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A Model for Special-Service Circuit Activity

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We describe a model for special-service circuit activity to assist in forecasting, provisioning, and "churn" studies. We assume that customers order a random number of circuits for an exponentially distributed period of time and that the rate of new connect orders grows exponentially with time. These assumptions yield simple formulae giving the means and variances of the number of active circuits at a future time and the total number of connected and disconnected circuits during a future period. Distributions of these variables can, in principle, also be computed. There are three important parameters characterizing the model: growth rate, disconnect rate, and batchiness; we describe their physical meaning and discuss methods to estimate them. This document describes the analytical portion of an effort to develop a model based on the physics of special-service circuit activity.

I. INTRODUCTION

The purpose of this paper is to describe a model for special-service circuit activity to assist in forecasting, provisioning, and "churn" studies, which can be summarized by a few parameters that have a physical interpretation. The calibration and measurement of the fit of this model to data in a New Jersey Bell database is being pursued simultaneously and will be reported elsewhere.

The model treated here is derived from a priori consideration of the physical behavior of customers. It is based on the assumption that the number of active circuits, although growing, is in some sense in

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equilibrium as well; that is, certain characteristics of the system are not changing. This is to be contrasted with a model proposed by Nucho in which transient analysis is fundamental.[1] The primary difference between these models is that the demand rate for new circuits is a function of the number of active circuits in the Nucho model, whereas it is considered to be an exogenous variable here. In the Nucho model, the variance to mean ratio of the number of active circuits increases indefinitely with time (since fluctuations tend to feed on themselves); in the model considered here this ratio remains constant. Another difference between the models is that the model described here allows an order to be for more than one circuit.

Here, we assume that (1) the arrivals of special-service circuit orders are given by a nonhomogeneous Poisson process with exponentially growing intensity, (2) each order is for a random number of circuits (a batch) with arbitrary distribution, and (3) the lifetime of an order is an exponentially distributed random variable, during which time the number of held circuits per order remains constant. Note that the last assumption implies that an order lifetime and a circuit lifetime have the same distribution.

We use three important parameters in special-services modeling, each with its own physical interpretation. These parameters may be described as growth rate, disconnect rate (per circuit), and batchiness.

The growth rate summarizes the rate at which the mean number of active circuits increases with time. It may be expressed in terms of proportion increase per unit of time; we denote it β. Thus the mean number of circuits at time t is proportional to $e^{\beta t}$. We actually assume that connect activity grows at rate β, but it turns out that the number of active circuits, the total connect rate, and the total disconnect rate are all proportional to $e^{\beta t}$ in this model. Of course, for small growth rates or short periods of time, exponential growth is very close to linear growth.

The disconnect rate, denoted μ, is the ratio of the number of disconnects per unit time (i.e., the total disconnect rate) to the number of active circuits. The mean circuit lifetime is then $1/\mu$. The distributions of circuit lifetimes have been shown to be well approximated by negative exponential distributions;[2] thus the disconnect rate does not vary with the age of a circuit.

The batchiness of the arrival process is related to the tendency of special service circuits to be ordered in multiples greater than one. We call the batchiness parameter ν and define it to be the ratio of the second moment to the first moment of the number of circuits in an order.

The ultimate goal of this modeling process is to provide a tool that can be used to predict special-services needs in the future. The model contained herein should be very useful in this regard. One should remember that the underlying process is stochastic so that there is a fundamental uncertainty even if one has exact specification of the parameters of the model. The standard deviation of future requirements can be quite large compared to the mean for small circuit groupings, and this presents a major problem for provisioning at the most detailed level. This problem cannot be surmounted with a better model and/or additional data collection. The present analysis allows quantification of the fundamental uncertainty of forecasting, an insight which is difficult to obtain purely by statistical methods. The only possible method to further decrease relative uncertainty is to aggregate demand, or to obtain advance knowledge of connect or disconnect activity (sometimes called "deterministic events").

The rest of the paper is organized as follows: Section II summarizes the important results of the paper, giving formulae for the means and variances of the number of active circuits in the future, the total number of connects in a future interval, and the total number of disconnects in a future interval; and giving statistical methods to estimate the fundamental parameters of the model such as growth rate, disconnect rate, and batchiness. The reader not interested in the derivation of these results may stop at this point.

The predictions (summarized in Section II) of the model are derived in Section IV. These derivations are primarily substitutions into formulae given in Section III. Section III describes and analyzes a much more general model than the one described in this introduction (we refer to the latter simply as "the model"). We have chosen to introduce this generalized model for two reasons. First, the analysis required for the treatment of the generalized model is little different in complexity from that required for treatment of the specific model. Second, the general results of Section III allow rapid exploration of the consequences of changes in assumptions of the model. For example, one can explore the effects of linear growth of demand, or the super-exponential growth in demand which follows introduction of a new service. However, we do feel that the original assumptions are appropriate in most circumstances. Thus, the consequences of this model are the only ones summarized in Section II, and it is this specific model which is being verified with respect to the New Jersey Bell Telephone Co. database. Thus, Section III is provided for reference in case of non-typical special service applications.

Section V derives the statistical methods (summarized in Section II) for estimation of the fundamental parameters of the model. Section VI is a summary.

Appendix A gives background information on the compound Poisson random variable, and Appendix B gives background information on the non-homogeneous Poisson process. These results are needed in Sections III and IV.

Table I presents values of a function useful in estimating growth (see Section II) and Table II lists the notation used in the paper.

II. SUMMARY OF KEY RESULTS

This section provides a summary of the important results of the paper derived in Sections IV and V.

