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Theory of Reflection From Antireflection Coatings

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The reflection that occurs when a beam, rather than a plane wave, is incident normally on a quarter-wavelength matching layer can be of vital importance in semiconductor laser design. An analysis in three dimensions is given for the general case of a field of arbitrary form and polarization incident on the matching layer. The field is represented as an angular spectrum of plane waves, each component plane wave being modified by the appropriate Fresnel reflection coefficient to give the field reflected back onto the diode structure. Brown's antenna reciprocity theorem is used to determine the amplitude of the corresponding mode traveling back down the diode.

I. INTRODUCTION

Antireflection coatings are used on one face of superluminescent diodes[1] and on both faces of diode-laser amplifiers.[2] The theoretical performance of such coatings has been analyzed by Clarke[3] using the technique of representing the emerging laser beam as an angular spectrum of plane waves, as originally applied by Reinhart et al.[4] and Gordon[5] to determine the reflectivity of an uncoated facet. Each plane wave was modified by the appropriate reflection coefficient of the uniform coating,[6] and Brown's antenna reciprocity theorem[7] used to calculate the amplitude of the wave coupled back into the device. The previous analysis[3] was restricted to two dimensions, on the grounds

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that the active region in the device would be a wide flat stripe, so that the emerging radiation would be a thin fan-shaped beam. Many important laser diode structures, particularly of the refractive index guided type, have relatively narrow active regions, hence the previous restriction is limiting. This restriction is removed in the present work, and the full three-dimensional analysis is presented.

II. FIELDS IN THE DIODE

The transverse electric field of a single mode traveling in the positive-z direction (see Fig. 1) along the length of the active-region stripe in a diode laser can be written in general as

$$\mathbf{E}_t^+(x, y, z) = [\mathbf{u}_x E_{tx}(x, y) + \mathbf{u}_y E_{ty}(x, y)]e^{-j\beta_m z}, \qquad (1)$$

where β_m is the phase constant of the mode and the time variation $\exp(j\omega t)$ has been suppressed. The field in this mode reflected back into the diode by the coating is

$$\mathbf{E}_t^-(x, y, z) = \rho[\mathbf{u}_x E_{tx}(x, y) + \mathbf{u}_y E_{ty}(x, y)]e^{+j\beta_m z}. \qquad (2)$$

The objective of this paper is to calculate the reflection coefficient ρ for arbitrary thickness h and refractive index n_2 of the coating. (Coupling to other modes is ignored here for the sake of simplicity.) It will be assumed that the beam eventually emerges into air, so that $n_3 = 1$, and that the refractive index of the diode has the effective value n_1, which is that of the active region in which the field is largely confined. (The surrounding bulk material has a refractive index that is some 10 percent below n_1.[1] A better choice of effective refractive index might therefore be a weighted average, as suggested by Kaplan's analysis.[8])

The field incident at the plane $z = 0$ can be represented as an

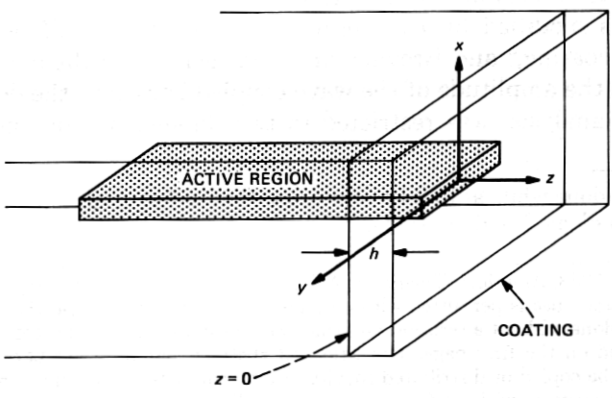

Fig. 1—Diode laser with coating.

angular spectrum of plane waves by the two spectrum functions $F_x(\alpha, \beta)$ and $F_y(\alpha, \beta)$, where (α, β, γ) are the direction cosines in the x-, y-, and z-directions.[9,10] Thus, the elemental plane wave incident in the direction (α, β) is $\mathbf{e}_{inc}(\alpha, \beta)d\alpha d\beta$, where

$$\mathbf{e}_{inc}(\alpha, \beta) = F_x(\alpha, \beta)\left(\mathbf{u}_x - \mathbf{u}_z \frac{\alpha}{\gamma}\right) + F_y(\alpha, \beta)\left(\mathbf{u}_y - \mathbf{u}_z \frac{\beta}{\gamma}\right) \tag{3}$$

with

$$F_x(\alpha, \beta) \leftrightarrow E_{tx}(x, y)$$

and

$$F_y(\alpha, \beta) \leftrightarrow E_{ty}(x, y), \tag{4}$$

in which $\leftrightarrow$ symbolizes a Fourier transform, such as

$$F_x(\alpha, \beta) = \frac{1}{\lambda_1^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_{tx}(x, y) \exp\{jk_1(\alpha x + \beta y\} dx dy. \tag{5}$$

The phase constant in the diode is $k_1 = 2\pi/\lambda_1 = n_1 k_0$, where k_0 is the phase constant of free space.

