h_4 + 6h_2^2h_3^2) + a_5(5h_1^4h_6 + 20h_1^3h_2h_5 + 20h_1^3h_3h_4$$
$$+ 30h_1^2h_2^2h_4 + 30h_1^2h_2h_3^2 + 20h_1h_2^3h_3 + h_2^5) + a_6(6h_1^5h_5$$
$$+ 30h_1^4h_2h_4 + 15h_1^4h_3^2 + 60h_1^3h_2^2h_3 + 15h_1^2h_2^4)$$
$$+ a_7(7h_1^6h_4 + 42h_1^5h_2h_3 + 35h_1^4h_2^3) + a_8(8h_1^7h_3$$
$$+ 28h_1^6h_2^2) + a_9(9h_1^8h_2) + a_{10}h_1^{10}. \qquad (25.10)$$

In general the coefficient of a_m in the expansion of F_n consists of sums of products of the h_k corresponding to partitions of n into m parts. The coefficient of the product is $m!$ divided by the product of the factorials of the exponents of the h_k (the multinomial theorem). For example, in F_{10} the coefficient of a_5 is found by writing down the partitions of 10 into 5 parts and proceeding thus (see Table 24.2, Ref. 5):

$1^4, 6$	$\rightarrow$ $h_1^4h_6$	coef. = $5!/4!$	$= 5$
$1^3, 2, 5$	$h_1^3h_2h_5$	$5!/3!$	$= 20$
$1^3, 3, 4$	$h_1^3h_3h_4$	$5!/3!$	$= 20$
$1^2, 2^2, 4$	$h_1^2h_2^2h_4$	$5!/2!2!$	$= 30$
$1^2, 2, 3^2$	$h_1^2h_2h_3^2$	$5!/2!2!$	$= 30$
$1, 2^3, 3$	$h_1h_2^3h_3$	$5!/3!$	$= 20$
2^5	h_2^5	$5!/5!$	$= 1.$

Now we may obtain a formal series solution (21) by successively solving for the h_k by requiring

$$HF_k(t) = 0, \qquad k = 1, 2, \cdots \qquad (26)$$

and setting

$$x_k(t) = BF_k(t) = F_k(t), \qquad k = 1, 2, \cdots . \qquad (27)$$

Recall that $h_1 = y$, the given bandlimited (low-pass) function, and all the other h_k are high-pass functions.† We have $Hh_k = h_k$, $k \geqslant 2$ and

$$HF_1 = a_1Hh_1 = 0 \qquad (28.1)$$

$$HF_2 = a_1h_2 + a_2Hh_1^2 = 0$$

$$h_2 = -\frac{a_2}{a_1}Hh_1^2 \qquad (28.2)$$

† Actually, for $k \geqslant 2$, h_k is a bandpass function whose Fourier transform vanishes over $(-\lambda, \lambda)$ and outside $[-k\lambda, k\lambda]$. This can be seen from (28.1)–(28.k).

$$HF_3 = a_1h_3 + 2a_2H(h_1h_2) + a_3Hh_1^3 = 0$$

$$h_3 = \frac{2a_2^2}{a_1^2} H(h_1 \cdot Hh_1^2) - \frac{a_3}{a_1} Hh_1^3 \tag{28.3}$$

$$HF_4 = a_1h_4 + a_2(2H(h_1h_3) + Hh_2^2) + 3a_3H(h_1^2h_2) + a_4Hh_1^4 = 0$$

$$h_4 = -\frac{4a_2^3}{a_1^3} H[h_1 \cdot H(h_1 \cdot Hh_1^2)] + \frac{2a_2a_3}{a_1^2} H(h_1 \cdot Hh_1^3)$$
$$- \left(\frac{a_2}{a_1}\right)^3 H[(Hh_1^2)^2] + \frac{3a_2a_3}{a_1^2} H(h_1^2 \cdot Hh_1^2) - \frac{a_4}{a_1} Hh_1^4 \tag{28.4}$$

$$HF_5 = a_1h_5 + a_2[2H(h_1h_4) + 2H(h_2h_3)] + a_3[3H(h_1^2h_3)$$
$$+ 3H(h_1h_2^2)] + 4a_4H(h_1^3h_2) + a_5Hh_1^5 = 0$$

$$h_5 = \frac{8a_2^4}{a_1^4} H\{h_1 \cdot H[h_1 \cdot H(h_1 \cdot Hh_1^2)]\} - \frac{4a_2^2a_3}{a_1^3} H[h_1 \cdot H(h_1 \cdot Hh_1^3)]$$
$$+ \frac{2a_2^4}{a_1^4} H[h_1 \cdot (Hh_1^2)^2] - \frac{6a_2^2a_3}{a_1^3} H[h_1 \cdot H(h_1^2 \cdot Hh_1^2)]$$
$$+ \frac{2a_2a_4}{a_1^2} H(h_1 \cdot Hh_1^4) + \frac{4a_2^4}{a_1^4} H[(Hh_1^2) \cdot H(h_1 \cdot Hh_1^2)]$$
$$- \frac{2a_2^2a_3}{a_1^3} H[(Hh_1^2) \cdot Hh_1^3] - \frac{6a_2^2a_3}{a_1^3} H[h_1^2 \cdot H(h_1 \cdot Hh_1^2)]$$
$$+ \frac{3a_3^2}{a_1^2} H(h_1^2 \cdot Hh_1^3) - \frac{3a_2^2a_3}{a_1^3} H[h_1 \cdot (Hh_1^2)^2]$$
$$+ \frac{4a_2a_4}{a_1^2} H(h_1^3 \cdot Hh_1^2) - \frac{a_5}{a_1} Hh_1^5. \tag{28.5}$$

Now replacing h_1 by y we have from (25), (27), and (28)

$$x_1 = BF_1 = a_1y \tag{29.1}$$

$$x_2 = BF_2 = a_2By^2 \tag{29.2}$$

$$x_3 = BF_3 = 2a_2B(yh_2) + a_3By^3$$
$$= -\frac{2a_2^2}{a_1} B(y \cdot Hy^2) + a_3By^3 \tag{29.3}$$

$$x_4 = BF_4 = a_2(2B(yh_3) + Bh_2^2) + 3a_3B(y^2h_2) + a_4By^4$$
$$= \frac{4a_2^3}{a_1^2} B[y \cdot H(y \cdot Hy^2)] - \frac{2a_2a_3}{a_1} B(y \cdot Hy^3)$$

$$+ \frac{a_2^3}{a_1^2} B(Hy^2)^2 - \frac{3a_2a_3}{a_1} B(y^2 \cdot Hy^2) + a_4By^4 \quad (29.4)$$

$$x_5 = BF_5 = a_2[2B(yh_4) + 2B(h_2h_3)] + 3a[B(y^2h_3) + B(yh_2^2)]$$
$$+ 4a_4B(y^3h_2) + a_5By^5$$
$$= -\frac{8a_2^4}{a_1^3} B\{y \cdot H[y \cdot H(y \cdot Hy^2)]\} + \frac{4a_2^2a_3}{a_1^2} B[y \cdot H(y \cdot Hy^3)]$$
$$- \frac{2a_2^4}{a_1^3} B[y \cdot H(Hy^2)^2] + \frac{6a_2^2a_3}{a_1^2} B[y \cdot H(y^2 \cdot Hy^2)]$$
$$- \frac{2a_2a_4}{a_1} B(y \cdot Hy^4) - \frac{4a_2^4}{a_1^3} B[(Hy^2) \cdot H(y \cdot Hy^2)]$$
$$+ \frac{2a_2^2a_3}{a_1^2} B[(Hy^2) \cdot (Hy^3)] + \frac{6a_2^2a_3}{a_1^2} B[y^2 \cdot H(y \cdot Hy^2)]$$
$$- \frac{3a_3^2}{a_1} B(y^2 \cdot Hy^3) + \frac{3a_2^2a_3}{a_1^2} B[y \cdot (Hy^2)^2]$$
$$- \frac{4a_2a_4}{a_1} B(y^3 \cdot Hy^2) + a_5By^5. \quad (29.5)$$

If in (29) we replace the H operator by $I - B$ and collect terms we obtain

$$x_1 = a_1y \quad (30.1)$$

$$x_2 = a_2By^2 \quad (30.2)$$

$$x_3 = \frac{2a_2^2}{a_1} B(y \cdot By^2) + \left(a_3 - \frac{2a_2^2}{a_1}\right) By^3 \quad (30.3)$$

$$x_4 = \frac{4a_2^3}{a_1^2} B[y \cdot B(y \cdot By^2)] - \left(\frac{4a_2^3}{a_1^2} - \frac{2a_2a_3}{a_1}\right) B(y \cdot By^3)$$
$$+ \frac{a_2^3}{a_1^2} B(By^2)^2 - \left(\frac{6a_2^3}{a_1^2} - \frac{3a_2a_3}{a_1}\right) B(y^2 \cdot By^2)$$
$$+ \left(a_4 - \frac{5a_2a_3}{a_1} + \frac{5a_2^3}{a_1^2}\right) By^4 \quad (30.4)$$

$$x_5 = \frac{8a_2^4}{a_1^3} B\{y \cdot B[y \cdot B(y \cdot By^2)]\} + \left(\frac{4a_2^2a_3}{a_1^2} - \frac{8a_2^4}{a_1^3}\right) B[y \cdot B(y \cdot By^3)]$$
$$+ \left(\frac{6a_2^2a_3}{a_1^2} - \frac{12a_2^4}{a_1^3}\right) B[y \cdot B(y^2 \cdot By^2)] + \frac{2a_2^4}{a_1^3}$$

$$\cdot B[y \cdot B(By^2)^2] + \left(\frac{3a_2^2 a_3}{a_1^2} - \frac{6a_2^4}{a_1^3}\right) B[y \cdot (By^2)^2]$$

$$+ \left(\frac{2a_2 a_4}{a_1} - \frac{10a_2^2 a_3}{a_1^2} + \frac{10a_2^4}{a_1^3}\right) B(y \cdot By^4) + \frac{4a_2^4}{a_1^3} B[(By^2)$$

$$\cdot B(y \cdot By^2)] + \left(\frac{6a_2^2 a_3}{a_1^2} - \frac{12a_2^4}{a_1^3}\right) B[y^2 \cdot B(y \cdot By^2)]$$

$$+ \left(\frac{2a_2^2 a_3}{a_1^2} - \frac{4a_2^4}{a_1^3}\right) B[(By^2) \cdot (By^3)]$$

$$+ \left(\frac{3a_3^2}{a_1} - \frac{12a_2^2 a_3}{a_1^2} + \frac{12a_2^4}{a_1^3}\right) B(y^2 \cdot By^3)$$

$$+ \left(\frac{4a_2 a_4}{a_1} - \frac{20a_2^2 a_3}{a_1^2} + \frac{20a_2^4}{a_1^3}\right) B(y^3 \cdot By^2)$$

$$+ \left(a_5 - \frac{6a_2 a_4}{a_1} + \frac{21a_2^2 a_3}{a_1^2} - \frac{14a_2^4}{a_1^3} - \frac{3a_3^2}{a_1}\right) By^5. \qquad (30.5)$$

Note that if y belongs to $\mathscr{B}_2(\lambda/n)$, then $By^k = y^k$ for $k = 1, 2, \cdots, n$. In this case we will have $x_n = a_n y^n$. So the sum of all coefficients in the expressions for x_n must be a_n. If ϕ is an odd function these formulas simplify considerably. It is rather curious that if $a_2 = 0$, $a_3 \neq 0$, the coefficient of $B(y^3 \cdot By^2)$ vanishes, whereas the coefficient of $B(y^2 \cdot By^3)$ does not. The coefficients in (30) are more simply expressed in terms of the coefficients in the power series for f as we see below.

