

## Some Traffic Overflow Problems with a Large Secondary Queue

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*When calls offered to a primary group of trunks find all of them busy, provisions are often made for these calls to overflow to other groups of trunks. Such traffic overflow systems have been of interest for a long time, but recently overflow systems that allow for some calls to be queued have been of importance. In this paper we analyze a traffic overflow system with queuing. The system consists of two groups, a primary and a secondary. We consider two cases which differ in the treatment of demands waiting in the primary queue. We present an analytical approach which is suitable if the secondary queue is large, or even infinite, and we contrast it with an earlier approach of ours which is more suitable if the secondary queue is not large. The analysis considerably reduces the dimensions of the problem and simplifies the calculation of various steady-state quantities of interest. Our results include expressions for the loss probabilities, the average waiting times in the queues, and the average number of demands in service in each group.*

### I. INTRODUCTION

In an earlier paper,<sup>1</sup> we analyzed a traffic overflow system with queuing. The system consists of two groups, a primary and a secondary, with  $n_k$  servers and  $q_k$  waiting spaces, which receive demands from independent Poisson sources  $S_k$  with arrival rates  $\lambda_k > 0$ ,  $k = 1$  and  $2$ , respectively, as depicted in Fig. 1. The service times of the demands are independent and exponentially distributed with mean service rate  $\mu > 0$ . If all  $n_2$  servers in the secondary are busy when a demand from  $S_2$  arrives, the demand is queued if one of the  $q_2$  waiting spaces is available; otherwise, it is lost (blocked and cleared from the system). Demands waiting in the secondary queue enter service (in some prescribed order) as servers in the secondary become free.

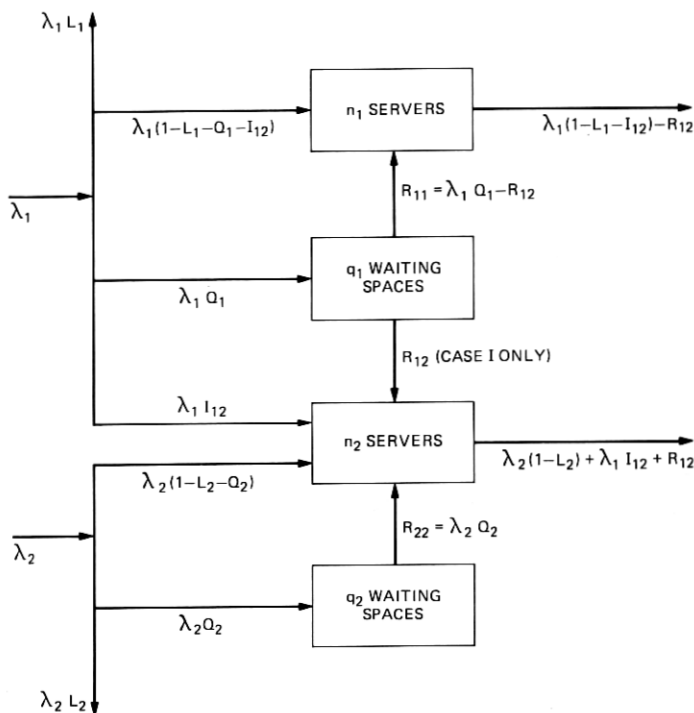


Fig. 1—Mean flow rates for an overflow system with queuing.

If all  $n_1$  servers in the primary are busy when a demand from  $S_1$  arrives and there is a free server in the secondary and no demands waiting in the secondary queue, the demand is served in the secondary. If there are no free servers, then the demand is queued in the primary queue, if one of the  $q_1$  waiting spaces is available; otherwise, it is lost. Two different cases were considered for the treatment of demands that are waiting in the primary queue. In case I, a demand waiting in the primary queue may enter service either in the primary, when a server becomes free, or in the secondary, if a server becomes free and no demands are waiting in the secondary queue. In case II, no overflow is permitted from the primary queue, so that a demand in the primary queue must wait for a server in the primary to become free.

In this paper, we use an alternate approach to analyze the same overflow system. This alternate approach is useful, in particular, when  $q_2$  is large, or even infinite. Which approach is preferable depends, essentially, on the relative size of  $q_2$  to  $n_1$ . Other approaches to the problem were discussed in the earlier paper.<sup>1</sup> Let  $p_{ij}$  denote the steady-state probability that there are  $i$  demands in the primary and  $j$  demands in the secondary, either in service or waiting. These proba-

bilities satisfy a set of generalized birth-and-death equations, which take the form of partial difference equations connecting nearest neighboring states. Here we carry out an analysis that reduces the dimensions of the problem, which may be considerable in cases of interest. An analogous reduction was obtained in the earlier analysis.<sup>1</sup>

The basic technique is to separate variables in regions away from certain boundaries of the state space, the elements of which are  $(i, j)$ . These regions are depicted in Fig. 2a. The analogous regions corresponding to the analysis of the earlier paper<sup>1</sup> are depicted in Fig. 2b, for comparison. The separation of variables leads to two sets of eigenvalue problems for the separation constant. The eigenvalues are roots of polynomial equations. The probabilities  $p_{ij}$  are then repre-