It should be noted for later reference that the above angular spectrum corresponds to a radiation far field (assuming that the subscript 1 region continues indefinitely but the active region stops in the plane $z = 0$), as $k_1 r \to \infty$, of[10]

$$\mathbf{E}(r, \theta, \phi) \simeq \frac{\exp(-jk_1 r)}{k_1 r} \mathbf{e}_{rad}(\alpha, \beta), \tag{6}$$

where the far-field vector pattern function is given in this case by

$$\mathbf{e}_{rad}(\alpha, \beta) = j2\pi\gamma \mathbf{e}_{inc}(\alpha, \beta) \tag{7}$$

and

$$\begin{aligned} \alpha &= \sin\theta \cos\phi \\ \beta &= \sin\theta \sin\phi \\ \gamma &= \cos\theta, \end{aligned} \tag{8}$$

where θ is the polar angle to the z-axis, ϕ is the azimuth angle in the x-y plane, and r is the distance to the point of observation.

III. REFLECTION AT THE COATING

The incident plane wave given by eq. (3) will be reflected by the coating. The amplitude reflection coefficient for a plane wave incident on such a uniform layer with its electric vector perpendicular to its plane of incidence (see Fig. 2) is[6]

$$R_\perp = \frac{P_1 \cos B + jP_3 \sin B}{P_2 \cos B + jP_4 \sin B} \tag{9}$$

and, with its electric vector parallel to its plane of incidence, the reflection coefficient is

$$R_\parallel = \frac{Q_1 \cos B + jQ_3 \sin B}{Q_2 \cos B + jQ_4 \sin B}, \tag{10}$$

where

$$P_{1,2} = n_2(1 - n_1^2 s^2/n_2^2)^{1/2}[n_1\gamma \mp (1 - n_1^2 s^2)^{1/2}]$$

$$P_{3,4} = n_1\gamma(1 - n_1^2 s^2)^{1/2} \mp n_2^2(1 - n_1^2 s^2/n_2^2) \tag{11}$$

$$Q_{1,2} = n_2(1 - n_1^2 s^2/n_2^2)^{1/2}[n_1(1 - n_1^2 s^2)^{1/2} \mp \gamma]$$

$$Q_{3,4} = n_1(1 - n_1^2 s^2/n_2^2) \mp n_2^2\gamma(1 - n_1^2 s^2)^{1/2} \tag{12}$$

with

$$s^2 = \alpha^2 + \beta^2 = 1 - \gamma^2 \tag{13}$$

and

$$B = (2\pi h/\lambda_2)(1 - n_1^2 s^2/n_2^2)^{1/2}, \tag{14}$$

where $\lambda_2 = \lambda_0/n_2$.

Note that when the magnitude of the sine of the angle of incidence $|s| > (n_1)^{-1}$, the wave will be totally internally reflected. In that case, the magnitude of the reflection coefficient will always be unity, but its phase will vary with the angle of incidence. But note also that the $\exp(j\omega t)$ sign convention adopted here means taking the negative

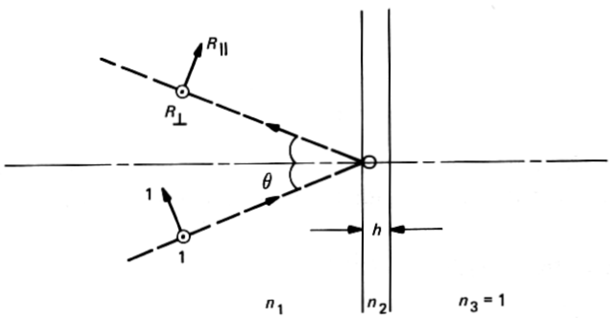

Fig. 2—Definition of the amplitude reflection coefficients $R_\perp$ and $R_\parallel$. Their phases are defined at O.

square root when the round-bracketed quantities in eqs. (11), (12), and (14) become negative.

The elemental plane wave given by the angular spectrum of eq. (3) consists, in general, of the sum of perpendicular and parallel polarized components, such that

$$\mathbf{e}_{\text{inc}} = \mathbf{e}_{\perp} + e_{\parallel}. \tag{15}$$

This resolution can be achieved by noting that the unit vector $\mathbf{u}_n$, which is both normal to the plane of incidence of the plane wave travelling in the direction (α, β) and also parallel to the bounding plane surface xOy, is

$$\mathbf{u}_n = \frac{1}{\sqrt{\alpha^2 + \beta^2}} [\mathbf{u}_x\beta - \mathbf{u}_y\alpha]. \tag{16}$$

Hence we may calculate

$$\mathbf{e}_{\perp} = \mathbf{u}_n[\mathbf{u}_n \cdot \mathbf{e}_{\text{inc}}) \tag{17}$$

and

$$\mathbf{e}_{\parallel} = \mathbf{e}_{\text{inc}} - \mathbf{u}_n(\mathbf{u}_n \cdot \mathbf{e}_{\text{inc}}). \tag{18}$$

The elemental plane wave $\mathbf{e}_{\text{refl}}(\alpha, \beta)d\alpha d\beta$ reflected by the coating is thus given by

$$\mathbf{e}_{\text{refl}}(\alpha, \beta) = R_{\perp}\mathbf{e}_{\perp}(\alpha, \beta) + R_{\parallel}\mathbf{u}_{\parallel}(\alpha, \beta) \tag{19}$$

and comes *from* the direction $(-\alpha, -\beta)$. (To avoid possible confusion it should be noted that the argument of $\mathbf{e}_{\text{refl}}(\alpha, \beta)$ denotes the direction of the incident wave.)