III. FORWARD SERIES METHOD

We can also solve (1) in the "forward" direction by writing

$$Bf(mx_1 + m^2 x_2 + m^3 x_3 + \cdots) = my, \qquad (31)$$

where

$$f(x) = \sum_1^\infty b_k x^k. \qquad (32)$$

Then applying the expansion (24) and (25) to $f(\sum m^k x_k)$ we have, equating coefficients of m^k,

$$Bb_1 x_1 = y \qquad (33.1)$$

$$Bb_1 x_2 + Bb_2 x_1^2 = 0 \qquad (33.2)$$

$$Bb_1 x_3 + Bb_2(2x_1 x_2) + Bb_3 x_1^3 = 0 \qquad (33.3)$$

$$Bb_1 x_4 + Bb_2(2x_1 x_3 + x_2^2) + Bb_3(3x_1^2 x_2) + Bb_4 x_1^4 = 0 \qquad (33.4)$$

$$Bb_1x_5 = Bb_2(2x_1x_4 + 2x_2x_3) + Bb_3(3x_1^2x_3 + 3x_1x_2^2) + Bb_4(4x_1^3x_2) + Bb_5x_1^5 = 0. \tag{33.5}$$

Then solving (33) successively for x_k, $(Bx_k = x_k)$, we have

$$x_1 = y/b_1 \tag{34.1}$$

$$x_2 = -\frac{b_2}{b_1^3}By^2 \tag{34.2}$$

$$x_3 = \frac{2b_2^2}{b_1^5}B(y\cdot By^2) - \frac{b_3}{b_1^4}By^3 \tag{34.3}$$

$$x_4 = -\frac{4b_2^3}{b_1^7}B[y\cdot B(y\cdot By^2)] + \frac{2b_2b_3}{b_1^6}B(y\cdot By^3) - \frac{b_2^3}{b_1^7}B(By^2)^2 + \frac{3b_2b_3}{b_1^6}B(y^2\cdot By^2) - \frac{b_4}{b_1^5}By^4 \tag{34.4}$$

$$\begin{aligned} x_5 = {} & \frac{8b_2^4}{b_1^9}B\{y\cdot B[y\cdot B(y\cdot By^2)]\} + \frac{4b_2^4}{b_1^9}B[(By^2)\cdot B(y\cdot By^2)] \\ & + \frac{2b_2^4}{b_1^9}B[y\cdot B(By^2)^2] - \frac{4b_2^2b_3}{b_1^8}B[y\cdot B(y\cdot By^3)] \\ & - \frac{6b_2^2b_3}{b_1^8}B[y\cdot B(y^2\cdot By^2)] - \frac{2b_2^2b_3}{b_1^8}B[(By^2)\cdot(By^3)] \\ & - \frac{6b_2^2b_3}{b_1^8}B[y^2\cdot B(y\cdot By^2)] - \frac{3b_2^2b_3}{b_1^8}B[y\cdot(By^2)^2] \\ & + \frac{2b_2b_4}{b_1^7}B(y\cdot By^4) + \frac{4b_2b_4}{b_1^7}B(y^3\cdot By^2) \\ & + \frac{3b_3^2}{b_1^7}B(y^2\cdot By^3) - \frac{b_5}{b_1^6}By^5. \end{aligned} \tag{34.5}$$

These correspond to the solutions for the h_k in (28) with H and B, and a's and b's interchanged, except here $x_1 = y/b_1$ as compared by $h_1 = y$ in (28). They agree with the formulas in (30) according to the identities in reversion of series (see 3.6.25, Ref. 5)

$$a_1b_1 = 1 \tag{35.1}$$

$$a_1^3b_2 = -a_2 \tag{35.2}$$

$$a_1^5b_3 = 2a_2^2 - a_1a_3 \tag{35.3}$$

$$a_1^7b_4 = 5a_1a_2a_3 - a_1^2a_4 - 5a_2^3 \tag{35.4}$$

$$a_1^9 b_5 = 6a_1^2 a_2 a_4 + 3a_1^2 a_3^2 + 14a_4^2 - a_1^3 a_5 - 21a_1 a_2^2 a_3 \tag{35.5}$$

$$a_1^{11} b_6 = 7a_1^3 a_2 a_5 + 7a_1^3 a_3 a_4 + 84a_1 a_1^3 a_3 - a_1^4 a_6$$
$$- 28a_1^2 a_2 a_3^2 - 42a_2^5 - 28a_1^2 a_2^2 a_4 \tag{35.6}$$

$$a_1^{13} b_7 = 8a_1^4 a_2 a_6 + 8a_1^4 a_3 a_5 + 120a_1^2 a_2^3 a_4$$
$$+ 180a_1^2 a_2^2 a_3^2 + 132a_2^6 - a_1^5 a_7 - 36a_1^3 a_2^2 a_5$$
$$- 72a_1^3 a_2 a_3 a_4 - 12a_1^3 a_2 a_3 a_4 - 12a_1^3 a_3^3 - 330a_1 a_1^4 a_3. \tag{35.7}$$

Of course the a's and b's may be interchanged in (35). Actually the expressions for x_k in (34) are analogous in a way to the coefficients a_k expressed in terms of the b_k as given by (35) (with the a's and b's interchanged). That is, if it were not for the B operator in eq. (33.k), we would have simply $x_k = a_k y^k$ according to the determining equations for the inverse coefficients. Because of the B operator, successive solutions for the x_k generate powers of y interposed with B operators and powers of $\{B(\cdot)\}$ in all combinations so that, for example, in x_5 we have a number of terms with coefficients $b_2^2 b_3 / b_1^8$ that combine only when B is replaced by I to give $-21 b_2^2 b_3 y^5 / b_1^8$, corresponding to the last term in (35.5). Note if all the $b_k = 1$, the sum of the integer coefficients in x_n is $(-1)^{n+1}$ as this is the case $y = f(x) = x/(1 - x)$, $x = \phi(y) = y/(1 + y)$. Note also that in the expression (34.5) for x_5, for example, there are five groups of functions with common "b" coefficients. These combine with certain weights, depending only on the coefficients of $f(x)$, to give x_5. Similarly, in the expression (29.5) obtained from the inverse function, there are again five groups of functions with common "a" coefficients that combine with certain weights, still depending only on the coefficients of $f(x)$, to give x_5. The interesting and rather puzzling fact is that the groups are not identical but overlap.

IV. A SOLUTION OF THE FORM $x(t) = B\phi\{z(t)\}$, z in $\mathscr{B}_2$

For the solution to (1) we have $x = \sum_1^\infty m^k x_k$, where according to (25) and (29), we have

$$x_1 = a_1 y$$

$$x_2 = a_1 h_2 + a_2 y^2 = a_2 B y^2$$

$$x_3 = a_1 h_3 + 2a_2 y h_2 + a_3 y^3 = 2a_2 B(y h_2) + a_3 B y^3.$$

Since

$$\phi(my) = a_1 my + a_2 m^2 y^2 + a_3 m^3 y^3 + \cdots,$$

we see that $x = B\phi(my) + \mathscr{O}(m^3)$, $m \to 0$. Then setting $m = 1$ (with y sufficiently small) we can conclude that

$$x = B\phi(y) + \mathscr{O}(y^3), \qquad y \to 0. \tag{36}$$

At least as $y \to 0$, $B\phi(y)$ is a better approximation than $\phi(y) = x + \mathscr{O}(y^2)$. Also $B\phi(y)$ could be a good approximation to x without y being small, as would be the case if y were a predominately low-frequency function (compared with its top frequency). This suggests that given y in (1) we determine a bandlimited function z, perhaps close to y, such that the solution to (1) is given by

$$x = B\phi(z). \tag{37}$$

To determine a series solution for z we set

$$z = \sum_1^\infty z_k m^k, \qquad Bz_k = z_k \tag{38}$$

and expand $\phi(z)$ in a power series in m:

$$\phi(mz_1 + m^2 z_2 + m^3 z_3 + \cdots) = \sum_1^\infty m^k F_k. \tag{39}$$

The F_k are given by (25) with z_k replacing h_k. The difference now is that the F_k are bandlimited to $[-k\lambda, k\lambda]$, i.e., $\phi(z)$ is not bandlimited (in general) with z in $\mathscr{B}_2$. However, we must have

$$BF_k = x_k, \tag{40}$$

where in terms of operations on y the x_k are given conveniently by (29). We have

$$Ba_1 z_1 = x_1 = a_1 y \tag{41.1}$$

$$B(a_1 z_2 + a_2 z_1^2) = x_2 = a_2 B y^2 \tag{41.2}$$

$$B(a_1 z_3 + 2a_2 z_1 z_2 + a_3 z_1^3) = x_3 = -\frac{2a_2^2}{a_1} B(y \cdot Hy^2) + a_3 B y^3 \tag{41.3}$$

$$\begin{aligned} B[a_1 z_4 + a_2(2z_1 z_3 + z_2^2) + a_3(3z_1^2 z_2) + a_4 z_1^4] &= x_4 \\ = \frac{4a_2^3}{a_1^2} B[y \cdot H(y \cdot Hy^2)] - \frac{2a_2 a_3}{a_1} B(y \cdot Hy^3) & \\ + \frac{a_2^3}{a_1^2} B(Hy^2)^2 - \frac{3a_2 a_3}{a_1} B(y^2 \cdot Hy^2) + a_4 B y^4 & \end{aligned} \tag{41.4}$$

$$\begin{aligned} &B[a_1 z_5 + a_2(2z_1 z_4 + 2z_2 z_3) + a_3(3z_1^2 z_3 + 3z_1 z_2^2) \\ &\quad + a_4(4z_1^3 z_2) + a_5 z_1^5] = x_5 = -\frac{8a_2^4}{a_1^3} B\{y \cdot H[y \cdot H(y \cdot Hy^2)]\} \\ &\quad + \frac{4a_2^2 a_3}{a_1^2} B[y \cdot H(y \cdot Hy^3)] - \frac{2a_2^4}{a_1^3} B[y \cdot H(Hy^2)^2] \end{aligned}$$

$$+ \frac{6a_2^2a_3}{a_1^2} B[y \cdot H(y^2 \cdot Hy^2)] - \frac{2a_2a_4}{a_1} B(y \cdot Hy^4)$$

$$- \frac{4a_2^4}{a_1^3} B[(Hy^2) \cdot H(y \cdot Hy^2)] + \frac{2a_2^2a_3}{a_1^2} B[(Hy^2) \cdot (Hy^3)]$$

$$+ \frac{6a_2^2a_3}{a_1^2} B[y^2 \cdot H(y \cdot Hy^2)] - \frac{3a_3^2}{a_1} B(y^2 \cdot Hy^3)$$

$$+ \frac{3a_2^2a_3}{a_1^2} B[y \cdot (Hy^2)^2] - \frac{4a_2a_4}{a_1} B(y^3 \cdot Hy^2) + a_5By^5. \tag{41.5}$$