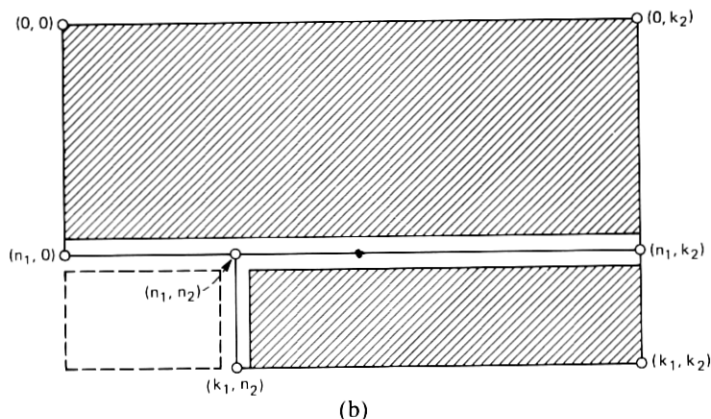
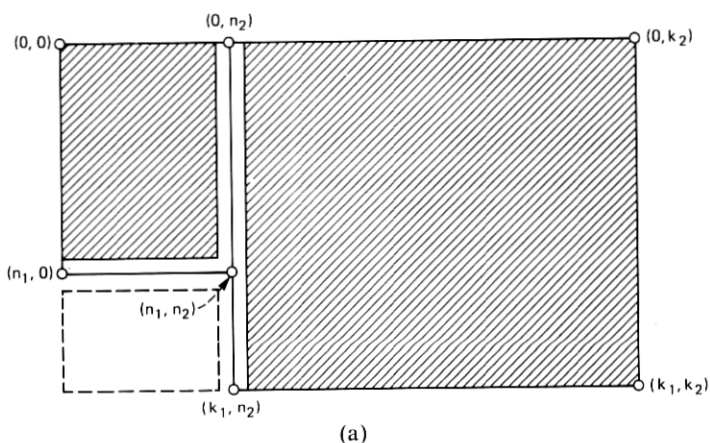


Fig. 2—Boundaries of regions in state space for the analysis of (a) this paper, and (b) the earlier paper.

sented in terms of the corresponding eigenfunctions. The constant coefficients in these representations are determined from the boundary conditions and the normalization condition that the sum of the probabilities is unity. In general, these constants have to be determined numerically. In case II, additional accessible states are in the region bordered by the broken lines in Fig. 2, since in this case demands wait in the primary queue even if a server is free in the secondary. In this additional region, the probabilities  $p_{ij}$  are expressed, as in the earlier analysis,<sup>1</sup> in terms of a fundamental solution of a partial difference equation.

There are various steady-state quantities of interest, which may be expressed in terms of the probabilities  $p_{ij}$ . The quantities include the loss (or blocking) probabilities, the average waiting times in the queues, and the average number of demands in service in each group. These quantities may be expressed directly in terms of the constant coefficients which occur in the representations for the probabilities  $p_{ij}$ . Thus the steady-state quantities of interest may be calculated directly, once the coefficients have been determined from the boundary and normalization conditions, without having to calculate the probabilities  $p_{ij}$ . Here again the reduction in the dimensions of the problem is valuable.

We first consider case I. The representation of the probabilities  $p_{ij}$  in terms of the eigenfunctions is discussed in Section II, and the boundary and normalization conditions are considered in Section III. Various steady-state quantities of interest are calculated in Section IV. Next, we consider case II and discuss the representation of the probabilities  $p_{ij}$  and the boundary and normalization conditions in Section V. The corresponding steady-state quantities of interest are considered in Section VI. Properties of the eigenfunctions which occur in the representations of the probabilities  $p_{ij}$  are given in the appendix.

## II. REPRESENTATION OF SOLUTION: CASE I

Let  $p_{ij}$  denote the steady-state probability that there are  $i$  demands in the primary and  $j$  demands in the secondary, either in service or waiting. These probabilities satisfy a set of generalized birth-and-death equations,<sup>2</sup> which may be derived in a straightforward manner. We define the traffic intensities

$$\alpha_1 = \lambda_1/\mu, \quad \alpha_2 = \lambda_2/\mu, \quad (1)$$

and let

$$k_1 = n_1 + q_1, \quad k_2 = n_2 + q_2. \quad (2)$$

It is convenient to introduce the function

$$\chi_\ell = \begin{cases} 1, & \ell \geq 0, \\ 0, & \ell < 0, \end{cases} \quad (3)$$



as well as the Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \quad (4)$$

Then,<sup>1</sup> for case I, it is found that

$$\begin{aligned} & [a_1(1 - \delta_{ik_1}\chi_{j-n_2}) + a_2(1 - \delta_{jk_2}) + \min(i, n_1) + \min(j, n_2)]p_{ij} \\ & = (1 - \chi_{i-1-n_1}\chi_{n_2-1-j})[a_1(1 - \delta_{i0})p_{i-1,j} \\ & \quad + (1 - \delta_{jk_2})\min(j + 1, n_2)p_{i,j+1}] \\ & \quad + (1 - \delta_{j0})[a_1\delta_{in_1}\chi_{n_2-j} + a_2(1 - \chi_{i-1-n_1}\chi_{n_2-j})]p_{i,j-1}, \\ & \quad + (1 - \delta_{ik_1})[(1 - \chi_{i-n_1}\chi_{n_2-1-j})\min(i + 1, n_1) + n_2\chi_{i-n_1}\delta_{jn_2}]p_{i+1,j}, \end{aligned} \quad (5)$$

for  $0 \leq i \leq k_1$ ,  $0 \leq j \leq k_2$ . These equations were constructed to imply that

$$p_{ij} = 0, \quad n_1 + 1 \leq i \leq k_1, \quad 0 \leq j \leq n_2 - 1, \quad (6)$$

since it is impossible for demands to be waiting in the primary queue when a server is free in the secondary. The normalization condition is

$$\sum_{i=0}^{n_1} \sum_{j=0}^{n_2-1} p_{ij} + \sum_{i=0}^{k_1} \sum_{j=n_2}^{k_2} p_{ij} = 1. \quad (7)$$