2.1 Churn

Our model depends on three physical parameters: growth rate (β), disconnect rate (μ), and batchiness (ν). The meaning of these parameters is described in Section I. Another physical parameter is "churn," which has been defined in many different ways. For any reasonable definition, the churn is determined by the growth and disconnect rates of the model. We define the churn to be the minimum of the disconnect rate per circuit and the connect rate per circuit, and denote it by γ. With this definition, it can be shown [see (75)] that

$$\gamma = \min(\mu, \mu + \beta). \tag{1}$$

The values of churn under other definitions are also readily available. For example, if one defines churn to be the ratio of the average total connect rate to the average rate of change of net active circuits, then this value of churn is $(1 - \gamma)^{-1}$. Under still another definition, the churn equals $\mu/(\mu + \beta)$.

2.2 Mean and variance of total active circuits at a future time

Here we give the mean $M(t)$ and the approximate variance $V(t)$ of the number of circuits in service at a given time t in the future. The mean and the variance depend on the present (at time $t = 0$) number k of circuits in service, the present instantaneous rate D_o of circuit demand due to new orders, and the three key parameters described previously: β, μ, and ν. We give these two relationships below:

$$M(t) = ke^{-\mu t} + \frac{D_o}{\mu + \beta}(e^{\beta t} - e^{-\mu t}), \tag{2}$$

and

$$V(t) = \nu\left[ke^{-\mu t}(1 - e^{-\mu t}) + \frac{D_o}{\mu + \beta}(e^{\beta t} - e^{-\mu t})\right]. \tag{3}$$

It is interesting that (2) and (3) together imply the relationship

$$V(t) = \nu(M(t) - ke^{-2\mu t}), \tag{4}$$

which relates the variance of a forecast to the mean of the forecast, the number of circuits currently active, k, and the parameters ν and μ.

2.3 Mean and variance of the total number of connected or disconnected circuits in the future

Similar results are available for the mean and variance of the total number of connected circuits (variables subscripted with a C) and the mean and variance of the total number of disconnected circuits (subscripted with a D) in an interval of length t beginning immediately:

$$M_C(t) = \frac{D_o}{\beta}(e^{\beta t} - 1), \tag{5}$$

$$V_C(t) = \nu\frac{D_o}{\beta}(e^{\beta t} - 1) = \nu M_C(t) \tag{6}$$

$$M_D(t) = k(1 - e^{-\mu t}) + \frac{D_o\mu}{\beta(\mu + \beta)}e^{\beta t} + \frac{D_o}{\mu + \beta}e^{-\mu t} - \frac{D_o}{\beta}, \tag{7}$$

and

$$V_D(t) = \nu\left\{ke^{-\mu t}(1 - e^{-\mu t}) + \frac{D_o\mu}{\beta(\mu + \beta)}e^{\beta t} + \frac{D_o}{\mu + \beta}e^{-\mu t} - \frac{D_o}{\beta}\right\} = \nu(M_D(t) - ke^{-2\mu t}). \tag{8}$$

In this case, the total numbers of connected and disconnected circuits are dependent random variables.

We may also obtain the coefficient of correlation ρ between the number of active circuits at different times

$$\rho[Y(t),\ Y(t + \tau)] = e^{-(\mu+\beta/2)\tau}, \tag{9}$$

where $Y(t)$ is the number of active circuits at time t.

2.4 Estimation of the model parameters

To use results such as (2) through (9), we must be able to estimate the parameters β, D_o, ν, and μ. These questions are addressed in Section V; we provide a brief summary here. Suppose that the system has been observed over the interval $[-\Theta, 0]$ and n connect orders are observed at times $t_1, \ldots, t_n$. Form the statistic

$$S = \sum_{i=1}^{n} t_i/n\Theta + 1, \tag{10}$$

and then the maximum likelihood estimator $\hat{\beta}$ for the growth rate β is

$$\hat{\beta} = \frac{1}{\Theta} f^{-1}(S), \tag{11}$$

where f is the function given in (79). Values of f^{-1} are available in Table I. Once $\hat{\beta}$ has been obtained from (11), the estimator $\hat{D}_o$ for the *instantaneous* present demand D_o (assumed to be at the end of the interval of observation $[-\Theta, 0]$) is

$$\hat{D}_o = \frac{n\hat{\beta}}{1 - e^{-\hat{\beta}\Theta}} \hat{N}, \tag{12}$$

where $\hat{N}$ is an estimator for the average number of circuits per order and is equal to the average number of circuits actually observed per order. The estimator $\hat{\nu}$ for the batchiness ν of the order size is

$$\hat{\nu} = \frac{\sum_{k=1}^{\infty} k^2 i_k}{\sum_{k=1}^{\infty} k i_k}, \tag{13}$$

where i_k is the observed number of existing orders of size k. The estimator $\hat{\mu}$ for the parameter μ can be obtained as the average disconnect rate for observed circuits

$$\hat{\mu} = \frac{m}{\tau}, \tag{14}$$

where m is the total number of disconnects observed, and τ is the sum of the observed connection times for all circuits; μ can also be obtained from estimators of the churn and growth rate through the use of (1).

Estimation of these parameters from data supplied by New Jersey Bell Telephone Co. is being investigated. Estimates of the disconnect rate $\hat{\mu}$ by service family are available in the Reed and Smith paper,[2] in which it is shown that the lifetimes of special-service circuits are well approximated by exponential functions with means dependent on the service families.

III. A GENERALIZED MODEL

This section treats a model that is more general than that which we propose for special-service activity in most cases. The analysis presented here will be applied to the specific model in Section IV.

3.1 Description of the generalized model

We examine an arbitrarily defined category of special-service circuits (for example, circuits of a particular service family in a given wire center) and divide the active circuits into independent groups. Possibly, each group is the demand from a single user, since it is reasonable that the activity of one user does not affect another. To facilitate this method of thinking we shall refer to the groups as "orders." Each order becomes nonzero for the first time at some point in time (referred to as the arrival or connect time of the order) and then has some history of changing size in some arbitrary manner before possibly becoming zero again indefinitely at some time (the departure or disconnect time of the order). The length of the interval between the arrival and departure of an order will be called the lifetime of the order. Obviously, the number of active circuits at any time equals the sum of the sizes of the existing orders at that time.