Solving these equations successively for z_k we find

$$z_1 = y \tag{42.1}$$

$$z_2 = 0 \tag{42.2}$$

$$z_3 = -\frac{2a_2^2}{a_1^2} B(y \cdot Hy^2) \tag{42.3}$$

$$z_4 = \frac{4a_2^3}{a_1^3} B[y \cdot H(y \cdot Hy^2)] - \frac{2a_2a_3}{a_1^2} B(y \cdot Hy^3) + \frac{a_2^3}{a_1^3} B(Hy^2)^2$$

$$- \frac{3a_2a_3}{a_1^2} B(y^2 \cdot Hy^2) + \frac{4a_2^3}{a_1^3} B[y \cdot B(y \cdot Hy^2)]. \tag{42.4}$$

The first and last terms in (42.4) combine $(H + B = I)$ to give $\frac{4a_2^3}{a_1^3}$ $B(y^2 \cdot Hy^2)$, which then combines with the fourth term to give

$$z_4 = \frac{a_2}{a_1^3} (4a_2^2 - 3a_1a_3)B(y^2 \cdot Hy^2) - \frac{2a_2a_3}{a_1^2} B(y \cdot Hy^3)$$

$$+ \frac{a_2^3}{a_1^3} B(Hy^2)^2 \tag{42.4a}$$

$$z_5 = -\frac{8a_2^4}{a_1^4} B\{y \cdot H[y \cdot H(y \cdot Hy^2)]\}$$

$$- \frac{8a_2^4}{a_1^4} B[y \cdot B(y^2 \cdot Hy^2)] + \frac{a_3}{a_1^3} (4a_2^2 - 3a_1a_3)B(y^2 \cdot Hy^3)$$

$$+ \frac{a_2^2}{a_1^4} (3a_1a_3 - 2a_2^2)B[y \cdot (Hy^2)^2]$$

$$+ \frac{4a_2}{a_1^3} (3a_2a_3 - a_1a_4)B(y^3 \cdot Hy^2) - \frac{2a_2a_4}{a_1^2} B(y \cdot Hy^4)$$

$$- \frac{4a_2^4}{a_1^3} B[(Hy^2)\cdot H(y\cdot Hy^2)] + \frac{2a_2^2 a_3}{a_1^3} B[(Hy^2)\cdot(Hy^3)]. \qquad (42.5)$$

The expressions for the third- and fourth-order terms, z_3 and z_4, are quite simple, owing to the fact that $z_1 = y$, $z_2 = 0$. Note in eq. (41.n) the "diagonal" term $a_n B z_1^n$ on the left is cancelled by the term $a_n B y^n$ appearing in x_n on the right. Also in case $a_2 = 0$ we have

$$z_1 = y \qquad (43.1)$$

$$z_2 = 0 \qquad (43.2)$$

$$z_3 = 0 \qquad (43.3)$$

$$z_4 = 0 \qquad (43.4)$$

$$z_5 = -\frac{3a_3^2}{a_1^2} B(y^2\cdot Hy^3). \qquad (43.5)$$

V. THE APPROXIMATE IDENTITY

The series development in the previous section suggests that as a practical expedient one might take

$$x \doteq B\phi(y) = B\phi\{Bf(x)\} \doteq \phi\{f(x)\} = x.$$

That is, what appears to be the naive thing to do may in fact be quite good, especially for odd functions ϕ (or f) that are not severely nonlinear. The interposition of the bandlimiting operator between a nonlinear function and its inverse and then subsequent bandlimiting is an interesting "approximate identity" that we examine further in the Appendix. One might ask how the interchange in the order of a particular function and its inverse in the transformation affects the approximate identity. The series expansion of the approximate identity may shed some light on the general problem. To keep track of the various orders, it is convenient to introduce the parameter m as before. We have

$$Bf(mx) = \sum_1^\infty m^k b_k B x^k, \qquad (44)$$

$$\phi\{Bf(mx)\} = \sum_1^\infty m^k F_k, \qquad (45)$$

where the F_k are given by (25) with $b_k B x^k$ replacing h_k, and $Bx = x$. Thus,

$$F_1 = a_1 b_1 x = x \qquad (46.1)$$

$$F_2 = a_1b_2Bx^2 + a_2b_1^2x^2 \tag{46.2}$$

$$F_3 = a_1b_3Bx^3 + 2a_2b_1b_2x\cdot Bx^2 + a_3b_1^3x^3 \tag{46.3}$$

$$F_4 = a_1b_4Bx^4 + 2a_2b_1b_3x\cdot Bx^3 + a_2b_2^2(Bx^2)^2 + 3a_3b_1^2b_2x^2\cdot Bx^2 + a_4b_1^4x^4 \tag{46.4}$$

$$F_5 = a_1b_5Bx^5 + 2a_2b_1b_4x\cdot Bx^4 + 2a_2b_2b_3(Bx^2)\cdot(Bx^3) + 3a_3b_1^2b_3x^2\cdot Bx^3 + 3a_3b_1b_2^2x\cdot(Bx^2)^2 + 4a_4b_1^3b_2x^3\cdot Bx^2 + a_5b_1^5x^5. \tag{46.5}$$

Now we set

$$B\phi\{Bf(mx)\} = \sum_1^\infty m^k u_k, \tag{47}$$

where

$$u_k = BF_k. \tag{48}$$

Now note in (46) that if all the terms in BF_k involving x were of the form Bx^k then BF_k would vanish identically for $k \geqslant 2$ because $\phi\{f(x)\} = x$. So we will introduce the high-pass operator $H = I - B$ to collect the terms Bx^k that cancel. For example, to collect terms Bx^3 in BF_3 we write

$$B(x\cdot Bx^2) = B(x^3 - x\cdot Hx^2) = Bx^3 - B(x\cdot Hx^2).$$

Thus,

$$u_1 = BF_1 = x \tag{49.1}$$

$$u_2 = BF_2 = a_1b_2Bx^2 + a_2b_1^2Bx^2 = 0 \tag{49.2}$$

$$\begin{aligned} u_3 &= BF_3 = a_1b_3Bx^3 + 2a_2b_1b_2B(x\cdot Bx^2) + a_3b_1^3Bx^3 \\ &= a_1b_3Bx^3 + 2a_2b_1b_2B(x^3 - x\cdot Hx^2) + a_3b_1^3Bx^3 \\ &= -2a_2b_1b_2B(x\cdot Hx^2) \end{aligned} \tag{49.3}$$

$$\begin{aligned} u_4 &= BF_4 = a_1b_4Bx^4 + 2a_2b_1b_3B(x\cdot Bx^3) + a_2b_2^2B(Bx^2)^2 \\ &\quad + 3a_3b_1^2b_2B(x^2\cdot Bx^2) + a_4b_1^4Bx^4 \\ &= a_1b_4Bx^4 + 2a_2b_1b_3B(x^4 - x\cdot Hx^3) + a_2b_2^2B(x^2 - Hx^2)^2 \\ &\quad + 3a_3b_1^2b_2B(x^4 - x^2\cdot Hx^2) + a_4b_1^4Bx^4 \\ &= -2a_2b_1b_3B(x\cdot Hx^3) - (2a_2b_2^2 + 3a_3b_1^2b_2)B(x^2\cdot Hx^2) \\ &\quad + a_2b_2^2B(Hx^2)^2 \end{aligned} \tag{49.4}$$

$$
\begin{aligned}
u_5 = BF_5 &= a_1b_5Bx^5 + 2a_2b_1b_4B(x\cdot Bx^4) \\
&\quad + 2a_2b_2b_3B[(Bx^2)\cdot(Bx^3)] \\
&\quad + 3a_3b_1^2b_3B(x^2\cdot Bx^3) + 3a_3b_1b_2^2B[x\cdot(Bx^2)^2] \\
&\quad + 4a_4b_1^3b_2B(x^3\cdot Bx^2) + a_5b_1^5Bx^5 \\
&= a_1b_5Bx^5 + 2a_2b_1b_4B(x^5 - x\cdot Hx^4) \\
&\quad + 2a_2b_2b_3B[(x^2 - Hx^2)\cdot(x^3 - Hx^3)] \\
&\quad + 3a_3b_1^2b_3B(x^5 - x^2\cdot Hx^3) \\
&\quad + 3a_3b_1b_2^2B[x\cdot(x^2 - Hx^2)^2] + 4a_4b_1^3b_2B(x^5 - x^3\cdot Hx^2) \\
&\quad + a_5b_1^5Bx^5 \\
&= -2a_2b_1b_4B(x\cdot Hx^4) - (2a_2b_2b_3 + 3a_3b_1^2b_3)B(x^2\cdot Hx^3) \\
&\quad - (2a_2b_2b_3 + 6a_3b_1b_2^2 + 4a_4b_1^3b_2)B(x^3\cdot Hx^2) \\
&\quad + 2a_2b_2b_3B[(Hx^2)\cdot(Hx^3)] + 3a_3b_1b_2^2B[x\cdot(Hx^2)^2]. \qquad (49.5)
\end{aligned}
$$

Now in order to assess the symmetry or lack of symmetry in interchanging ϕ and f we can use the identities (35) to express the mixed coefficients of u_n in terms of the b_k or the a_k alone. We have

$$u_1 = x \qquad (50.1)$$

$$u_2 = 0 \qquad (50.2)$$

$$u_3 = {}_3C_1B(x\cdot Hx^2) \qquad (50.3)$$

$$u_4 = {}_4C_1B(x\cdot Hx^3) + {}_4C_2B(x^2\cdot Hx^2) + {}_4C_3B(Hx^2)^2 \qquad (50.4)$$

$$
\begin{aligned}
u_5 &= {}_5C_1B(x\cdot Hx^4) + {}_5C_2B(x^2\cdot Hx^3) + {}_5C_3B(x^3\cdot Hx^2) \\
&\quad + {}_5C_4B[(Hx^2)\cdot(Hx^3)] + {}_5C_5B[x\cdot(Hx^2)^2], \qquad (50.5)
\end{aligned}
$$

where

$${}_3C_1 = -2a_2b_1b_2 = \frac{2b_2^2}{b_1^2} = \frac{2a_2^2}{a_1^4} \qquad (50.3a)$$

$${}_4C_1 = -2a_2b_1b_3 = \frac{2b_2b_3}{b_1^2} = \frac{2a_2a_3}{a_1^5} - \frac{4a_2^3}{a_1^6} \qquad (50.4a)$$

$${}_4C_2 = -(2a_2b_2^2 + 3a_3b_1^2b_2) = -\frac{4b_2^3}{b_1^3} + \frac{3b_2b_3}{b_1^2} = -\frac{2a_2^3}{a_1^6} + \frac{3a_2a_3}{a_1^6}$$

$${}_4C_3 = a_2b_2^2 = -\frac{b_2^3}{b_1^3} = \frac{a_2^3}{a_1^6}$$

$$ {}_5C_1 = -2a_2b_1b_4 = \frac{2b_2b_4}{b_1^2} = -\frac{10a_2^2a_3}{a_1^7} + \frac{2a_2a_4}{a_1^6} + \frac{10a_2^4}{a_1^8} \tag{50.5a} $$

$$ {}_5C_2 = -(2a_2b_2b_3 + 3a_3b_1^2b_3) = -\frac{4b_2^2b_3}{b_1^3} + \frac{3b_3^2}{b_1^2} $$

$$ = (2a_2^2 - a_1a_3)(2a_2^2 - 3a_1a_3)/a_1^8 $$

$$ {}_5C_3 = -(2a_2b_3 + 6a_3b_1b_2^2 + 4a_4b_1^3b_2) $$

$$ = \frac{8b_2^4}{b_1^4} - \frac{12b_2^2b_3}{b_1^3} + \frac{4b_2b_4}{b_1^2} = \frac{4a_2^4}{a_1^7} - \frac{8a_2^2a_3}{a_1^7} + \frac{4a_2a_4}{a_1^3} $$

$$ {}_5C_4 = 2a_2b_2b_3 = -\frac{2b_2^2b_3}{b_1^3} = \frac{2a_2^2a_3}{a_1^7} - \frac{4a_2^4}{a_1^8} $$

$$ {}_5C_5 = 3a_3b_1b_2^2 = \frac{6b_2^4}{b_1^4} - \frac{3b_2^2b_3}{b_1^3} = \frac{3a_2^2a_3}{a_1^7}. $$