We assume that  $q_2 \geq 1$ . Then, for  $0 \leq i \leq k_1$  and  $n_2 + 1 \leq j \leq k_2$ , the variables in (5) may be separated, and there are solutions of the form  $\alpha_i\beta_j$  where

$$\begin{aligned} & [a_1(1 - \delta_{ik_1}) + \min(i, n_1) + \kappa]\alpha_i \\ & = a_1(1 - \delta_{i0})\alpha_{i-1} + (1 - \delta_{ik_1})\min(i + 1, n_1)\alpha_{i+1} \end{aligned} \quad (8)$$

for  $0 \leq i \leq k_1$ , and

$$[a_2(1 - \delta_{jk_2}) + n_2 - \kappa]\beta_j = a_2\beta_{j-1} + n_2(1 - \delta_{jk_2})\beta_{j+1} \quad (9)$$

for  $n_2 + 1 \leq j \leq k_2$ , and  $\kappa$  is a separation constant. The solution of (9) may be expressed in terms of Chebyshev polynomials of the second kind,<sup>3</sup>  $U_\ell(x)$ . It is convenient to define

$$\Psi_\ell(\kappa) = \left(\frac{n_2}{a_2}\right)^{\ell/2} U_\ell\left(\frac{a_2 + n_2 - \kappa}{2\sqrt{a_2n_2}}\right), \quad (10)$$

and

$$\phi_j(\kappa) = \Psi_{k_2-j}(\kappa) - \Psi_{k_2-j-1}(\kappa). \quad (11)$$

The properties of these functions which we will need are given in the appendix. We note here, however, that  $U_0(x) \equiv 1$ ,  $U_{-1}(x) \equiv 0$  and

$\phi_{k_2}(\kappa) \equiv 1$ . It follows from (9) and (102) that  $\beta_j$  is proportional to  $\phi_j(\kappa)$  for  $n_2 \leq j \leq k_2$ .

For  $0 \leq i \leq n_1 - 1$ , (8) implies that

$$(a_1 + i + \kappa)\alpha_i = a_1(1 - \delta_{i0})\alpha_{i-1} + (i + 1)\alpha_{i+1}. \quad (12)$$

The solution of (12) may be expressed in terms of Poisson-Charlier polynomials.<sup>4,5</sup> We here denote the solution of (12) for which  $\alpha_0 = 1$  by  $s_i(\kappa, a_1)$ . The properties of  $s_i(\kappa, a)$  which we will need are given in the appendix. For  $n_1 \leq i \leq k_1$ , (8) implies that

$$[a_1(1 - \delta_{ik_1}) + n_1 + \kappa]\alpha_i = a_i\alpha_{i-1} + n_1(1 - \delta_{ik_1})\alpha_{i+1}. \quad (13)$$

Corresponding to (10) and (11), we define

$$\Omega_\ell(\kappa) = \left(\frac{n_1}{a_1}\right)^{\ell/2} U_\ell\left(\frac{a_1 + n_1 + \kappa}{2\sqrt{a_1 n_1}}\right) \quad (14)$$

and

$$\theta_i(\kappa) = \Omega_{k_1-i}(\kappa) - \Omega_{k_1-i-1}(\kappa). \quad (15)$$

It follows from (13) and (111) that  $\alpha_i$  is proportional to  $\theta_i(\kappa)$  for  $n_1 - 1 \leq i \leq k_1$ , and we note that  $\theta_{k_1}(\kappa) \equiv 1$ .

Consequently, we take

$$\alpha_i = \begin{cases} s_i(\kappa, a_1)\theta_{n_1}(\kappa), & 0 \leq i \leq n_1, \\ s_{n_1}(\kappa, a_1)\theta_i(\kappa), & n_1 - 1 \leq i \leq k_1, \end{cases} \quad (16)$$

where

$$s_{n_1-1}(\kappa, a_1)\theta_{n_1}(\kappa) = s_{n_1}(\kappa, a_1)\theta_{n_1-1}(\kappa). \quad (17)$$

With the help of (15), (96), (97), and (110), this equation may be written in the form

$$\kappa[s_{n_1}(1 + \kappa, a_1)\Omega_{q_1}(\kappa) - s_{n_1-1}(1 + \kappa, a_1)\Omega_{q_1-1}(\kappa)] = 0. \quad (18)$$

The expression in the square brackets in (18) is a polynomial in  $\kappa$  of degree  $k_1 = n_1 + q_1$ . It was shown<sup>1</sup> that its zeros are negative and distinct, and we denote them by  $\kappa_r$ ,  $r = 1, \dots, k_1$ . We also let  $\kappa_0 = 0$ .