We assume that there is a large pool of customers (or potential orders) so that the arrival of an order has little effect on the potential arrival of others. Thus, the arrival of orders can be modeled by a nonhomogeneous Poisson process, whose intensity at time t is given by some function $\lambda(t)$. For background on this process see Ross[3] or Karlin and Taylor.[4] Denote the probability that an arriving order at time t is initially of size m as $q_m(t)$, and let $P^*_{mn}(t, x)$ be the probability that an order arriving at time t of initial size m as becomes size n at time $x \geqslant t$.

3.2 Distribution of the number of active circuits at a given time

Since the orders are noninterfering it can easily be seen (see Appendix B) that the number of orders of size n at time x is Poisson distributed with mean $\alpha_n(x)$, where

$$\alpha_n(x) = \sum_m \int_{-\infty}^{x} \lambda(t) q_m(t) P^*_{mn}(t, x) dt, \tag{15}$$

and that the numbers of orders of different sizes at time x are independent of each other. If $Y(x)$ is the total number of active special-services circuits in the category of interest at time x, then $Y(x)$ has a compound Poisson distribution (see Appendix A), and

$$E[Y(x)] = \sum_{n=1}^{\infty} n\alpha_n(x), \tag{16}$$

and

$$\text{var}[Y(x)] = \sum_{n=1}^{\infty} n^2\alpha_n(x). \tag{17}$$

3.3 Distribution of future active circuits due to present orders

The transient behavior of this model is easily derived if one has knowledge of the distribution of *order sizes* at a given time. We treat this case first and then consider the more difficult case where only the total number of *active circuits* at a given time is known. In either case, we will find the distribution of the number of active circuits at time y resulting from the orders observed to be active at time x. The total circuits active at time y is the sum of this with the number of circuits at time y resulting from orders arriving between x and y.

Case 1: Order sizes known

Given an order is of size n at time x, the conditional density that it arrived as an order of size m at time t is $\rho_{mn}(t, x)$, where

$$\rho_{mn}(t, x) = \frac{\lambda(t)q_m(t)P^*_{mn}(t, x)}{\alpha_n(x)} \tag{18}$$

and if several orders of size n are present at x their arriving times and sizes may be considered to be conditionally independent (see Appendix B). Thus an order of size n at time x becomes an order of size l at time $y \geqslant x$ with probability $r_{nl}(x, y)$ where

$$r_{nl}(x, y) = \alpha_n^{-1}(x) \sum_{m=1}^{\infty} \int_{-\infty}^{x} \lambda(t)q_m(t)P^*_{mn}(t, x)q_{mnl}(t, x, y)dt, \tag{19}$$

and $q_{mnl}(t, x, y)$ is the conditional probability that an order arriving as size m at time t which is of size n at time x becomes size l at time y. Note that q_{mnl} is not available solely from P^*.

Equation (19) allows us to compute the distribution of the total number of circuits at time y that were due to orders *observed* at time x, since all orders behave independently. Evaluating these distributions explicitly can be quite difficult. We can, however, easily evaluate the moments. Let $M_n(x, y)$ be the mean order size at time y for an order observed to be size n at time x, and let $V_n(x, y)$ be the mean order size at time y for an order observed to be size n at time x, and let $V_n(x, y)$ be the analogously defined variance. Then

$$M_n(x, y) = \sum_{l=1}^{\infty} l r_{nl}(x, y), \tag{20}$$

$$V_n(x, y) = \sum_{l=1}^{\infty} l^2 r_{nl}(x, y) - M_n^2(x, y). \tag{21}$$

If $i_n(x)$ is the number of orders of size n observed at time x, and $M(x, y)$ and $V(x, y)$ denote the mean and variance of the number of circuits at time y due to orders observed at time x, then

$$M(x, y) = \sum_{n} i_n(x)M_n(x, y), \tag{22}$$

and

$$V(x, y) = \sum_n i_n(x) V_n(x, y). \tag{23}$$

Note that there is a potential problem if orders can become size zero and then become nonzero later, since determination of i_0, the number of active orders of size 0, may be an impossible task.

Case 2: Order sizes unknown

We now examine the more difficult case where we observe the total number of active circuits at time x (call this k), without observing the distribution of the order sizes. The conditional probability that there are j_1 orders of size 1, j_2 orders of size 2, etc., given that k total circuits are observed at time x, written $\delta_{k,x}(j_1, j_2, \ldots)$, is easily found to be

$$\delta_{k,x}(j_1, j_2, \ldots) = \frac{\prod_i [\alpha_i(x)^{j_i}/j_i!]}{\sum_{j_1+2j_2+\cdots=k} \prod_i [\alpha_i(x)^{j_i}/j_i!]}, \tag{24}$$

provided that $j_1 + 2j_2 + \cdots = k$. Let the conditional first and second moment of the number of circuits at time y due to orders observed at time x, given that a total of k circuits were observed at time x, be $M_{k,x}(y)$ and $M^{(2)}{}_{k,x}(y)$ respectively. Then

$$M_{k,x}(y) = \sum E(J_i) M_i(x, y), \tag{25}$$

where J_i is a random variable with the same distribution as the conditional number of orders of size i at time x, so that the expectation is the expectation with respect to the probability distribution given in eq. (24). Also,

$$M^{(2)}_{k,x}(y) = \sum E(J_i) V_i(x, y) + E((\sum J_i M_i(x, y))^2), \tag{26}$$

where the expectation is in the same sense as before. Needless to say, these expectations with respect to the distribution in (24) are very difficult to evaluate for substantial k.