Now if we interchange the order of ϕ and f in (45) and write

$$ Bf\{B\phi(mx)\} = \sum_1^\infty m^k v_k, \tag{51} $$

we obtain the v_k by replacing u_k by v_k in (49.k) and then interchanging the a's and b's. We have $u_1 = v_1 = x$ and $u_2 = v_2 = 0$. We should compare u_k and v_k, $k \geqslant 3$, for $f'(0) = 1 = a_1 = b_1$. Then we have

$$ u_k = v_k \qquad k = 1, 2, 3. \tag{52} $$

But we have, for example (with $a_1 = b_1 = 1$),

$$ u_4 = 2b_2b_3B(x\cdot Hx^3) + (3b_2b_3 - 4b_2^3)B(x^2\cdot Hx^2) - b_2^3B(Hx^2)^2 \tag{53.1} $$

$$ v_4 = (2b_2b_3 - 4b_2^3)B(x\cdot Hx^3) $$
$$ + (3b_2b_3 - 2b_2^3)B(x^2\cdot Hx^2) + b_2^3B(Hx^2)^2. \tag{53.2} $$

If, however, $b_2 = 0$ ($a_2 = 0$) we have

$$ u_k = v_k = 0, \qquad k = 2, 3, 4, \tag{53.3} $$

and if $a_1 = b_1 = 1$,

$$ u_5 = v_5 = 3b_3^2B(x^2\cdot Hx^3). \tag{53.4} $$

In case $b_2 = b_4 = b_6 = 0$, ($a_2 = a_4 = a_6 = 0$), we have

$$ u_7 = \frac{3b_3b_5}{b_1^2} B(x^2\cdot Hx^5) + \left(\frac{5b_3b_5}{b_1^2} - \frac{9b_3^3}{b_1^3}\right) B(x^4\cdot Hx^3) $$
$$ - \frac{3b_3^3}{b_1^3} B[x\cdot(Hx^3)^2], \tag{54} $$

where the coefficients expressed in terms of the a's are

$$\frac{3b_3b_5}{b_1^2} = \frac{3a_3a_5}{a_1^8} - \frac{9a_3^3}{a_1^9},$$

$$\frac{5b_3b_5}{b_1^2} - \frac{9b_3^3}{b_1^3} = \frac{5a_3a_5}{a_1^8} - \frac{6a_3^3}{a_1^9} - \frac{3b_3^3}{b_1^3} = \frac{3a_3^3}{a_1^9}.$$

So we do not have, in general, $u_7 = v_7$ for odd functions $f(x)$ with $f'(0) = 1$.

VI. APPLICATION TO COMPATIBLE SINGLE-SIDEBAND TRANSMISSION

The mathematical problem of compatible single-sideband transmission was formulated in Ref. 4. Given a signal $y(t)$ in $\mathscr{B}_2$ the problem is to determine m such that the equation

$$B\{\sqrt{(1 + s(t))^2 + \hat{s}(t)^2}\} = my(t) + 1 \tag{55}$$

has a solution $s(t)$ in $\mathscr{B}_2$. In (55) $\hat{s}(t)$, sometimes called the quadrature signal, is the Hilbert transform of $s(t)$, and B is operating on the envelope of the single-sideband signal. The idea is to transmit a single-sideband signal that is compatible with receivers designed for double-sideband (AM) reception. Setting

$$2s(t) + s^2(t) + \hat{s}^2(t) = x(t), \tag{56}$$

we may write (55) as

$$Bf\{x(t)\} = my(t), \tag{57}$$

where

$$f(x) = \sqrt{1 + x} - 1, \qquad x \geqslant -1. \tag{58}$$

Then $s(t)$ may be found from the solution $x(t)$ of (57). (This requires factoring $1 + x(t)$ in the form $g(t)\bar{g}(t)$, where the bandwidth of g is half the bandwidth of x.) Then with $y = f(x)$ we have for the inverse

$$x = \phi(y) = 2y + y^2. \tag{59}$$

Setting

$$x = \sum_1^{\infty} m^k x_k, \tag{60}$$

we have from (29) with $a_1 = 2$, $a_2 = 1$, $a_k = 0$ for $k \geqslant 3$,

$$x_1 = 2y \tag{61.1}$$

$$x_2 = By^2 \tag{61.2}$$

$$x_3 = -B(y \cdot Hy^2) \tag{61.3}$$

$$x_4 = B[y \cdot H(y \cdot Hy^2)] + 1/4B(Hy^2)^2 \quad (61.4)$$

$$x_5 = -B\{y \cdot H[y \cdot H(y \cdot Hy^2)]\} - 1/4B[y \cdot H(Hy^2)^2] - 1/2B[(Hy^2) \cdot H(y \cdot Hy^2)]. \quad (61.5)$$

Replacing H by $I - B$ we have from (30) the alternate forms

$$x_3 = B(y \cdot By^2) - By^3 \quad (61.3a)$$

$$x_4 = B[y \cdot B(y \cdot By^2)] - B(y \cdot By^3) + 1/4B(By^2)^2 - 3/2B(y^2 \cdot By^2) + 5/4By^4 \quad (61.4a)$$

$$\begin{aligned} x_5 = {} & B\{y \cdot B[y \cdot B(y \cdot By^2)]\} - B[y \cdot B(y \cdot By^3)] \\ & - 3/2B[y \cdot B(y^2 \cdot By^2)] + 1/4B[y \cdot (By^2)^2] \\ & - 3/4B[y \cdot (By^2)^2] + 5/4B(y \cdot By^4) \\ & + 1/2B[By^2) \cdot B(y \cdot By^2)] \\ & - 3/2B[y^2 \cdot B(y \cdot By^2)] - 1/2B[(By)^2 \cdot (By^3)] \\ & + 3/2B(y^2 \cdot By^3) \\ & + 5/2B(y^3 \cdot By^2) - 7/4By^5. \end{aligned} \quad (61.5a)$$

The factoring of $1 + x$ can be avoided by developing a series solution for s, $Bs = s$. We have

$$x = 2s + s^2 + \hat{s}^2$$

$$x = mx_1 + m^2x_2 + m^3x_3 + \cdots.$$

Then setting

$$s = ms_1 + m^2s_2 + m^3s_3 + \cdots \quad (62)$$

$$\hat{s} = m\hat{s}_1 + m^2\hat{s}_2 + m^3\hat{s}_3 + \cdots, \quad (63)$$

we have

$$s^2 = m^2s_1^2 + m^3 2s_1s_2 + m^4(2s_1s_3 + s_2^2) + m^5(2s_1s_4 + 2s_2s_3) + \cdots \quad (64)$$

$$\hat{s}^2 = m^2\hat{s}_1^2 + m^3 2\hat{s}_1\hat{s}_2 + m^4(2\hat{s}_1\hat{s}_3 + \hat{s}_2^2) + m^5(2\hat{s}_1\hat{s}_4 + 2\hat{s}_2\hat{s}_3) + \cdots. \quad (65)$$

Note that if s belongs to $\mathscr{B}_2(\lambda)$, then the Fourier transform* of $(s + i\hat{s})$ vanishes outside $[0, \lambda]$ and that of its complex conjugate $(s - i\hat{s})$

---

*Here the Fourier transform of $\hat{s}$ is $-i(sgn\omega)S(\omega)$ where $S(\omega)$ is the Fourier transform of s.

vanishes outside $[-\lambda, 0]$. Thus the Fourier transform of $e^{-i\lambda t/2}(s + i\hat{s})$ vanishes outside $[-(\lambda/2), \lambda/2]$. It follows that the Fourier transform of $s^2 + \hat{s}^2$ vanishes outside $[-\lambda, \lambda]$, and hence that the sums of the coefficients of m^n in (64) and (65) are functions whose Fourier transforms vanish outside $[-\lambda, \lambda]$.

Now we can solve successively for s_k. It is convenient to introduce the Hilbert transform (Quadrature) operator Q

$$\hat{g} = Qg \tag{66}$$

to indicate the "hat" of complicated expressions.

Equating coefficients of m_k in

$$x = 2s + s^2 + \hat{s}^2,$$

we have

$$s_1 = x_1/2 = y \tag{67.1}$$

$$\begin{aligned} s_2 &= 1/2x_2 - 1/2(s_1^2 + \hat{s}_1^2) \\ &= 1/2By^2 - 1/2(y^2 + \hat{y}^2), \end{aligned} \tag{67.2}$$

which may be written, using $Bs_2 = s_2$, as

$$s_2 = -\ 1/2B\hat{y}^2 \tag{67.2a}$$

$$\begin{aligned} s_3 &= 1/2x_3 - (s_1s_2 + \hat{s}_1\hat{s}_2) \\ &= -\ 1/2B(y\cdot Hy^2) + 1/2yB\hat{y}^2 + 1/2\hat{y}\cdot QB\hat{y}^2. \end{aligned} \tag{67.3}$$

Here we may write

$$yB\hat{y}^2 = y\cdot\hat{y}^2 - y\cdot H\hat{y}^2$$

and then use $Bs_3 = s_3$ to obtain

$$\begin{aligned} s_3 &= -\ 1/2B(y\cdot Hy^2) - 1/2B(y\cdot H\hat{y}^2) + 1/2B(y\cdot\hat{y}^2) + 1/2B(\hat{y}\cdot QB\hat{y}^2) \\ &= -\ 1/2B[y\cdot H(y^2 + \hat{y}^2)] + 1/2B(y\cdot\hat{y}^2) + 1/2B(\hat{y}\cdot QB\hat{y}^2). \end{aligned}$$