It was also shown<sup>1</sup> that the zeros of  $\phi_{n_2}(\kappa)$  are positive. Hence,  $\phi_{n_2}(\kappa_r) \neq 0$ ,  $r = 0, \dots, k_1$ . Moreover, it follows from (118) that

$$\lim_{q_2 \rightarrow \infty} \frac{\phi_j(\kappa_r)}{\phi_{n_2}(\kappa_r)} = \left[ \sqrt{\frac{a_2}{n_2}} \frac{1}{(\epsilon_r + \sqrt{\epsilon_r^2 - 1})} \right]^{j-n_2}, \quad r = 1, \dots, k_1, \quad (19)$$

for fixed  $j$ , where

$$\epsilon_r = \frac{(a_2 + n_2 - \kappa_r)}{2\sqrt{a_2 n_2}}, \quad (20)$$

and the positive square root of  $\epsilon_r^2 - 1$  is taken in (19). We note, from (109), that

$$\epsilon_r + \sqrt{\epsilon_r^2 - 1} > \sqrt{\frac{a_2}{n_2}}, \quad r = 1, \dots, k_1, \quad (21)$$

since  $\kappa_r < 0$ ,  $r = 1, \dots, k_1$ . On the other hand, from (105),

$$\frac{\phi_j(0)}{\phi_{n_2}(0)} = \left(\frac{a_2}{n_2}\right)^{j-n_2}, \quad (22)$$

and, as expected, the condition  $a_2 < n_2$  is necessary for a steady-state solution when  $q_2 = \infty$ .

In view of the above results, we may represent the probabilities  $p_{ij}$ , for  $n_2 \leq j \leq k_2$ , in the form

$$p_{ij} = \begin{cases} \sum_{r=0}^{k_1} b_r s_i(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \theta_{n_1}(\kappa_r) \frac{\phi_j(\kappa_r)}{\phi_{n_2}(\kappa_r)}, & 0 \leq i \leq n_1, \\ \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \theta_i(\kappa_r) \frac{\phi_j(\kappa_r)}{\phi_{n_2}(\kappa_r)}, & n_1 \leq i \leq k_1, \end{cases} \quad (23)$$

where the constants  $b_r$  are to be determined. The reason for the factor  $s_{n_2}(-\kappa_r, a_2)$  will be apparent shortly. We note, from (94), that  $s_{n_2}(-\kappa_r, a_2) > 0$ , since  $\kappa_r \leq 0$ ,  $r = 0, \dots, k_1$ .

For  $0 \leq i \leq n_1 - 1$  and  $0 \leq j \leq n_2 - 1$ , the variables in (5) may be separated, and there are solutions of the form  $\gamma_i \delta_j$  where

$$(a_1 + i + \eta)\gamma_i = a_1(1 - \delta_{i0})\gamma_{i-1} + (i + 1)\gamma_{i+1}, \quad 0 \leq i \leq n_1 - 1, \quad (24)$$

and

$$(a_2 + j - \eta)\delta_j = a_2(1 - \delta_{j0})\delta_{j-1} + (j + 1)\delta_{j+1}, \quad 0 \leq j \leq n_2 - 1, \quad (25)$$

and  $\eta$  is a separation constant. It follows from (93) that  $\gamma_i$  is proportional to  $s_i(\eta, a_1)$  for  $0 \leq i \leq n_1$  and  $\delta_j$  is proportional to  $s_j(-\eta, a_2)$  for  $0 \leq j \leq n_2$ . Now  $s_{n_2}(-\eta, a_2)$  is a polynomial in  $\eta$  of degree  $n_2$ . It was shown<sup>1</sup> that its zeros are positive and distinct, and we denote them by  $\eta_\ell$ ,  $\ell = 1, \dots, n_2$ . Then

$$s_{n_2}(-\eta_\ell, a_2) = 0, \quad \ell = 1, \dots, n_2. \quad (26)$$

Consequently, we represent  $p_{ij}$ , for  $0 \leq i \leq n_1$  and  $0 \leq j \leq n_2$ , in the form

$$p_{ij} = \sum_{r=0}^{k_1} b_r s_i(\kappa_r, a_1) s_j(-\kappa_r, a_2) \theta_{n_1}(\kappa_r) + \sum_{\ell=1}^{n_2} c_\ell s_i(\eta_\ell, a_1) s_j(-\eta_\ell, a_2), \quad (27)$$

where the constants  $c_\ell$  are also to be determined. Note that the representations in (23) and (27) agree for  $j = n_2$ ,  $0 \leq i \leq n_1$ , in view of (26).

It remains to satisfy the boundary conditions at  $i = n_1$ ,  $0 \leq j \leq n_2 - 1$  and at  $j = n_2$ ,  $0 \leq i \leq k_1$ , as well as the normalization condition (7). This is done in the next section.

### III. BOUNDARY CONDITIONS: CASE I

From (5), the boundary conditions at  $i = n_1$  imply that

$$(a_1 + a_2 + n_1 + j)p_{n_1, j} = a_1 p_{n_1-1, j} + (a_1 + a_2)(1 - \delta_{j0})p_{n_1, j-1} + (j+1)p_{n_1, j+1}, \quad (28)$$

for  $0 \leq j \leq n_2 - 1$ . Also, the boundary conditions at  $j = n_2$  imply that

$$(a_1 + a_2 + i + n_2)p_{i, n_2} = a_1(1 - \delta_{i0})p_{i-1, n_2} + a_2 p_{i, n_2-1} + (i+1)p_{i+1, n_2} + n_2 p_{i, n_2+1}, \quad (29)$$

for  $0 \leq i \leq n_1 - 1$ ,

$$[a_1(1 - \delta_{q_1, 0}) + a_2 + n_1 + n_2]p_{n_1, n_2} = a_1 p_{n_1-1, n_2} + (a_1 + a_2)p_{n_1, n_2-1} + (n_1 + n_2)(1 - \delta_{q_1, 0})p_{n_1+1, n_2} + n_2 p_{n_1, n_2+1}, \quad (30)$$

and, if  $q_1 \geq 1$ ,

$$[a_1(1 - \delta_{ik_1}) + a_2 + n_1 + n_2]p_{i, n_2} = a_1 p_{i-1, n_2} + (n_1 + n_2)(1 - \delta_{ik_1})p_{i+1, n_2} + n_2 p_{i, n_2+1}, \quad (31)$$

for  $n_1 + 1 \leq i \leq k_1$ .