Things simplify somewhat if

$$M_i(x, y) = i\theta(x, y), \tag{27}$$

that is, if the conditional means are proportional to the size of the order. In this case, (25) and (26) give

$$M_{k,x}(y) = k\theta(x, y), \tag{28}$$

and

$$M^{(2)}_{k,x}(y) = \sum E(J_i) V_i(x, y) + k^2\theta^2(x, y), \tag{29}$$

so that

$$V_{k,x}(y) = \sum E(J_i) V_i(x, y), \tag{30}$$

where $V_{k,x}(y)$ denotes the conditional variance. Equation (30) can be approximated by using the following approximation which is intuitively reasonable for k near $\sum i\alpha_i(x)$,

$$E(J_i) \approx k \frac{\alpha_i(x)}{\sum i\alpha_i(x)}. \tag{31}$$

In this case, (30) and (31) give the following useful approximation:

$$V_{k,x}(y) \approx k \frac{\sum \alpha_i(x) V_i(x, y)}{\sum i\alpha_i(x)}. \tag{32}$$

3.4 Distribution of future active circuits due to future orders

In section 3.3, we found the mean and the variance of the number of circuits at time y from orders observed at time x. To obtain the total number of circuits at time y, we need to add to this the (independent) number of circuits due to orders arriving between time x and time y. The number of orders of size n at time y that arrived between times x and y is easily seen to be Poisson with mean $\alpha_n(x, y)$, where

$$\alpha_n(x, y) = \sum_{m=1}^{\infty} \int_x^y \lambda(t) q_m(t) P^*_{mn}(t, y) dt, \tag{33}$$

and the number of orders of different sizes are independent of each other (see Appendix B). Thus, the number of circuits at time y due to arrivals occurring between x and y has a compound Poisson distribution (see Appendix A) with mean and variance denoted $M^*(x, y)$ and $V^*(x, y)$, where

$$M^*(x, y) = \sum_n n\alpha_n(x, y), \tag{34}$$

and

$$V^*(x, y) = \sum_n n^2 \alpha_n(x, y). \tag{35}$$

3.5 Mean and variance of future active circuits

To find expressions for the mean or variance of the total number of active circuits at time y, we merely add together the appropriate means or variances from the circuits active at time y due to orders observed at time x and from the circuits active at time y due to arrivals between x and y, since these are independent. For example, eqs. (28) and (34) give

$$M^T_{k,x}(y) = k\theta(x, y) + \sum_n n\alpha_n(x, y), \tag{36}$$

where $M^T_{k,x}(y)$ is the total mean number of circuits observed at time y given k circuits are observed at time x [and assuming relationship (27)]. Also, eqs. (32) and (35) give the following approximation:

$$V^T_{k,x}(y) \approx k \frac{\sum \alpha_i(x) V_i(x, y)}{\sum i\alpha_i(x)} + \sum n^2 \alpha_n(x, y), \tag{37}$$

where $V^T_{k,x}(y)$ is the similarly defined variance.

3.6 Churn

We have previously defined churn as the minimum of the disconnect rate per circuit and the connect rate per circuit. Values of churn from other definitions are also easily obtained. We will here derive the churn, which happens to be a function of time in this case. To compute churn we need to know the probability measure for the individual order histories. Let $U_m(t, x)$ be the expected number of connects for an order of size m arriving at time t in the interval $[t, x]$ (thus $U_m(t, t) = m$). The expected total connect rate at time x, denoted $U(x)$, is then found to be:

$$U(x) = \frac{d}{dx}\left(\sum_m \int_{-\infty}^{x} \lambda(t) q_m(t) U_m(t, x) dt\right), \tag{38}$$

and similarly for the disconnects using the variable D,

$$D(x) = \frac{d}{dx}\left(\sum_m \int_{-\infty}^{x} \lambda(t) q_m(t) D_m(t, x) dt\right), \tag{39}$$

and thus we obtain the churn at time x, $\gamma(x)$:

$$\gamma(x) = \min\{D(x)/E[Y(x)],\ U(x)/E[Y(x)]\}, \tag{40}$$

where $E[Y(x)]$ is given by (16).

IV. THE MODEL FOR SPECIAL-SERVICE CIRCUIT ACTIVITY

Here, we assume that the demand rate grows exponentially and that the behavior of orders is not dependent on the time of arrival. Specifically, we assume,

$$\lambda(t) = \lambda_o e^{\beta t}, \tag{41}$$

$$q_m(t) = q_m, \tag{42}$$

and

$$P^*_{mn}(t, x) = P_{mn}(x - t). \tag{43}$$

Later we will assume a specific form for P_{mn}.

Assumptions (41) through (43) are equivalent to:

1. exponential growth in the rate of new orders at rate β (new orders occur as a nonhomogeneous Poisson process),

2. the probability that a new order is for m circuits is q_m, and

3. an order initially for m circuits requires a total of n circuits after z units of time with probability $P_{mn}(z)$. We will shortly further specify P_{mn} to represent unchanging orders of exponential lifetime.

We now explore the consequences of (41) to (43) in the analysis presented in Section III. Substituting into (15) we find that the number of orders of size n at time x is Poisson distributed with mean $\alpha_n(x)$, where

$$\alpha_n(x) = \alpha_n e^{\beta x}, \tag{44}$$

and

$$\alpha_n = \lambda_o \sum_m q_m \int_o^\infty e^{-\beta y} P_{mn}(y) dy. \tag{45}$$

(The total number of circuits required at any time has a compound Poisson distribution, see Appendix A.) Thus, the mean and variance of the number of circuits at time x, $Y(x)$, are growing exponentially at the same rate, and the ratio remains fixed:

$$E[Y(x)] = e^{\beta x} \sum_{n=1}^{\infty} n\alpha_n, \tag{46}$$

$$\operatorname{var}[Y(x)] = e^{\beta x} \sum_{n=1}^{\infty} n^2\alpha_n, \tag{47}$$

or

$$\operatorname{var}[Y(x)] = \nu E[Y(x)], \tag{48}$$

where

$$\nu = \frac{\sum_{n=1}^{\infty} n^2\alpha_n}{\sum_{n=1}^{\infty} n\alpha_n}. \tag{49}$$