Then since $H(y^2 + \hat{y}^2) = 0$, we have

$$s_3 = 1/2\ B(y\cdot\hat{y}^2) + 1/2\ B(\hat{y}\cdot QB\hat{y}^2) \tag{67.3a}$$

$$\begin{aligned} s_4 &= 1/2\ x_4 - (s_1s_3 + 1/2\ s_2^2) - (\hat{s}_1\hat{s}_3 + 1/2\ \hat{s}_2^2) \\ &= 1/2\ B[y\cdot H(y\cdot Hy^2)] + 1/4\ B(Hy^2)^2 \\ &\quad -\ 1/2\ yB(y\cdot\hat{y}^2) - 1/2\ \hat{y}QB(y\cdot\hat{y}^2) \\ &\quad -\ 1/8\ (B\hat{y}^2)^2 - 1/8\ (QB\hat{y}^2)^2. \end{aligned} \tag{67.4}$$

There appears to be no simplification here. One may prefer the alternate expression (61.4a) for x_4 to eliminate the H operator. Note that QB can be replaced by the bandlimiting quadrature operator $\hat{B}$ where

$$\hat{B}g = (\hat{B}g)(t) = \int_{-\infty}^{\infty} g(s)\,\frac{1 - \cos\lambda(t - s)}{\pi(t - s)}\,ds. \tag{68}$$

VII. THE COMPLEX-VALUED COMPANDING PROBLEM

The c.v. companding problem is considerably more complicated than the r.v. companding problem, even for the same analytic companding function. For example, if

$$f(x) = \tan^{-1}x + \epsilon x, \qquad (\epsilon > 0),$$

we know from the Landau-Miranker theory that the r.v. companding problem

$$Bf(x) = y$$

has a unique solution x in $\mathscr{B}_2$ corresponding to every real-valued y in $\mathscr{B}_2$. However, in the case of complex-valued y, this may not be true because x must then take complex values which, if the norm of y is not restricted, may be singularities of f. In addition, we are confronted with the problem of establishing the uniqueness of the solution, which may require still more severe restrictions on the norm of y.

Beurling's uniqueness proof (see Ref. 1) for the r.v. companding problem is elegant and simple: Suppose $f(x) = \mathscr{O}(|x|)$, $|x| \to 0$, and is monotone increasing, and further that

$$Bf(x_1) = y \qquad \text{and} \qquad Bf(x_2) = y,$$

with x_1, x_2, (and y) in $\mathscr{B}_2$. Then $f(x_1)$ and $f(x_2)$ belong to L_2 and $B\{f\cdot(x_1) - f(x_2)\} = 0$, i.e., the Fourier transform of $\{f(x_1) - f(x_2)\}$ vanishes over $(-\lambda, \lambda)$, and therefore $\{f(x_1) - f(x_2)\}$ must be orthogonal to $(x_1 - x_2)$. But this is impossible unless $x_1 \equiv x_2$, for otherwise $(x_1 - x_2)\cdot\{f(x_1) - f(x_2)\}$, which is everywhere non-negative, will be positive everywhere on the real axis, except at the isolated zeros (if any) of $(x_1 - x_2)$.

For establishing uniqueness in the c.v. companding problem, it would seem that the weakest analogue of monotocity should be "schlichtness" of f, i.e., that x should be confined to a region, where $f(x_1) = f(x_2)$ implies $x_1 = x_2$. This suffices to establish uniqueness of the solution in the special case where x has a one-sided Fourier transform, but we are not able to see that it suffices in the general case. We can establish the following analogue of Buerling's theorem, where, without loss of generality, we assume $f'(0)$ is positive.

Theorem 1: *Suppose $f(0) = 0$, $f'(0) > 0$, and $f(z)$ is analytic in a convex region G including the origin, wherein*

$$Re\{f'(z)\} > 0.$$

If $x_1(t)$ and $x_2(t)$ belong to $\mathscr{B}_2$ and are confined to G for all real t, then

$$Bf(x_1) = y \qquad and \qquad Bf(x_2) = y$$

imply

$$x_1(t) \equiv x_2(t).$$

Proof of Theorem 1: Since G is convex, any two points x_1 and x_2 in G can be connected by a straight line segment in G. Suppose $(x_1 - x_2) = re^{i\theta}$, where $r > 0$. Then integrating f' along the connecting line segment, we have

$$f(x_1) - f(x_2) = e^{i\theta} \int_0^r f'(x_2 + se^{i\theta})ds,$$

and hence

$$\text{Re}\, \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \text{Re}\, \frac{1}{r} \int_0^r f'(x_2 + se^{i\theta})ds > 0.$$

In case $x_1 \rightarrow x_2$, the limit is Re $f'(x_2) > 0$.

Now $\{f[x_1(t)] - f[x_2(t)]\}$ belongs to L_2 and must be orthogonal to all members of $\mathscr{B}_2$, in particular to $\{x_1(t) - x_2(t)\}$; i.e., setting

$$P(t) = (\bar{x}_1 - \bar{x}_2)\{f(x_1) - f(x_2)\} = |x_1 - x_2|^2 \frac{\{f(x_1) - f(x_2)\}}{x_1 - x_2},$$

the integral of $P(t)$ must vanish. However, we see that the real part of $P(t)$ is non-negative everywhere on the real axis, and vanishes only where $x_1 = x_2$. Since the integral of P is zero, its real part vanishes a.e. Thus the function $\{x_1(t) - x_2(t)\}$ in $\mathscr{B}_2$ vanishes a.e., and hence everywhere. □

Now we can establish that the c.v. companding problem,

$$Bf(x) = y,$$

will have a solution x, which will take values in a disk centered on the origin, wherein Re$\{f'(z)\} > 0$, provided $\|y\|_2$ is sufficiently small. Then the uniqueness of the solution follows from Theorem 1.

An objectionable, but inherent, feature of companding problems (as formulated here) is that a restriction on $\|y\|_\infty$ is not sufficient to give a corresponding restriction on $\|x\|_\infty$. We can, however, establish that $\|x\|_2$ will be small if $\|y\|_2$ is small, and hence that $\|x\|_\infty$ will be small, according to the inequality (given in the introduction) for a function g in $\mathscr{B}_2(\lambda)$,

$$\|g\|_\infty \leqslant \sqrt{\lambda/\pi}\, \|g\|_2. \tag{69}$$

In the sequel, we assume, for convenience and without loss of generality, that $f'(0) = 1$,

$$f(z) = z + \sum_{2}^{\infty} b_k z^k, \qquad |z| < R_0, \tag{70}$$

where R_0 (perhaps ∞) is the radius of convergence of the series. We exclude the trivial case $f(z) = z$, and define

$$M(r) = \max_{|z=r|} |f'(z) - 1| \leqslant \sum_{2}^{\infty} k|b_k|r^{k-1}, \qquad r < R_0, \tag{71}$$

which increases steadily from 0 to ∞, allowing us to define p uniquely by

$$M(p) = 1. \tag{72}$$

Then it is clear that

$$\mathrm{Re}\{f'(z)\} > 0 \qquad \text{for} \quad |z| < p. \tag{73}$$

We are now able to establish the following result.

Theorem 2: *Let $y(t)$ be any complex-valued function in $\mathscr{B}_2 = \mathscr{B}_2(\lambda)$, satisfying*

$$\sqrt{\lambda/\pi}\, \| y \|_2 \leqslant \max_{0<r<p} \{r[1 - M(r)]\} = r_0[1 - M(r_0)]. \tag{74}$$

Then the companding problem

$$Bf(x) = y$$

has a unique solution $x = x(t)$ in $\mathscr{B}_2$.

Proof of Theorem 2: We can use the method of Landau and Miranker to obtain a Cauchy sequence $\{x_n\}$ converging to the solution x, provided we restrict $\| y \|_2$, in the end, to be sufficiently small that all the approximants satisfy $|x_n| < p$.

Assuming the norm of y to be sufficiently small, we take

$$x_1 = y = By, \tag{75}$$

which should be a good approximation to x for small y. Then we set, following Landau and Miranker,

$$x_{n+1} = x_n + y - Bf(x_n), \qquad n \geqslant 1, \tag{76}$$

so that, by induction, $Bx_n = x_n$, i.e., x_n belongs to $\mathscr{B}_2$. We have, writing the same equation for $n - 1$ and subtracting,

$$\begin{aligned} x_{n+1} - x_n &= x_n - Bf(x_n) - \{x_{n-1} - Bf(x_{n-1})\} \\ &= B[x_n - f(x_n) - \{x_{n-1} - f(x_{n-1})\}]. \end{aligned} \tag{77}$$

Now we write

$$f(x_n) - x_n - \{f(x_{n-1}) - x_{n-1}\} = \int_{x_{n-1}}^{x_n} \{f'(z) - 1\}dz. \tag{78}$$

Then assuming that

$$|x_n| \leqslant r < p \qquad \text{for all} \quad n \geqslant 1 \tag{79}$$

[and all t, suppressed in the notation $x_n = x_n(t)$] we have in (78)

$$\left| \int_{x_{n-1}}^{x_n} \{f'(z) - 1\}dz \right| \leqslant M(r)|x_n - x_{n-1}|, \tag{80}$$

where

$$M(r) < M(p) = 1.$$

Substituting in (77) the inequality (80) for the magnitude of the function in (78), we obtain

$$\| x_{n+1} - x_n \|_2 \leqslant M(r) \| x_n - x_{n-1} \|_2. \tag{81}$$

So, under the assumption (79), $\{x_n\}$ forms a Cauchy sequence converging to x in $\mathscr{B}_2$ [cf. Landau, Ref. 1]. It follows from (76) that

$$Bf(x) = y. \tag{82}$$

Now we would like to see how large $\| y \|_2$ may be in order that (79) hold, giving the conclusion in (82). We write

$$x_n = x_1 + (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_n - x_{n-1}) \tag{83}$$

from which follows

$$\| x_n \|_2 \leqslant \sum_1^n \| x_k - x_{k-1} \|_2, \tag{84}$$

where

$$x_1 = y \qquad \text{and} \qquad x_0 = 0.$$

Applying (81) to (84), we have

$$\| x_n \|_2 \leqslant \frac{1 - \alpha^n}{1 - \alpha} \| y \|_2, \tag{85}$$

where $\alpha = M(r) < 1$, provided (79) holds. This will be the case, according to (69), if

$$\| x_n \|_2 \leqslant \sqrt{\pi/\lambda}\, r \qquad \text{for all} \quad n \geqslant 1, \tag{86}$$

which, in turn, will hold if in (85) we have

$$\sqrt{\pi/\lambda}\, \| y \|_2 \leqslant r[1 - M(r)]. \tag{87}$$

Here we are free to take the maximum over r. Thus the problem will have a solution x satisfying the hypotheses, provided the norm of y satisfies

$$\sqrt{\pi/\lambda}\, \| y \|_2 \leqslant \max_{0<r<p} \{r(1 - M(r)\} = r_0[1 - M(r_0)], \tag{88}$$

and the solution is unique according to Theorem 1. □

We note that if (88) is satisfied, then the solution x satisfies, according to (85) and (69),

$$\| x \|_\infty \leqslant r_0 < p. \tag{89}$$

So, in fact, the restriction (88) on the norm of y is too severe. We obtain a slightly better result later, using a different method.

We now wish to show that if y_1 and y_2 are close to each other, then the corresponding solutions, x_1 and x_2, are also close to each other.