If we substitute (27) into (28), we find, after reduction with the help of the recurrence relations in the appendix, that

$$\sum_{r=0}^{k_1} b_r \theta_{n_1}(\kappa_r) [\kappa_r s_{n_1}(1 + \kappa_r, a_1) s_j(-\kappa_r, a_2) + a_1 s_{n_1}(\kappa_r, a_1) s_j(-1 - \kappa_r, a_2)] + \sum_{\ell=1}^{n_2} c_\ell [\eta_\ell s_{n_1}(1 + \eta_\ell, a_1) s_j(-\eta_\ell, a_2) + a_1 s_{n_1}(\eta_\ell, a_1) s_j(-1 - \eta_\ell, a_2)] = 0, \quad (32)$$

for  $0 \leq j \leq n_2 - 1$ . We remark that the first sum in (32) may be written in a different form with the help of the relationship

$$\kappa_r [s_{n_1}(1 + \kappa_r, a_1) \theta_{n_1}(\kappa_r) + s_{n_1}(\kappa_r, a_1) \Omega_{q_1-1}(\kappa_r)] = 0, \quad (33)$$

which follows from (15), (97), and the fact that  $\kappa_r$ ,  $r = 0, \dots, k_1$ , are the roots of (18).

If we make use of (23) and (27) in (29), we find after reduction that

$$a_2 \sum_{r=0}^{k_1} b_r s_i(\kappa_r, a_1) \theta_{n_1}(\kappa_r) \left[ s_{n_2}(-\kappa_r, a_2) \frac{\phi_{n_2-1}(\kappa_r)}{\phi_{n_2}(\kappa_r)} - s_{n_2-1}(-\kappa_r, a_2) \right] - a_2 \sum_{\ell=1}^{n_2} c_\ell s_i(\eta_\ell, a_1) s_{n_2-1}(-\eta_\ell, a_2) = 0 \quad (34)$$

for  $0 \leq i \leq n_1 - 1$ . The first sum in (34) may be written in a different form with the help of the relationship

$$a_2[s_{n_2}(-\kappa_r, a_2)\phi_{n_2-1}(\kappa_r) - s_{n_2-1}(-\kappa_r, a_2)\phi_{n_2}(\kappa_r)] \\ = -\kappa_r[s_{n_2}(1 - \kappa_r, a_2)\Psi_{q_2}(\kappa_r) - s_{n_2-1}(1 - \kappa_r, a_2)\Psi_{q_2-1}(\kappa_r)], \quad (35)$$

which follows from (11) and the recurrence relations in the appendix. We note that  $\kappa_0 = 0$ . Also, for  $q_1 \geq 1$ , if we make use of (23) in (31), it is found after reduction that

$$\sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \left[ a_2 \theta_i(\kappa_r) \frac{\phi_{n_2-1}(\kappa_r)}{\phi_{n_2}(\kappa_r)} \right. \\ \left. - n_2(1 - \delta_{ik_1})\theta_{i+1}(\kappa_r) \right] = 0, \quad (36)$$

for  $n_1 + 1 \leq i \leq k_1$ .

If we substitute (23) and (27) into (30), and make use of (17), (102), and (111), we find that

$$\sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \left[ a_2 \theta_{n_1}(\kappa_r) \frac{\phi_{n_2-1}(\kappa_r)}{\phi_{n_2}(\kappa_r)} - n_2(1 - \delta_{q_1,0})\theta_{n_1+1}(\kappa_r) \right] \\ - (a_1 + a_2) \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2-1}(-\kappa_r, a_2) \theta_{n_1}(\kappa_r) \\ - (a_1 + a_2) \sum_{\ell=1}^{n_2} c_\ell s_{n_1}(\eta_\ell, a_1) s_{n_2-1}(-\eta_\ell, a_2) = 0. \quad (37)$$

This condition may be shown to be redundant, by summing (32) from  $j = 0$  to  $n_2 - 1$ , (34) from  $i = 0$  to  $n_1 - 1$  and, for  $q_1 \geq 1$ , (36) from  $i = n_1 + 1$  to  $k_1$ , and adding.

The constants  $b_r$ ,  $r = 0, \dots, k_1$  and  $c_\ell$ ,  $\ell = 1, \dots, n_2$  are determined by (32), (34), and (36) only to within a multiplicative constant, which is determined from the normalization condition (7). But, from (23), with the help of (15), (18), (97), and (98), it is found that

$$\sum_{i=0}^{k_1} p_{ij} = b_0 s_{n_2}(0, a_2) \frac{\phi_j(0)}{\phi_{n_2}(0)} [s_{n_1}(1, a_1) \Omega_{q_1}(0) - s_{n_1-1}(1, a_1) \Omega_{q_1-1}(0)], \quad (38)$$

for  $n_2 \leq j \leq k_2$ . Then, from (7) and (27), with the help of (11) and (98), it follows that

$$b_0 s_{n_2}(0, a_2) \frac{\Psi_{q_2}(0)}{\phi_{n_2}(0)} [s_{n_1}(1, a_1) \Omega_{q_1}(0) - s_{n_1-1}(1, a_1) \Omega_{q_1-1}(0)] \\ + \sum_{r=0}^{k_1} b_r s_{n_1}(1 + \kappa_r, a_1) s_{n_2-1}(1 - \kappa_r, a_2) \theta_{n_1}(\kappa_r) \\ + \sum_{\ell=1}^{n_2} c_\ell s_{n_1}(1 + \eta_\ell, a_1) s_{n_2-1}(1 - \eta_\ell, a_2) = 1. \quad (39)$$