Further results are possible if the behavior for orders over time is specified. We assume that the order size does not change over its lifetime, which has a common distribution with c.d.f. F independent of size. Later we will assume that F is an exponential distribution. Although in practice the number of circuits per order does change with time, it is conceivable that this movement is relatively unimportant; or even if important, that the general form of eqs. (2) and (3) will hold, although the parameter μ may then have a different physical meaning than we will associate here. In the model suggested here, $P_{mo}(y) = F(y)$; $P_{mm}(y) = 1 - F(y)$; $P_{mn}(y) = 0$, $n \neq 0$, $n \neq m$. In this case we may compute α_n more explicitly. Substituting into (45), we obtain

$$\alpha_n = q_n \left(\frac{\lambda_o}{\beta} [1 - \tilde{F}(\beta)] \right), \tag{50}$$

where

$$\tilde{F}(\beta) = \int_o^\infty e^{-\beta y} dF(y). \tag{51}$$

When the lifetimes are exponentially distributed with mean $1/\mu$, i.e. $F(x) = 1 - e^{-\mu x}$,

$$\alpha_n = q_n \lambda_o (\mu + \beta)^{-1}. \tag{52}$$

Also, the batchiness ν is related to the order-size distribution; substitution into (49) yields

$$\nu = \frac{\sum n^2 q_n}{\sum n q_n}. \tag{53}$$

The assumption that the order size does not change throughout its lifetime also allows more explicit representation of the mean and variance of the future requirements for circuits. Our development here parallels that of Section III. We first compute the probability that an observed order will change size during the period of observation. Recall that $q_{mnl}(t, x, y)$ is the conditional probability that an order is of size l at time y given that it was of size n at time x and arrived as size m at time t. We easily obtain:

$$q_{mmm}(t, x, y) = \frac{\bar{F}(y - t)}{\bar{F}(x - t)}, \tag{54}$$

and

$$q_{mmo}(t, x, y) = 1 - q_{mmm}(t, x, y),$$

where

$$\bar{F}(x) = 1 - F(x),$$

and $q_{mml}(t, x, y) = 0$, $l \neq 0$, $l \neq m$. The value of q is irrelevant for $n \neq m$.

We next find the probability that an order of size n at time x becomes of size l at time y, which we denote $r_{nl}(x, y)$. Substitution into

(19) gives

$$r_{nl}(x, y) = 0, \qquad n \neq l, \qquad n \neq 0; \tag{55}$$

$$r_{nn}(x, y) = \bar{G}(y - x), \tag{56}$$

where

$$\bar{G}(t) = \frac{\int_0^\infty e^{-\beta z}\bar{F}(z + t)dz}{\int_0^\infty e^{-\beta z}\bar{F}(z)dz}; \tag{57}$$

and

$$r_{n0}(x, y) = 1 - \bar{G}(y - x).$$

Note that in the exponential-lifetime case, where $\bar{F}(z) = e^{-\mu z}$,

$$\bar{G}(y) = e^{-\mu y}. \tag{58}$$

We next find the mean and variance of the number of circuits in an order at time y, which was observed to be of size n at time x, denoted $M_n(x, y)$ and $V_n(x, y)$, respectively. Substitutions of (55) and (56) into (20) and (21) give:

$$M_n(x, y) = n\bar{G}(y - x), \qquad \text{and} \tag{59}$$

$$V_n(x, y) = n^2\bar{G}(y - x)[1 - \bar{G}(y - x)]. \tag{60}$$

Notice that the conditional means are proportional to the size of the order, i.e., (59) implies (27).

We now focus on the mean and variance of the number of circuits at time y due to orders which were observed at time x, given that k circuits were observed at time x. These quantities are denoted $M_{k,x}(y)$ and $V_{k,x}(y)$ respectively. Equation (28) gives

$$M_{k,x}(y) = k\bar{G}(y - x). \tag{61}$$

We also conclude that, given the approximation in (32),

$$V_{k,x}(y) \approx k\bar{G}(y - x)[1 - \bar{G}(y - x)] \frac{\sum n^2 q_n}{\sum n q_n}, \tag{62}$$

thus

$$V_{k,x} \approx \nu[1 - \bar{G}(y - x)]M_{k,x}(y), \tag{63}$$

where use has been made of (53) and (61).

We next find the expected number of orders of size n at time y that arrived during the interval (x, y) denoted $\alpha_n(x, y)$. Use of (3.3) yields

$$\alpha_n(x, y) = \lambda_o e^{\beta y} q_n \int_o^{y-x} e^{-\beta z} \bar{F}(z) dz, \tag{64}$$

where use is made of the fact $P^*_{mn}(t, x) = q_{mmn}(t, t, x)$, which follows from (54). When the lifetime distribution for orders is exponential, (64) becomes

$$\alpha_n(x, y) = \lambda_o e^{\beta y} q_n \left(\frac{1 - e^{-(\beta+\mu)(y-x)}}{\beta + \mu} \right). \tag{65}$$

The mean and variance of all circuits at time y due to orders arriving in the interval (x, y), $M^*(x, y)$ and $V^*(x, y)$, respectively, can be obtained by substitution of (64) into (34) and (35) yielding

$$M^*(x, y) = \lambda_o \sum n q_n e^{\beta y} \int_o^{y-x} e^{-\beta z} \bar{F}(z) dz, \tag{66}$$

and

$$V^*(x, y) = \nu M^*(x, y), \tag{67}$$

while for exponential lifetimes,

$$M^*(x, y) = \frac{D_o}{\beta + \mu} (e^{\beta t} - e^{-\mu t}), \tag{68}$$

where

$$D_o = \lambda_o e^{\beta x} \sum n q_n, \tag{69}$$

and

$$t = y - x.$$

Note that (61) and (68) [or (36)] give eq. (2), and (63) and (67) [or (37)] give (3), since the total number of active circuits at time y is the sum of the number of active circuits due to orders present at time x and the number of active circuits due to order arrivals between times x and y, and these random variables are independent. Equations (5) through (9) can easily be derived by the methods described in the paper, although we omit the details here.