Lemma: Let y_1 and y_2 satisfy the hypotheses on y in Theorem 2. Then the solutions of

$$Bf(x_1) = y_1 \qquad \textit{and} \qquad Bf(x_2) = y_2$$

satisfy

$$\| x_1 - x_2 \|_2 \leqslant \frac{\| y_1 - y_2 \|_2}{1 - M(r_0)} \leqslant 2\sqrt{\pi/\lambda}\, r_0 \tag{90}$$

$$\| x_1 - x_2 \|_\infty \leqslant \sqrt{\pi/\lambda}\, \frac{\| y_1 - y_2 \|_2}{1 - M(r_0)} \leqslant 2r_0. \tag{91}$$

Proof of the Lemma: We have

$$x_1 - x_2 = y_1 - y_2 - B[f(x_1) - x_1 - f(x_2) + x_2], \tag{92}$$

giving

$$\| x_1 - x_2 \|_2 \leqslant \| y_1 - y_2 \|_2 + \| B[\cdot] \|_2 \leqslant \| y_1 - y_2 \|_2 + \| [\cdot] \|_2. \tag{93}$$

Also, since, according to (89),

$$| x_1 | \leqslant r_0 < p \qquad \text{and} \qquad | x_2 | \leqslant r_0 < p,$$

we have from (78) and (80),

$$\| f(x_1) - x_1 - f(x_2) + x_2 \|_2 \leqslant M(r_0) \| x_1 - x_2 \|_2, \tag{94}$$

which with (93) gives

$$\| x_1 - x_2 \|_2 \leqslant \frac{\| y_1 - y_2 \|_2}{1 - M(r_0)}. \tag{95}$$

This, with (69) and the assumptions on y_1 and y_2, establishes the lemma. □

With this Lemma and Theorem 2 we can show that the problem

$$Bf\{x(t;m)\} = my(t),$$

for fixed y in $\mathscr{B}_2$, has a unique solution in $\mathscr{B}_2$ for all complex m of sufficiently small magnitude, the solution $x(t;m)$ being a continuous function of the complex variable m in a certain disk centered on the origin. To establish for each t that $F(m;t) = x(t;m)$ is an analytic function of m in that disk, we show that $F(m)$ has a derivative (nondirectional) there. Working with the derivative we are able to improve on Theorem 2. It is convenient now to set $\sqrt{\lambda/\pi}\,\|y\|_2 = 1$ so that $|y(t)| \leqslant 1$.

Theorem 3: Let $y(t)$ be any complex-valued function in $\mathscr{B}_2 \equiv \mathscr{B}_2(\lambda)$ satisfying

$$\sqrt{\lambda/\pi}\,\|y\|_2 = 1.$$

Then the problem

$$Bf\{x(t;m)\} = my(t)$$

has a unique solution $x(t;m)$ in $\mathscr{B}_2$ for all complex m satisfying

$$|m| \leqslant \alpha(p) = \int_0^p [1 - M(r)]dr, \tag{96}$$

where $M(r)$ and p are defined in (71) and (72). Furthermore, for each fixed real t, $x(t;m)$ is an analytic function of m, $|m| \leqslant \alpha(p)$, and hence, since $x(t;0) \equiv 0$,

$$x(t;m) = \sum_1^\infty m^k x_k(t), \qquad |m| \leqslant \alpha(p), \qquad (-\infty < t < \infty) \tag{97}$$

where the $x_k(t)$ depend only on $y(t)$ and f.

We note, before proving Theorem 3, that in Theorem 2, $0 < r_0 < p$, and in Theorem 3

$$\alpha(p) = \int_0^{r_0} [1 - M(r)]dr + \int_{r_0}^{p} [1 - M(r)]dr,$$

where $M(r)$ increases from 0 to 1 over $(0, p)$. Thus

$$\int_0^{r_0} [1 - M(r)]dr > r_0[1 - M(r_0)],$$

and hence

$$r_0[1 - M(r_0)] < \alpha(p) < p. \tag{98}$$

Then, according to Theorem 3, the c.v. companding problem

$$Bf(x) = y$$

has solutions for y of larger norm than specified in Theorem 2.

Proof of Theorem 3: We first consider the solutions $x_1 = x(t;m_1)$, $x_2 = x(t;m_2)$, corresponding to $y_1 = m_1y(t)$, $y_2 = m_2y(t)$, where

$$m_2 = m_1 + \epsilon, \qquad \epsilon = |\epsilon|e^{i\theta} \tag{99}$$

and

$$|m_1| + |\epsilon| \leqslant r_0[1 - M(r_0)], \tag{100}$$

so that Theorem 2 and the Lemma apply.

We have

$$y_2 - y_1 = \epsilon y(t), \tag{101}$$

and hence, from the Lemma,

$$\| x_2 - x_1 \|_2 \leqslant \frac{|\epsilon|}{1 - M(r_0)} \cdot \| y \|_2 = \frac{|\epsilon| \sqrt{\pi/\lambda}}{1 - M(r_0)}. \tag{102}$$

Now

$$B\{f(x_2) - f(x_1)\} = \epsilon y, \tag{103}$$

which we rewrite as

$$\frac{x_2 - x_1}{\epsilon} = y - B\left\{\frac{f(x_2) - f(x_1) - (x_2 - x_1)}{\epsilon}\right\}. \tag{104}$$

We intend to let $\epsilon \rightarrow 0$ (with any argument) and show that the quantity on the left tends to a limit, independent of $\arg(\epsilon)$; viz., $F'(m_1;t)$, where

$$F'(m;t) = \frac{\partial}{\partial m} x(t;m), \qquad |m| \leqslant r_0[1 - M(r_0)]. \tag{105}$$

From (78), (80), and (102) we have

$$\| f(x_2) - f(x_1) - (x_2 - x_1) \|_2 \leqslant M(r_0) \| x_2 - x_1 \|_2$$
$$\leqslant \frac{|\epsilon| \sqrt{\pi/\lambda} M(r_0)}{1 - M(r_0)}. \tag{106}$$

So

$$\frac{f(x_2) - f(x_1) - (x_2 - x_1)}{\epsilon} \quad \text{belongs to} \quad L_2. \tag{107}$$

We also have from (102), or the Lemma,

$$\| x_2 - x_1 \|_\infty \leqslant \frac{|\epsilon|}{1 - M(r_0)}. \tag{108}$$

Thus we may write

$$x_2 = x_1 + \epsilon u, \tag{109}$$

where

$$u = u(t;m_1, \epsilon) \quad \text{belongs to } \mathscr{B}_2 \quad \text{and} \quad |u| = \mathscr{O}(1) \quad \text{as} \quad \epsilon \to 0. \tag{110}$$

Then

$$\begin{aligned} f(x_2) - f(x_1) &= f(x_1 + \epsilon u) - f(x_1) \\ &= \epsilon u f'(x_1) + \mathscr{O}(\epsilon^2 u^2). \end{aligned}$$

So (104) may be rewritten as

$$u = y - B\{uf'(x_1) - u + \mathscr{O}(\epsilon u^2)\}. \tag{111}$$

Now letting $\epsilon \to 0$ and replacing m_1 by m and x_1 by $x(t;m)$, we obtain, setting $u(t;m, o) = u(t;m)$, the equation

$$\begin{aligned} u(t;m) &= y(t) - B\{u(t;m)[f'\{x(t;m)\} - 1]\}, \\ |m| &\leqslant A < r_0[1 - M(r_0)]. \end{aligned} \tag{112}$$

Here we make the identification

$$u(t;m) = F'(m;t) = \frac{\partial}{\partial m} x(t;m), \qquad |m| \leqslant A \tag{113}$$

by verifying that (112) has a solution $u(t;m)$ in $\mathscr{B}_2$, in fact, for $|m|$ larger than $r_0[1 - M(r_0)]$. We observe, since $x(t;0) \equiv 0$, and $f'(0) = 1$, that

$$u(t;0) = y(t). \tag{114}$$

Actually, we can obtain better estimates for $|x(t;m)|$ by integrating its partial derivative from 0 to m.

We consider the equation for u,

$$u = y - B\{u \cdot [f'(x) - 1]\}, \tag{115}$$

assuming $x = x(t; m)$ is known and satisfies

$$\| x \|_\infty \leqslant r < p, \tag{116}$$

so that

$$|f'(x) - 1| \leqslant M(r) < M(p) = 1. \tag{117}$$

Using this inequality in (115) we obtain

$$\| u \|_2 \leqslant \| y \|_2 + M(r) \| u \|_2, \tag{118}$$

or

$$\| u \|_2 \leqslant \frac{\| y \|_2}{1 - M(r)} = \frac{\sqrt{\pi/\lambda}}{1 - M(r)}. \tag{119}$$

The last inequality implies that (115) has a solution u in $\mathscr{B}_2$ (obtained iteratively), in fact, for $\| x \|_\infty < p$, since $M(r) < 1$ for $r < p$. We note further that the inequality (119) is crude, with equality possible only for $r = 0$, for we cannot have

$$| f'\{x(t;m)\} - 1 | \equiv M(r), \qquad (-\infty < t < \infty)$$

unless $x(t;m) \equiv 0$. Therefore, in (115) we have

$$\| B\{u[f'(x) - 1]\|_2 < M(r) \| u \|_2 \quad \text{for} \quad 0 < \| x \|_\infty \leqslant r < p.$$

So we have strict inequality in (119) for $0 < r < p$. Hence,

$$\| u \|_\infty < \frac{1}{1 - M(r)} \quad \text{for} \quad 0 < \| x \|_\infty \leqslant r < p. \tag{120}$$

Now let us set

$$m = \alpha e^{i\theta}, \qquad \alpha > 0 \tag{121}$$

and

$$r(\alpha) = \max_\theta \| x(t;\alpha e^{i\theta}) \|_\infty. \tag{122}$$

We want to see how large we can make α, say $\alpha(p)$, and have $r(\alpha) < p$. Using (120) in

$$x(t;m) = \int_0^m u(t;\xi) d\xi, \tag{123}$$

we obtain the inequality

$$r(\alpha) < \int_0^\alpha \frac{d\xi}{1 - M[r(\xi)]}, \qquad 0 < \alpha \leqslant \alpha(p). \tag{124}$$

Then, after defining

$$s(\alpha) = \int_0^\alpha \frac{d\xi}{1 - M[s(\xi)]}, \qquad 0 < \alpha \leqslant \alpha(p), \tag{125}$$

it is clear, since $M(r)$ is an increasing function of r, that we will have

$$r(\alpha) < s(\alpha), \qquad 0 < \alpha \leqslant \alpha(p). \tag{126}$$

Differentiating (125) with respect to α, we obtain the simple equation

$$s'(\alpha)\{1 - M[s(\alpha)]\} = 1,$$

or, considering α as a function of s,

$$\frac{d\alpha}{ds} = 1 - M(s).$$

Thus

$$\alpha(s) = \int_0^s [1 - M(r)]dr, \qquad 0 < s \leqslant p. \tag{127}$$

We have $s(\alpha(p)) = p$ and $r(\alpha) < s(\alpha)$ for $\alpha > 0$. So we will have

$$\| x(t;m) \|_\infty < p \quad \text{for} \tag{128}$$

$$|m| \leqslant \alpha(p) = \int_0^p [1 - M(r)]dr. \tag{129}$$

According to (120) and (128), the partial derivative $u(t;m)$ will exist for $|m|$ somewhat larger than $\alpha(p)$. This completes the proof of Theorem 3. □

VIII. AN ILLUSTRATIVE EXAMPLE

It will be shown in a future paper that the r.v. companding problem

$$B\left\{\frac{x}{1 - x}\right\} = y, \qquad x < 1, \qquad x, y \quad \text{in} \quad \mathscr{B}_2(\lambda), \tag{130}$$

is equivalent to finding the reproducing kernel for a certain Hilbert space of bandlimited functions. The specific problem with $\lambda = 2$ (for convenience) and

$$y = m\,\frac{\sin 2t}{2t} \tag{131}$$

is quite easily solved. For real $m > -2$, the solution is

$$x(t;m) = 2\beta\,\frac{\sin 2t}{2t} - \beta^2\left(\frac{\sin t}{t}\right)^2, \tag{132}$$

where

$$\beta = m/(2 + m).$$

We need not be concerned here with the derivation of this solution, as we will later show directly that it satisfies

$$B\left\{\frac{x(t;m)}{1 - x(t;m)}\right\} = m\,\frac{\sin 2t}{2t} \tag{133}$$

for all m in a certain region of the complex plane, but for no other m.