We note, from (104) and (105), that

$$\lim_{q_2 \rightarrow \infty} \frac{\Psi_{q_2}(0)}{\phi_{n_2}(0)} = \frac{n_2}{(n_2 - a_2)}, \quad \text{for } a_2 < n_2. \quad (40)$$

Once the constants  $b_r$  and  $c_r$  have been determined, the steady-state probabilities  $p_{ij}$  may be calculated from (23) and (27). We remark that the number of constants to be determined is only  $k_1 + 1 + n_2$ , whereas the number of probabilities  $p_{ij}$  is  $(q_2 + 1)(k_1 + 1) + n_2(n_1 + 1)$ , which in general is considerably larger. It is of interest to compare the approach taken in this paper with that adopted in the earlier paper<sup>1</sup> (see Fig. 2). There the variables in (5) were separated in the region  $0 \leq i \leq n_1 - 1$  and  $0 \leq j \leq k_2$ , leading to  $k_2 + 1$  eigenvalues and eigenfunctions, and, for  $q_1 \geq 1$ , in the region  $n_1 + 1 \leq i \leq k_1$  and  $n_2 + 1 \leq j \leq k_2$ , leading to  $q_1$  eigenvalues with corresponding eigenfunctions vanishing at  $i = n_1$ . In the corresponding representation of the probabilities  $p_{ij}$  in terms of the eigenfunctions, there are thus  $k_2 + 1 + q_1$  constants to be determined. In view of (2), which approach is preferable depends on the relative size of  $q_2$  to  $n_1$ . The approach adopted in this paper is suitable if  $q_2$  is large, or even infinite.

#### IV. SOME STEADY-STATE QUANTITIES: CASE I

We proceed now to the calculation of various steady-state quantities of interest. These quantities are depicted in Fig. 1, which indicates the mean flow rates. The loss probabilities  $L_1$  and  $L_2$  are given by

$$L_1 = \sum_{j=n_2}^{k_2} p_{k_1, j}, \quad L_2 = \sum_{i=0}^{k_1} p_i, \quad k_2, \quad (41)$$

and the probabilities that a demand from the primary, or secondary, source is queued on arrival are

$$Q_1 = (1 - \delta_{q_1, 0}) \sum_{i=n_1}^{k_1-1} \sum_{j=n_2}^{k_2} p_{ij}, \quad Q_2 = \sum_{i=0}^{k_1} \sum_{j=n_2}^{k_2-1} p_{ij}. \quad (42)$$

The probability that a demand arriving from the primary source overflows immediately is

$$I_{12} = \sum_{j=0}^{n_2-1} p_{n_1, j}. \quad (43)$$

Since the mean service rate is  $\mu$ , the mean departure rate from the primary queue to the primary servers is

$$R_{11} = n_1 \mu (1 - \delta_{q_1, 0}) \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} p_{ij}, \quad (44)$$

while the mean rate of overflow from the primary queue to the

secondary servers is

$$R_{12} = n_2 \mu (1 - \delta_{q_1,0}) \sum_{i=n_1+1}^{k_1} p_{i,n_2}. \quad (45)$$

The mean departure rate from the secondary queue is

$$R_{22} = n_2 \mu \sum_{i=0}^{k_1} \sum_{j=n_2+1}^{k_2} p_{ij}. \quad (46)$$

It may be verified from (5) that

$$R_{11} + R_{12} = \lambda_1 Q_1, \quad R_{22} = \lambda_2 Q_2. \quad (47)$$

These relationships hold since, in the steady state, the departure rates from the queues are equal to the arrival rates to them.

The average number of demands in the primary and secondary queues are

$$V_1 = \sum_{i=n_1}^{k_1} \sum_{j=n_2}^{k_2} (i - n_1) p_{ij}, \quad V_2 = \sum_{i=0}^{k_1} \sum_{j=n_2+1}^{k_2} (j - n_2) p_{ij}. \quad (48)$$

Also, the average number of demands in service in the two groups are

$$X_1 = \sum_{i=0}^{n_1} \sum_{j=0}^{k_2} i p_{ij} + n_1 (1 - \delta_{q_1,0}) \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} p_{ij} \quad (49)$$

and

$$X_2 = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2-1} j p_{ij} + n_2 \sum_{i=0}^{k_1} \sum_{j=n_2}^{k_2} p_{ij}. \quad (50)$$

If we apply Little's theorem<sup>2</sup> to the primary and secondary queues, we find that the average waiting times of the demands which are queued in the primary or in the secondary are given by

$$W_1 = \frac{V_1}{\lambda_1 Q_1} (q_1 \geq 1), \quad W_2 = \frac{V_2}{\lambda_2 Q_2}, \quad (51)$$

respectively, independently of the order of service within each queue. Also, if we apply Little's theorem to the primary and secondary groups of servers, we obtain

$$\lambda_1 (1 - L_1 - I_{12}) - R_{12} = \mu X_1, \quad \lambda_2 (1 - L_2) + \lambda_1 I_{12} + R_{12} = \mu X_2, \quad (52)$$

since the mean service rate is  $\mu$ .