Next, turning our attention to churn for the specific model of this section, we find that the expected number of connects in the interval $[t, x]$ for an order arriving at time t of size m, denoted $U_m(t, x)$, is given by

$$U_m(t, x) = m, \tag{70}$$

and similarly,

$$D_m(t, x) = mF(x - t), \tag{71}$$

where the variable D represents disconnects. The total connect rate, total disconnect rate, and churn rate at time x, $U(x)$, $D(x)$, and $\gamma(x)$, respectively, can be obtained from (38) through (40), yielding

$$U(x) = \lambda_o e^{\beta x} \sum m q_m, \tag{72}$$

$$D(x) = \lambda_o e^{\beta x} \sum m q_m \tilde{F}(\beta), \tag{73}$$

and

$$\gamma(x) = \gamma = \beta \tilde{F}(\beta)/(1 - \tilde{F}(\beta)), \qquad \beta \geqslant 0;$$

$$\gamma(x) = \gamma = \beta/(1 - \tilde{F}(\beta)), \qquad \beta < 0. \tag{74}$$

In the special case where lifetimes are exponentially distributed,

$$\gamma = \mu, \qquad \beta \geqslant 0;$$

$$\gamma = \mu + \beta, \qquad \beta < 0. \tag{75}$$

V. ESTIMATION OF THE PARAMETERS OF INTEREST

In this section, we describe the methodology that can be used to estimate the three key parameters of the model; β, the growth rate; μ, the disconnect rate; and ν, the batchiness.

5.1 Estimation of β

Suppose that we wish to estimate β on the basis of observed arrivals of orders, which by assumption occur according to a nonhomogeneous Poisson process with intensity $\lambda_o e^{\beta t}$. Suppose that the system is observed over the interval $[-\Theta, 0]$ and arrivals have been noted at times $t_1, \ldots, t_n$. We show how to obtain the maximum-likelihood estimator for β. (For a discussion of maximum-likelihood estimation, see any elementary book on statistics such as Mood & Graybill.)[5] The log-likelihood function, $\ln L(n, t_1, \ldots, t_n)$, is easily seen to be

$$\ln L(n, t_1, \ldots, t_n) = n \ln\lambda_o + \beta \sum_{i=1}^{n} t_i - \lambda_o \left(\frac{1 - e^{\beta\Theta}}{\beta}\right). \tag{76}$$

Differentiating with respect to λ_0 and β we find the necessary conditions for a maximum:

$$n/\lambda_o = \frac{1 - e^{-\beta\Theta}}{\beta}, \tag{77}$$

$$\sum_{i=1}^{n} t_i - \frac{\lambda_o \Theta}{\beta} e^{-\beta\Theta} + \lambda_o \left(\frac{1 - e^{-\beta\Theta}}{\beta^2}\right) = 0. \tag{78}$$

Using (77) to eliminate λ_o in (78), we obtain

$$S = \frac{x e^x - e^x + 1}{x(e^x - 1)} \equiv f(x), \tag{79}$$

where

$$S = \frac{\sum t_i}{n\Theta} + 1 \tag{80}$$

and

$$x = \beta\Theta. \tag{81}$$

The function f defined in (79) can be seen to be strictly monotonic with range between 0 and 1. Therefore, eq. (79) allows us to solve for $\beta\Theta$ as $f^{-1}(S)$, where S is the statistic defined in (80) equal to the proportion of the interval (*after* $-\ \Theta$) at which the average time of arrival occurs. Thus, the maximum-likelihood estimator for β, written $\hat{\beta}$, is given by

$$\hat{\beta} = \frac{f^{-1}(S)}{\Theta}. \tag{82}$$

The function f has the properties:

$$f(-\infty) = 0, f(0) = 1/2, f(\infty) = 1, \quad \text{and} \quad f(x) + f(-x) = 1.$$

Thus,

$$\begin{aligned} f^{-1}(0) &= -\infty, \\ f^{-1}(1/2) &= 0, \\ f^{-1}(1) &= \infty, \end{aligned}$$

and

$$f^{-1}(1/2 - x) = -f^{-1}(1/2 + x).$$

The function f^{-1} is tabulated in Table I.

For small x, $f(x)$ may readily be expanded in the power series:

$$f(x) = 1/2\left[1 + (1/6)x - \frac{1}{360}x^3 \cdots\right]$$

so that

$$f^{-1}(1/2 + y) = 12y + 28.8y^3 \cdots. \tag{83}$$

Similarly, a large f expansion yields

$$f^{-1}(1 - 1/y) \approx y - y^2 e^{-y}. \tag{84}$$

We may also determine the mean and variance of the statistic S given the correct parameter β and the number of observed arrivals. It is well known that the distribution of the arrival times for a nonhomogeneous Poisson process, conditioned on a given number of arrivals

Table I—Values of the function f^{-1} useful in estimating the growth rate β, and several approximations for the function [see eq. (79) and following]