We know from Theorem 3 that (133) has a unique solution for sufficiently small $|m|$, and that the solution is an analytic function of m. It follows that (132) is the solution of (133) for all complex m, $|m| < |m_0|$, for some $|m_0| > 0$. Since $m = -2$ is the only point where $x(t;m)$ is not analytic, we might suppose $|m_0| = 2$. The series expansion of $x(t;m)$ certainly converges (uniformly in t) for all $|m| < 2$, but it is not a solution of (133) for all such m. For example, we have

$$x\left(\pm \frac{\pi}{2};m\right) = -\left(\frac{2\beta}{\pi}\right)^2$$

and

$$m = \frac{2\beta}{1-\beta}.$$

Then for $\beta = \pm i\pi/2$, we have

$$m = \frac{-2}{1 \pm i\dfrac{2}{\pi}}, \qquad \text{and} \quad x\left(\pm \frac{\pi}{2};m\right) = 1.$$

Therefore, the meromorphic function of t,

$$f\{x(t;m)\} = \frac{x(t;m)}{1 - x(t;m)}$$

will have poles at $t = \pm \pi/2$ for $m = -2/(1 \pm i2/\pi)$. Thus we have here an example, $|m| < 2$, for which (132) is not a solution of (133). However, it is, according to Theorem 3, for all m satisfying $|m| \leqslant 3 - 2\sqrt{2}$. This, as it turns out, is an overly conservative upper bound for $|m|$.

We now turn to the problem of determining precisely those m for which (132) is a solution of (133).

First we can easily show that the r.v. problem has no solution for $m \leqslant -2$ by convolving both sides of (133) with $1/\pi K(t)$, where $K(t) = (\sin t)^2/t^2$. The result is

$$\int_{-\infty}^{\infty} \frac{x(s;m)}{1 - x(s;m)} \cdot \frac{1}{\pi} K(t-s)ds = \frac{m}{2} K(t). \tag{134}$$

Since $x/(1-x) > -1$, and $K(t) \geqslant 0$, we have

$$\int_{-\infty}^{\infty} \frac{x(s;m)}{1 - x(s;m)} \cdot \frac{1}{\pi} K(t-s)ds > -\frac{1}{\pi} \int_{-\infty}^{\infty} K(t)dt = -1$$

This gives, setting $t = 0$ in (134),

$$m > -2. \tag{135}$$

Now to proceed towards our stated goal, we first write

$$1 - x(t;m) = \left(1 - \beta e^{it} \frac{\sin t}{t}\right)\left(1 - \beta e^{-it} \frac{\sin t}{t}\right). \tag{136}$$

Then in order for $x(t;m)$ to be a solution of the problem, the Fourier transform of the function

$$h(t;m) = \frac{x(t;m)}{1 - x(t;m)} - m \frac{\sin 2t}{2t} \tag{137}$$

must vanish over $(-2,2)$. With a bit of manipulation we arrive at the expression

$$h(t;m) = g(t;m) + g(-t;m), \tag{138}$$

where

$$g(t;m) = \frac{-\beta^2}{1 - \beta} \frac{e^{2it}\left(1 - e^{it} \frac{\sin t}{t}\right)}{2it\left(1 - \beta e^{it} \frac{\sin t}{t}\right)} \tag{139}$$

and

$$\beta = \frac{(m/2)}{1 + (m/2)} \neq 1.$$

We now introduce the complex variable $\tau = t + iu$, and observe that

$$|g(t + iu;\, m)| = \mathcal{O}\left\{\frac{e^{-2u}}{|t + iu|}\right\}, \qquad u \to +\infty. \tag{140}$$

Then if the denominator satisfies the condition,

$$\left(1 - \beta e^{i\tau} \frac{\sin \tau}{\tau}\right) \textit{is zero-free} \quad \text{for} \quad u \geqslant 0, \tag{141}$$

it is easy to see (by contour integration in the upper half-plane) that

$$G(\omega;m) = \int_{-\infty}^{\infty} g(t;m)e^{-i\omega t}dt = 0 \quad \text{for} \quad \omega < 2. \tag{142}$$

On the other hand, if the function in (141) has zeros τ_k in the upper half plane $u > 0$, (it must have no real zeros in order for g to have a Fourier transform) we will have, by the calculus of residues,

$$G(\omega;m) = \sum_1^n c_k e^{-i\omega\tau_k}, \qquad c_k \neq 0, \tag{143}$$

where n, depending on β, is finite, since it is clear that the function in (141) can have only a finite number of zeros in the upper half-plane. Since the Fourier transform of $h(t;m)$ is given by

$$H(\omega, m) = G(\omega;m) + G(-\omega;m) \tag{144}$$

it will vanish over $(-2,2)$ if, and only if, the condition (141) is satisfied.

Now the values taken by $e^{i\tau} \sin \tau/\tau$ in the upper half-plane, $u \geqslant 0$ are precisely those values on the boundary and interior of the cardioid-like region whose boundary is described parametrically by

$$e^{it} \frac{\sin t}{t}, \qquad -\pi \leqslant t \leqslant \pi.$$

(Some values are taken more than once.) Then $x(t;m)$ will be a solution to the problem except for those values of m such that $1/\beta$ is a point on the boundary or in the interior of the cardioid-like region. By the mapping

$$m = \frac{2}{\dfrac{1}{\beta} - 1},$$

$x(t;m)$ is a solution to the problem for precisely those (finite) m lying in the region to the right of the boundary line described parametrically by

$$m = \frac{2}{e^{it} \dfrac{\sin t}{t} - 1}, \qquad -\pi \leqslant t \leqslant \pi. \tag{145}$$

This region (see Fig. 1) includes the half-plane $\text{Re}\{m\} \geqslant -4/3$, its boundary indenting more to the left near the real axis, having a cusp at its leftmost point, $m = -2$, where it is tangent to the real axis. It is found that the distance from the origin to the boundary is minimal (see circle in Fig. 1) at the point m_0 and its conjugate, where

$$m_0 = (-2 + \xi) + i\xi,$$
$$\xi = \frac{1}{t_0} \doteq .4895273114 \doteq (2.42786943)^{-1}, \tag{146}$$

t_0 is the smallest positive root of $\sin t/t = \cos t + \sin t$,

$$|m_0| = \frac{\sqrt{2}}{\sin t_0} \doteq 1.58781760, \ \arg\{m_0\} = \frac{\pi}{4} + t_0.$$

So $|m_0|$ is the largest number such that $x(t;m)$ is a solution for *all* m

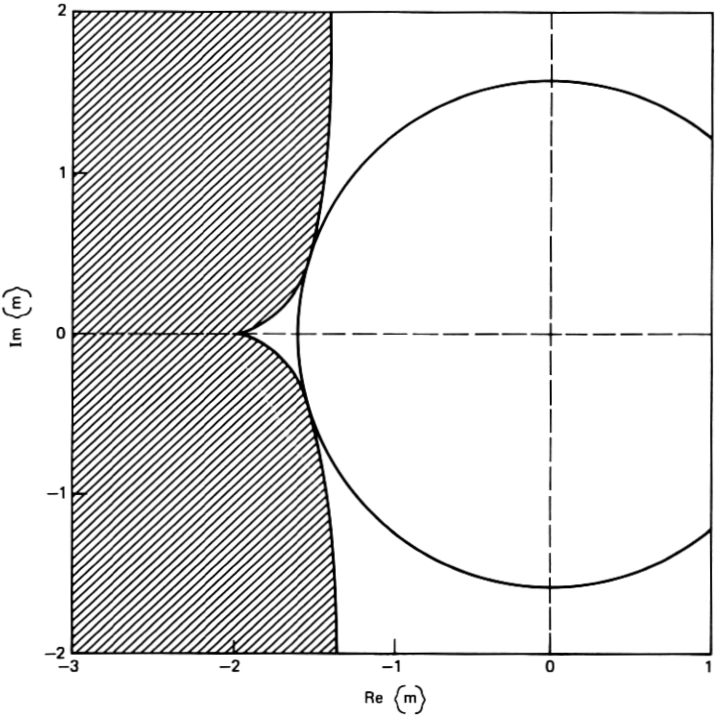

Fig. 1—Open region in m-plane (unshaded) for which eq. (132) is a solution of eq. (133). Shown is the largest disk (centered on the origin) contained in the region.

satisfying $|m| < |m_0|$. Also,

$$|m_0| = \min_t R_0(t),$$

where $R_0(t)$ is the radius of convergence of the series

$$\frac{x(t;m)}{1 - x(t;m)} = m \frac{\sin 2t}{2t} + m^2 h_2(t) + m^3 h_3(t) + \cdots,$$

the minimum occurring for $t = \pm t_0$.

IX. CONCLUSION

The expressions for the nth order components $x_n(t)$ of the series solution to the companding problem become so complicated that, for practical purposes, only the first few are of interest. These should be useful in correcting small distortions in nonlinear transmission systems which fit the companding model. It would appear that the corrections applied internally to the inverse function (the z_k in Section V) would be more effective for correcting larger distortions, especially if the lower frequency components are predominant in the signals. In this connection, the simpler "approximate identity" should be quite effective for correcting small to moderate distortions of a more general nature, as evidenced by the inequality given in the appendix. Experimental evidence of the effectiveness of these correction schemes would be desirable.