The steady-state quantities of interest may be expressed in terms of the constants  $b_r$  and  $c_r$  with the help of the representations in (23) and (27). From (41), it is found, with the help of (11), since  $\theta_{k_1}(\kappa) \equiv 1$ , that

$$L_1 = \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)}, \quad (53)$$

and, with the help of (22) and (38), that

$$L_2 = b_0 \left( \frac{a_2}{n_2} \right)^{q_2} s_{n_2}(0, a_2) [s_{n_1}(1, a_1) \Omega_{q_1}(0) - s_{n_1-1}(1, a_1) \Omega_{q_1-1}(0)]. \quad (54)$$

From (42), it is found, with the help of (11) and (15), that

$$Q_1 = \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) [\Omega_{q_1}(\kappa_r) - 1] \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)}, \quad (55)$$

and, with the help of (38), that

$$Q_2 = b_0 \frac{[\Psi_{q_2}(0) - 1]}{\phi_{n_2}(0)} \cdot s_{n_2}(0, a_2) [s_{n_1}(1, a_1) \Omega_{q_1}(0) - s_{n_1-1}(1, a_1) \Omega_{q_1-1}(0)]. \quad (56)$$

Next, from (43), with the help of (98), it is found that

$$I_{12} = \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2-1}(1 - \kappa_r, a_2) \theta_{n_1}(\kappa_r) + \sum_{\ell=1}^{n_2} c_\ell s_{n_1}(\eta_\ell, a_1) s_{n_2-1}(1 - \eta_\ell, a_2). \quad (57)$$

Also, from (44) and (45), if we make use of (11) and (15), we obtain

$$R_{11} = n_1 \mu \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \Omega_{q_1-1}(\kappa_r) \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)} \quad (58)$$

and

$$R_{12} = n_2 \mu \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \Omega_{q_1-1}(\kappa_r). \quad (59)$$

From (46), with the help of (11), (38), (56), and (104), it follows that  $R_{22} = \lambda_2 Q_2$ , as expected. The first relationship in (47) provides a useful numerical check, in view of the expressions in (55), (58), and (59).

From (48), with the help of (11), (114), and (115), it is found that

$$V_1 = b_0 s_{n_1}(0, a_1) s_{n_2}(0, a_2) \left( \frac{n_1}{a_1} \right)^{q_1} \Lambda_{q_1} \left( \frac{a_1}{n_1} \right) \frac{\Psi_{q_2}(0)}{\phi_{n_2}(0)} + \sum_{r=1}^{k_1} \frac{b_r}{\kappa_r} s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \cdot \{a_1 [\Omega_{q_1}(\kappa_r) - 1] - n_1 \Omega_{q_1-1}(\kappa_r)\} \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)}, \quad (60)$$

where

$$\Lambda_q(\xi) = \sum_{\ell=1}^q \ell \xi^\ell = \begin{cases} \xi [1 - (q+1)\xi^q + q\xi^{q+1}] / (1-\xi)^2, & \xi \neq 1 \\ \frac{1}{2}q(q+1), & \xi = 1. \end{cases} \quad (61)$$



Also, with the help of (22) and (38), it is found that

$$V_2 = b_0 \Lambda_{q_2} \left( \frac{a_2}{n_2} \right) s_{n_2}(0, a_2) [s_{n_1}(1, a_1) \Omega_{q_1}(0) - s_{n_1-1}(1, a_1) \Omega_{q_1-1}(0)]. \quad (62)$$

It follows from (51), (56), and (62) that

$$W_2 = \frac{\Lambda_{q_2}(a_2/n_2) \phi_{n_2}(0)}{\lambda_2 [\Psi_{q_2}(0) - 1]}. \quad (63)$$

Next, from (49), with the help of (7), (11), (98), and (99), it is found that

$$\begin{aligned} X_1 = n_1 - \sum_{\ell=1}^{n_2} c_\ell s_{n_1-1}(2 + \eta_\ell, a_1) s_{n_2-1}(1 - \eta_\ell, a_2) \\ - \sum_{r=0}^{k_1} b_r s_{n_1-1}(2 + \kappa_r, a_1) \theta_{n_1}(\kappa_r) \\ \cdot \left[ s_{n_2-1}(1 - \kappa_r, a_2) + s_{n_2}(-\kappa_r, a_2) \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)} \right]. \quad (64) \end{aligned}$$

Similarly, from (50), it is found that

$$\begin{aligned} X_2 = n_2 - \sum_{\ell=1}^{n_2} c_\ell s_{n_1}(1 + \eta_\ell, a_1) s_{n_2-1}(2 - \eta_\ell, a_2) \\ - \sum_{r=0}^{k_1} b_r s_{n_1}(1 + \kappa_r, a_1) s_{n_2-1}(2 - \kappa_r, a_2) \theta_{n_1}(\kappa_r). \quad (65) \end{aligned}$$

In view of (53), (54), (57), (59), (64), and (65), the relationships in (52) provide a useful numerical check.

From (107) to (109), it follows that

$$\lim_{q_2 \rightarrow \infty} \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)} = \frac{(\epsilon_r + \sqrt{\epsilon_r^2 - 1})}{(\epsilon_r + \sqrt{\epsilon_r^2 - 1} - \sqrt{a_2/n_2})}, \quad r = 1, \dots, k_1, \quad (66)$$

where  $\epsilon_r$  is given by (20). With the help of (40) and (66), we obtain the results for the limiting case  $q_2 = \infty$ , with  $a_2 < n_2$ , in (53), (55), (58), (60), and (64). From (54), as expected,  $L_2 \rightarrow 0$  as  $q_2 \rightarrow \infty$ , for  $a_2 < n_2$ . Also, since  $\phi_{k_2}(0) = 1$ , it follows from (22) and (40) that

$$\lim_{q_2 \rightarrow \infty} \frac{[\Psi_{q_2}(0) - 1]}{\phi_{n_2}(0)} = \frac{n_2}{(n_2 - a_2)}, \quad \text{for } a_2 < n_2, \quad (67)$$

and from (61) that

$$\lim_{q_2 \rightarrow \infty} \Lambda_{q_2} \left( \frac{a_2}{n_2} \right) = \frac{n_2 a_2}{(n_2 - a_2)^2}, \quad \text{for } a_2 < n_2. \quad (68)$$