X	$f^{-1}(X)$	12X-6	Approximation in (83)	Approximation in (84)
0.50	0.0000	0.0000	0.0000	
0.52	0.2402	0.2400	0.2402	
0.54	0.4819	0.4800	0.4818	
0.56	0.7263	0.7200	0.7262	
0.58	0.9751	0.9600	0.9747	
0.60	1.2299	1.2000	1.2288	
0.62	1.4926	1.4400	1.4898	
0.64	1.7654	1.6800	1.7590	
0.66	2.0507	1.9200	2.0380	
0.68	2.3517		2.3280	
0.70	2.6721		2.6308	
0.72	3.0168		2.9467	
0.74	3.3920			
0.76	3.8060			
0.78	4.2703			
0.80	4.8010			4.8316
0.82	5.4219			5.4362
0.84	6.1691			6.1746
0.86	7.1010			7.1025
0.88	8.3164			8.3166
0.90	9.9954			9.9955
0.92	12.4994			12.4994
0.94	16.6667			16.6667
0.96	25.0000			25.0000
0.98	50.0000			50.0000
1.00	∞			∞

in the interval, is the same as the order statistics from n i.i.d. random variables with probability density proportional to the arrival rate. Thus S has the distribution of the average of n i.i.d. random variables, Y_i on [0, 1] with density $g(\rho)$, where

$$g(\rho) = \frac{xe^{\rho x}}{e^x - 1}$$

and

$$x = \beta\Theta.$$

It is easily seen that

$$E(Y) = f(x) \tag{85}$$

and

$$\text{var}(Y) = \frac{1}{x^2} - \frac{1}{e^x + e^{-x} - 2}. \tag{86}$$

Equation (86) is valid if $x \neq 0$; when $x = 0$, var$(Y) = 1/12$, the limit of (86) as x goes to 0. The expression for the variance is symmetric in x and takes its maximum value at $x = 0$.

Thus, for a given value of the growth rate β, and a (large) given number of observations n, the observed statistic S is approximately normally distributed with mean $f(x)$ and variance less than or equal to $1/12n$. This observation can be readily translated into confidence intervals through the use of elementary statistical theory. For example, a 95-percent confidence interval for $f(x)$ (assuming normality of the statistic) is

$$S - 1.96 \sqrt{\frac{1}{12n}} \leq f(x) \leq S + 1.96 \sqrt{\frac{1}{12n}}, \tag{87}$$

which translates to

$$f^{-1}\left(S - 1.96 \sqrt{\frac{1}{12n}}\right) \leq \beta\Theta \leq f^{-1}\left(S + 1.96 \sqrt{\frac{1}{12n}}\right). \tag{88}$$

If S is close to 0.5, then we can use $f^{-1}(x) \approx 12x - 6$ [see (83)] to obtain for the 95-percent confidence interval for $\beta\Theta$

$$\beta\Theta = 12S - 6 \pm \frac{6.79}{\sqrt{n}}. \tag{89}$$

5.2 Estimation of ν

There are several possible statistics for the measurement of the batchiness ν. We shall take as our starting point eq. (49) which defines the batchiness in terms of the distribution of the order size at (any) point in time. This is preferable and is more robust than using the distribution of the order size on arrival, although the two happen to equal when order sizes do not change with time. Thus, if ν is to be estimated based on observation of the system at a given point in time at which i_n orders of size n are observed, then a reasonable estimator for ν, which we write $\hat{\nu}$, is:

$$\hat{\nu} = \frac{\sum n^2 i_n}{\sum n i_n}. \tag{90}$$

When the number of circuits does change during the lifetime of an order, it is possible that the form of (3) still holds. In this case, it is likely that the parameter ν which multiplies each of the two terms in (3) is different. Equation (90) is a reasonable estimator for the multiplier of the second term. The multiplier of the first term should be estimated by other methods.

Table II—Notation

D_o	Present ($t = 0$) circuit demand rate due to new orders.
$D(x)$	Expected total disconnect rate at time x.
$D_m(t, x)$	Expected number of disconnects in the interval $[t, x]$ for an order of size m arriving at time t.
$i_n(x)$	Observed number of orders of size n at time x.
J_i	Conditional number of orders of size i due to orders observed at time x given that k circuits were observed at time x.
$M(x, y)$	Mean total number of circuits at time y due to orders.
$M^*(x, y)$	Mean of the total number of circuits at time y due to orders arriving between x and y.
$M_{k,x}^T(y)$	Mean of the number of circuits at time y given k are observed at time x.
$M_{k,x}(y)$	Conditional expectation of the number of circuits at time y due to orders observed at time x given that k circuits are observed at time x.
$M_{k,x}^{(2)}(y)$	Conditional second moment of the number of circuits at time y due to orders observed.
$M_n(x, y)$	Mean order size at time y for an order observed to be of size n at time x.
$P_{mn}(t)$	Probability that an order of initial size m becomes of size n, t time units after arrival.
$P_{mn}^*(t, x)$	Probability that an order arriving at time t of initial size m becomes of size n at time x.
q_m	$q_m(t)$ when there is no dependence on t.
$q_m(t)$	Probability that an order at time t is initially for m circuits.
$q_{mnl}(t, x, y)$	Conditional probability that an order arriving as size m at time t and of size n at time x becomes of size l at time y.
$r_{nl}(x, y)$	Probability that an order of size n at time x becomes an order of size l at time y.
$U(x)$	Expected total connect rate at time x.
$U_m(t, x)$	Expected number of connects for an order of size m arriving at time t in the interval $[t, x]$.
$V(x, y)$	Variance of the total number of circuits at time y due to orders observed active at time x.
$V^*(x, y)$	Variance of the total number of circuits at time y due to orders arriving between x and y.
$V_{k,x}^T(y)$	Variance of the number of circuits at time y given that k are observed at time x.
$V_{k,x}(y)$	Conditional variance of the total number of circuits at time y due to orders observed at x given that k circuits are observed at time x.
$V_n(x, y)$	Variance of order size at time y for an order observed to be of size n at time x.
$Y(x)$	Total number of active special service circuits at time x.
α_n	Constant of proportionality for the exponential growth of $\alpha_n(x)$.
$\alpha_n(x)$	Mean number of orders of size n at time x.
$\alpha_n(x, y)$	The number of orders of size n at time y which arrived between times x and y.
β	Growth rate.
γ	Churn.
$\delta_{k,x}(j_1, j_2, \ldots)$	Conditional probability that there are j_1 orders of size 1, j_2 orders of size 2, etc., at time x given that k total circuits are observed at time x.
$\theta(x, y)$	Defined in (27).
Θ	Length of observation period.
λ_o	Present ($t = 0$) arrival rate of orders.
$\lambda(t)$	Instantaneous arrival rate of orders at time t.
μ	Disconnect rate (per circuit).
ν	Batchiness.
$\rho_{mn}(t, x)$	The conditional probability density that an order arrived at time t of initial size m given that it is of size n at time x.