The question of the convergence of the series solution is a matter of little practical concern, but the fact that it does attaches more mathematical significance to the results. To settle this question we had to show that the complex-valued companding problem has solutions for functions of sufficiently small norm. This generalizes the result for functions of one-sided spectra; and whether or not the general result will ever find practical application, it is an interesting addition to a theory, though still incomplete in many respects. For example, it is doubtful that the condition that $x(t)$ be confined to a convex region G where $\mathrm{Re}\{f'\} > 0$ is a necessary condition for uniqueness of the solution. In this connection, one could probably use analytic continuation arguments to show that the specific problem examined in Section VIII has solutions only for those values of m for which the (particular) solution given is the only solution, this being unique for sufficiently small $|m|$, and being an analytic function of m having no branch points. Also there is the difficult question of determining for what $y(t)$ the companding problem has a solution, where particular interest is attached to the real-valued problem with analytic companding functions. It can be shown for the case $f(x) = x/(1 - x)$, $x < 1$, that the problem has a solution for every (r.v.) y in $\mathscr{B}_2$ satisfying $y > -1$. This suggests (conjecture) that the r.v. companding problem with $f(x) = x/(1 - x^2)$, $-1 < x < 1$, has a solution for every (r.v.) y in $\mathscr{B}_2$, or more generally for monotone $f(x)$ defined on $(-1, 1)$ having singularities as strong as poles at ± 1. In general, it is not enough for $f(x)$ to increase from $-\infty$ to $+\infty$ over its range of definition in order to draw the same conclusion; e.g., $f(x) = \log(1 + x)$, $x > -1$. The questions raised here are certainly worthy of future consideration.

In connection with the series solution, one naturally inquires whether an explicit formula (albeit complicated and involving partitions of various kinds) can be given for the general term x_n. Perhaps the combinatorics experts will consider this question.

We note that the solution $x = B\phi(y)$, valid for y (of sufficiently small norm) whose Fourier transforms vanish outside $[0, \lambda]$, is verified by the fact that in (29.n) the expression reduces to $x_n = a_n B y^n$, the other terms vanishing because B is operating on functions whose Fourier transforms vanish over $(-\infty, \lambda)$. The same reduction occurs in the expression (34.n), because, in this case, B is operating on functions whose Fourier transforms vanish over $(-\infty, 0)$ and agree over $[0, \lambda]$; i.e.,

$$B[\cdots] = By^n$$

holds for each term in (34.n), the sum of all the coefficients being a_n.

Some interesting identities are obtained by equating the expressions for x_n in the series solution of the general problem to those obtained from the explicit solution (given in the introduction) to the special problem

$$B \log(1 + x) = my,$$

which involve $\hat{y}$, the Hilbert transform of y, which does not appear in the more general expressions. For example, we find from (34.3) that x_3 in the series solution of this problem is given by

$$x_3 = 1/2B(y \cdot By^2) - 1/3By^3,$$

and, from the series expansion of the explicit solution, by

$$x_3 = 1/8yB(y^2 - \hat{y}^2) - 1/8B(y\hat{y}^2) + 1/24\,By^3 + 1/4\hat{y}B(y\hat{y}).$$

It is an interesting exercise to show directly that these two expressions are identical.

Finally, since truly bandlimited signals exist only as mathematical abstractions, some attention should be given to developing a mathematical theory of practical companding problems,

$$\int_{-\infty}^{\infty} f\{x(s)\}k(t - s)ds = y(t),$$

where $k(t)$ is the (absolutely-integrable) impulse response of a practical low-pass filter, so that the theory may be extended to signals that are merely bounded. Here one may not be interested, for various reasons, in the exact solution of this problem but, instead, a compromise problem, where the equation is nearly satisfied with both $x(t)$ and $y(t)$ being close to bandlimited functions. For example, in many cases $f\{x(t)\}$ is given (say) by an nth order differential operator acting on $y(t)$. Then the (exact) solution $x(t) = \phi\{f(x)\}$ may be far from a bandlimited function. However, if $y(t)$ is close to a bandlimited function there should be an approximate solution which is also close to a bandlimited function. A case in point is found in Landau's simulation of the iterative solution of the companding problem (Ref. 2), where, in fact, the equation he was obtaining approximate solutions to was the case $y(t) = k(t)$, (approximately bandlimited) for which the unique solution is (a multiple of) the Dirac delta function.

X. ACKNOWLEDGMENT

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APPENDIX

Suppose $f(x)$ is a monotone increasing function of the real variable x, satisfying

$$(i) \qquad f(0) = 0$$

$$(ii) \qquad 0 < m_1 \leqslant f'(x) \leqslant m_2 < \infty, \qquad (-\infty < x < \infty).$$

Then f has an inverse ϕ

$$(iii) \qquad x = \phi\{f(x)\}, \qquad (-\infty < x < \infty),$$

satisfying, since $1 = f'(x)\phi'\{f(x)\}$,

$$(iv) \qquad 0 < \frac{1}{m_2} \leqslant \phi'(y) \leqslant \frac{1}{m_1} < \infty, \qquad (-\infty < y < \infty).$$

Now let $x = x(t)$ be any function in $\mathscr{B}_2$. We wish to establish

$$\| x - B\phi\{Bf(x)\} \|_2 \leqslant \gamma \| x \|_2, \tag{147}$$

where
$$\gamma = \frac{\epsilon^2}{4(1 + \epsilon)}, \qquad \epsilon = \frac{m_2}{m_1} - 1.$$

Set

$$y(t) = y = Bf(x). \tag{148}$$

Then

$$f(x) = y + h, \qquad Bh = 0. \tag{149}$$

Now set

$$x_1 = B\phi(y) \tag{150}$$

so that, since $x = \phi(y + h) = Bx = B\{\phi(y + h)\}$,

$$x - x_1 = B\{\phi(y + h) - \phi(y)\}. \tag{151}$$

Since $|\phi(y+h)-\phi(y)| \leqslant \frac{1}{m_1} h$, we see that

$$\|\phi(y+h)-\phi(y)\|_2 \leqslant \frac{1}{m_1}\|h\|_2$$

and hence that

$$\|x-x_1\|_2 \leqslant \frac{1}{m_1}\|h\|_2,$$

but we can improve this inequality by writing

$$\phi(y+h)-\phi(y) = \int_y^{y+h} \{\phi'(\xi)-\alpha\}d\xi + \alpha h, \tag{152}$$

where
$$\alpha = \frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right).$$

Then, since

$$|\phi'(\xi)-\alpha| \leqslant \frac{1}{2}\left(\frac{1}{m_1}-\frac{1}{m_2}\right), \tag{153}$$

we have

$$\phi(y+h)-\phi(y) = u+\alpha h, \tag{154}$$

where $u = u(t)$ and

$$|u| \leqslant \frac{1}{2}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)|h|. \tag{155}$$

Thus

$$x - x_1 = B(u+\alpha h) = Bu, \tag{156}$$

and hence

$$\|x-x_1\|_2 \leqslant \|u\|_2 \leqslant \frac{1}{2}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\|h\|_2. \tag{157}$$

Now we need an inequality of the form $\|h\|_2 \leqslant c\|x\|_2$. We have

$$h = Hf(x), \tag{158}$$

where $H = I - B$ is the high-pass operator. So, clearly

$$\|h\|_2 \leqslant \|f(x)\|_2 \leqslant m_2\|x\|_2.$$

We can improve this inequality by setting

$$v(t) = v = f(x) - \beta x, \tag{159}$$

where
$$\beta = 1/2(m_1+m_2).$$

Then

$$|v| \leqslant 1/2(m_2 - m_1)|x|, \tag{160}$$

and, since $Hx = 0$,

$$h = Hf(x) = H\{f(x) - \beta x\} = Hv. \tag{161}$$

Hence

$$\| h \|_2 \leqslant \| v \|_2 \leqslant 1/2(m_2 - m_1)\| x \|_2. \tag{162}$$

This, with (157) gives

$$\| x - x_1 \|_2 \leqslant \frac{1}{4}\left(\frac{1}{m_1} - \frac{1}{m_2}\right)(m_2 - m_1)\| x \|_2 = \gamma \| x \|_2, \tag{163}$$

which is the result (147). The number γ in the inequality is invariant under the interchange of ϕ and f in the approximate identity (147), as we would expect from using only (*ii*) and (*iv*).

So $x_1 = B\phi\{Bf(x)\}$ is a good approximation to x if γ is small. The manipulations leading to the inequality (163) suggest an iterative scheme for solving, given y in $\mathscr{B}_2$,

$$BF(x) = y, \qquad x \quad \text{in} \quad \mathscr{B}_2, \tag{164}$$

provided $\gamma < 1$, which will be the case if $(m_2/m_1) < 3 + 2\sqrt{2}$.

We set

$$x_n = B\phi(y + h_{n-1}), \qquad n \geqslant 1, \tag{165}$$

where

$$\mathrm{h}_n = \mathrm{Hf}(\mathrm{x}_n), \qquad \mathrm{n} \geqslant 0, \tag{166}$$

and $\mathrm{x}_0 = \mathrm{h}_0 = 0$,

giving $\mathrm{x}_1 = \mathrm{B}\phi(\mathrm{y})$ as in (150).

Now we wish to show that

$$\| x - x_n \|_2 \leqslant \gamma^n \| x \|_2, \qquad n \geqslant 1. \tag{167}$$

We have

$$x - x_n = B\{\phi(y + h) - \phi(y + h_{n-1})\}. \tag{168}$$

Following the previous pattern we write

$$\phi(y + h) - \phi(y + h_{n-1}) = u_n + \alpha(h - h_{n-1}), \tag{169}$$

where

$$u_n = \int_{y+h_{n-1}}^{y+h} \{\phi'(\xi) - \alpha\} d\xi,$$

and hence

$$|u_n| \leqslant \frac{1}{2}\left(\frac{1}{m_1} - \frac{1}{m_2}\right)|h - h_{n-1}|. \tag{170}$$

Then

$$x - x_n = B\{u_n + \alpha(h - h_{n-1})\} = Bu_n, \tag{171}$$

giving

$$\|x - x_n\|_2 \leqslant \frac{1}{2}\left(\frac{1}{m_1} - \frac{1}{m_2}\right)\|h - h_{n-1}\|_2. \tag{172}$$

Now

$$h - h_{n-1} = H\{f(x) - f(x_{n-1})\}, \quad n \geqslant 1, \quad \text{with } x_0 = h_0 = 0. \tag{173}$$

Here we write

$$\begin{aligned} f(x) - f(x_{n-1}) &= \int_{x_{n-1}}^{x} \{f'(\xi) - \beta\}d\xi + \beta(x - x_{n-1}) \\ &= v_n + \beta(x - x_{n-1}), \end{aligned} \tag{174}$$

where

$$|v_n| \leqslant \frac{m_2 - m_1}{2}|x - x_{n-1}|. \tag{175}$$

Then

$$h - h_{n-1} = H\{v_n + \beta(x - x_{n-1})\} = Hv_n, \tag{176}$$

giving, with (175),

$$\|h - h_{n-1}\|_2 \leqslant \frac{m_2 - m_1}{2}\|x - x_{n-1}\|_2. \tag{177}$$

This, with (172), gives

$$\|x - x_n\|_2 \leqslant \gamma\|x - x_{n-1}\|_2, \tag{178}$$

whence follows, with $x_0 = 0$,

$$\|x - x_n\|_2 \leqslant \gamma^n\|x\|_2, \qquad n \geqslant 1. \tag{179}$$

Note that there is a bonus attached to $x_1 = B\phi(y)$, in that only one filtering operation is required to obtain it. Thereafter, two filtering operations are required to obtain x_n from y and x_{n-1}.

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