IV. COUPLING BACK INTO THE DIODE

Brown's antenna reciprocity theorem states that if a plane wave of vector amplitude $\mathbf{e}_p$ is incident from the direction $\mathbf{u}_p$ on a linear, reciprocal device, which when radiating has the far-field vector pattern function (see eq. 6) of $\mathbf{e}_{\text{rad}}(\mathbf{u})$, then the coupling ratio

$$c = \frac{\lambda^2}{j4\pi Z P_0} \mathbf{e}_p \cdot \mathbf{e}_{\text{rad}}(\mathbf{u}_p) \tag{20}$$

gives the complex ratio of the single-mode amplitude when receiving to that when transmitting a total power P_0.[7,11] Z and λ are the characteristic impedance and wavelength in the radiating medium. Equation (20) is a precise result, being a consequence ultimately of the Lorentz reciprocity theorem.

In the present instance the incident plane wave $\mathbf{e}_p$ is the elemental reflected plane wave $\mathbf{e}_{\text{ref}}d\alpha d\beta$ given by eq. (19), and so, integrating over all directions in the forward hemisphere, the reflection coefficient describing the returning mode amplitude is

$$\rho = \frac{\lambda^2}{j4\pi ZP_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{e}_{\text{refl}}(\alpha, \beta) \cdot \mathbf{e}_{\text{rad}}(-\alpha, -\beta) d\alpha d\beta \tag{21}$$

or

$$\rho = \frac{\lambda^2}{2ZP_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{R_{\parallel}\mathbf{e}_{\text{inc}}(\alpha, \beta) + \mathbf{u}_n(R_{\perp} - R_{\parallel}) [\mathbf{u}_n \cdot \mathbf{e}_{\text{inc}}(\alpha, \beta)]\} \cdot \mathbf{e}_{\text{inc}}(-\alpha, -\beta)\gamma d\alpha d\beta \tag{22}$$

with $\mathbf{e}_{\text{inc}}(\alpha, \beta)$ given by eq. (3) and $\mathbf{u}_n$ by eq. (16). The total radiated power P_0, when the radiation is specified by the two spectrum functions $F_x(\alpha, \beta)$ and $F_y(\alpha, \beta)$, is given by[12]

$$P_0 = \frac{\lambda^2}{2Z} \int_D \int \left[\frac{1 - \beta^2}{\gamma} |F_x(\alpha, \beta)|^2 + \frac{1 - \alpha^2}{\gamma} |F_y(\alpha, \beta)|^2\right] d\alpha d\beta, \tag{23}$$

where D is the domain of (α, β) such that $\alpha^2 + \beta^2 = \leq 1$.

V. APPLICATION TO A *Y*-POLARIZED LASER MODE

In order to see what this result means, consider a guided mode in the laser whose tangential electric field is wholly y-directed, for which therefore $F_x \equiv 0$. Then

$$\mathbf{e}_{\text{inc}}(\alpha, \beta) = F_y(\alpha, \beta) \left(\mathbf{u}_y - \mathbf{u}_z \frac{\beta}{\gamma}\right) \tag{24}$$

and

$$\mathbf{u}_n = \frac{1}{\sqrt{\alpha^2 + \beta^2}} (\mathbf{u}_x\beta - \mathbf{u}_y\alpha). \tag{25}$$

Consequently, the reflection coefficient in this case is

$$\rho = \frac{\lambda^2}{2ZP_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[R_{\parallel}\left(1 + \frac{\beta^2}{\gamma^2} - \frac{\alpha^2}{\alpha + \beta^2}\right) + R_{\perp} \frac{\alpha^2}{\alpha^2 + \beta^2}\right] \gamma F_y(\alpha, \beta) F_y(-\alpha, -\beta) d\alpha d\beta. \tag{26}$$

Then finally, assuming that the beam spread is vanishingly narrow in the y–z plane compared to the x–y plane,

$$\rho = \frac{\lambda^2 K}{2ZP_0} \int_{-\infty}^{\infty} \gamma R_{\perp} F_y(\alpha, 0) F_y(-\alpha, 0) d\alpha, \tag{27}$$

where K depends on the β-dependence of F_y. Equation (27) is the two-dimensional result used previously.[3]

VI. CONCLUSIONS

A complete three-dimensional analysis has been presented for the calculation of reflection from antireflection coatings. It reduces to the two-dimensional result given previously where the incident beam was assumed to be narrowly confined in one of the principal planes. The form and polarization of the field incident on the coating can be arbitrarily specified.

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