5.3 *Estimation of* μ

The estimation of μ is relatively straightforward. The maximum-likelihood estimator is given in (14) and further details including estimated values by service family are given in Reed and Smith.[2]

5.4 *Estimation of* D_o

Equation (77) allows the MLE estimator of λ_o,

$$\hat{\lambda}_o = \frac{n\hat{\beta}}{1 - e^{-\hat{\beta}\Theta}}, \tag{91}$$

where the estimator $\hat{\beta}$ has been previously described in (82). The estimator of D_o (the instantaneous demand at the end of the observation interval of length Θ), $\hat{D}_o$ then is

$$\hat{D}_o = \lambda_o \hat{N}, \tag{92}$$

where $\hat{N}$ is an estimate of the average batch size. The previous expectation can be estimated from the order sizes at arrival epochs, or more crudely from the general distribution of order sizes at an arbitrary point of time.

Interestingly enough, $D_o/(\mu + \beta)$ can be estimated solely from the number of active circuits at a point of time. For simplicity, we assume that the orders are solely of size one, although the analysis could be repeated for other distributions. In this case, the following analysis is applicable.

Suppose that the number of active circuits at time t is a Poisson random variable with mean $\lambda e^{\beta t}$. The time that the mean is between x and $x + dx$ is $dx/x\beta$. The expected time that the mean is between x and $x + dx$ and a total of k active circuits are observed is $\frac{dx}{x\beta}\frac{x^k}{k!}e^{-x}$. Thus, if k active circuits are observed, the conditional distribution of the mean (in our case this is $D_o/(\mu + \beta)$) has density proportional to $x^{k-1}e^{-x}$, or is a standard gamma random variable with k degrees of freedom. This random variable has mean and variance equal to k. Thus, if k circuits are observed, the conditional distribution of $D_o/(\mu + \beta)$ has mean k and variance k, and is approximately normally distributed if k is large. This information can be used to modify eqs. (2) and (3), to take into account the variability of $D_o/(\mu + \beta)$ to obtain:

$$M_k(t) = ke^{\beta t}, \tag{93}$$

and

$$V_k(t) = \nu k[e^{\beta t} - e^{-2\mu t}] + k[e^{\beta t} - e^{-\mu t}]^2, \tag{94}$$

where the subscript k on the variables on the left-hand side indicates conditional means and variances knowing β, μ, and ν but not knowing D_o. Note that the variance-to-mean ratio is unbounded for increasing t, since errors in estimation of D_o accumulate indefinitely.

VI. SUMMARY

We have described a model for special-services activity useful in forecasting special-services requirements. It requires three physical characterizations of the process (growth rate, disconnect rate, and batchiness) and two instantaneous measurements (the current number of active circuits and the instantaneous rate of new connects). We give means and variances for the numbers of active circuits at a given point in the future and for the total number of connects or disconnects during a future period. The distribution of these variables can be computed by the methodology described in the paper. We also describe general techniques for estimation of the required parameters.

Work is being undertaken to verify and calibrate the model with the New Jersey Bell Telephone Co. database and will be reported elsewhere.

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APPENDIX A

The Compound Poisson Random Variable

Briefly, a random variable is said to be a compound Poisson random variable if it can be thought of as the sum of a Poisson number of independent identically distributed positive integer-valued random variables. Thus Y is a compound Poisson random variable if

$$Y = \sum_{i=1}^{N} X_i,$$

where N is a Poisson random variable with mean α, N and the X_i are independent and

$$P\{X_i = n\} = p_n.$$

We have rather easily

$$E(Y) = \alpha E(X) = \alpha \sum n p_n$$

$$\mathrm{Var}(Y) = \alpha E(X^2) = \alpha \sum n^2 p_n.$$

An alternative (and equivalent) way of specifying a compound Poisson random variable is

$$Y = \sum_{i=1}^{\infty} i Z_i,$$

where

Z_i are independent Poisson random variables with $E(Z_i) = \alpha p_i \equiv \alpha_i$. In this case it is convenient to think of Z_i as the number of batches or orders of size n that are aggregated to give the total number denoted Y.

APPENDIX B

The Nonhomogeneous Poisson Process

A process which counts events is a nonhomogeneous Poisson process (see, for example, Ross, Ref. 5) with intensity $\lambda(t) \geqslant 0$ if the number of events in the interval $[x, y]$ is a Poisson random variable with mean $\int_x^y \lambda(t)dt$, and the number of events in disjoint intervals are independent.

Fact: If events from a nonhomogeneous Poisson process are of two types, and an event at time t is of Type 1 with probability $p(t)$, then the process which counts Type 1 events is a nonhomogeneous Poisson process with intensity $\lambda(t)p(t)$, and it is independent of the counting process which counts Type 2 events [which is a nonhomogeneous Poisson process intensity $\lambda(t)(1 - p(t))$].

Fact: Suppose that a nonhomogeneous Poisson process is observed over the interval $[x, y]$ and n events are observed. If the times of these events are arranged in random order, their distribution is identical to that of n independent identically distributed random variables whose density at t is

$$\frac{\lambda(t)}{\displaystyle\int_x^y \lambda(z)dZ}$$

if $t \in [x, y]$ and is zero otherwise.

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