From (67) and (68), we obtain the limiting results in (56) and (62), and

from (1) and (63), we have

$$\lim_{q_2 \rightarrow \infty} W_2 = \frac{1}{\mu(n_2 - a_2)}, \quad \text{for } a_2 < n_2. \quad (69)$$

We remark that the steady-state quantities of interest may be calculated directly, once the constants  $b_r$  and  $c_r$  have been determined, without having to calculate the probabilities  $p_{ij}$ .

## V. REPRESENTATION AND BOUNDARY CONDITIONS: CASE II

We now consider the second case, in which no overflow is permitted from the primary queue, so that a demand in the primary queue must wait for a server in the primary to become free. Since case II differs from case I only in the treatment of demands waiting in the primary queue, we assume that  $q_1 \geq 1$ , as well as  $q_2 \geq 1$ . The steady-state probabilities  $p_{ij}$  now satisfy the equations<sup>1</sup>

$$\begin{aligned} & [a_1(1 - \delta_{ik_1}\chi_{j-n_2}) + a_2(1 - \delta_{jk_2}) + \min(i, n_1) + \min(j, n_2)]p_{ij} \\ & = a_1(1 - \delta_{i0})(1 - \chi_{i-n_1-1}\chi_{n_2-1-j})p_{i-1,j} \\ & \quad + (1 - \delta_{j0})(a_1\chi_{i-n_1}\chi_{n_2-j} + a_2)p_{i,j-1} \\ & \quad + (1 - \delta_{ik_1})\min(i+1, n_1)p_{i+1,j} \\ & \quad + (1 - \delta_{jk_2})\min(j+1, n_2)p_{i,j+1}, \quad (70) \end{aligned}$$

for  $0 \leq i \leq k_1$ ,  $0 \leq j \leq k_2$ . The normalization condition is

$$\sum_{i=0}^{k_1} \sum_{j=0}^{k_2} p_{ij} = 1. \quad (71)$$

Now (5) and (70) are identical for  $0 \leq i \leq n_1 - 1$ ,  $0 \leq j \leq n_2$  and for  $0 \leq i \leq k_1$ ,  $n_2 + 1 \leq j \leq k_2$ . It follows that the representations in (23) and (27) are still valid, although the constants  $b_r$  and  $c_r$  will, of course, be different. However, the boundary conditions (29), which correspond to  $j = n_2$  for  $0 \leq i \leq n_1 - 1$ , are still satisfied, and hence (34) still holds, for  $0 \leq i \leq n_1 - 1$ . The remaining boundary conditions are different. Also, from (70), for  $n_1 + 1 \leq i \leq k_1$  and  $0 \leq j \leq n_2 - 1$ , we have

$$\begin{aligned} & (a_1 + a_2 + n_1 + j)p_{ij} \\ & = (a_1 + a_2)(1 - \delta_{j0})p_{i,j-1} + n_1(1 - \delta_{ik_1})p_{i+1,j} + (j+1)p_{i,j+1}. \quad (72) \end{aligned}$$

We define<sup>1</sup> the quantities  $\Pi_{mj}$ , for  $m, j = 0, 1, \dots$ , as the solutions of the equations

$$\begin{aligned} & (a_1 + a_2 + n_1 + j)\Pi_{mj} = (a_1 + a_2)(1 - \delta_{j0})\Pi_{m,j-1} \\ & \quad + n_1(1 - \delta_{m0})\Pi_{m-1,j} + (j+1)\Pi_{m,j+1}, \quad (73) \end{aligned}$$

which satisfy the initial conditions

$$\Pi_{m0} = \delta_{m0}, \quad m = 0, 1, \dots \quad (74)$$

It follows from (73) and (74), by induction on  $j$ , that

$$\Pi_{mj} = 0, \quad m > j. \quad (75)$$

It may be verified, from (72) to (74), that

$$p_{ij} = \sum_{m=i}^{k_1} p_{m0} \Pi_{m-i,j}, \quad n_1 + 1 \leq i \leq k_1, \quad 0 \leq j \leq n_2. \quad (76)$$

The quantities  $\Pi_{mj}$  may be calculated sequentially from the recurrence relations (73), with the help of (75) and the initial condition  $\Pi_{00} = 1$ .

In order that the representations in (23) and (76) agree for  $j = n_2$ , we must have

$$\sum_{m=i}^{k_1} p_{m0} \Pi_{m-i,n_2} = \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \theta_i(\kappa_r) \quad (77)$$

for  $n_1 + 1 \leq i \leq k_1$ . But, from (73), (74), and (93), it follows that  $\Pi_{0,n_2} = s_{n_2}(n_1, a_1 + a_2) > 0$ . Hence, (77) may be used to solve successively for  $p_{k_1,0}, \dots, p_{n_1+1,0}$  in terms of  $b_r, r = 0, \dots, k_1$ . It remains to satisfy the boundary conditions at  $i = n_1, 0 \leq j \leq n_2 - 1$  and at  $j = n_2, n_1 \leq i \leq k_1$ , as well as (34) for  $0 \leq i \leq n_1 - 1$ , and the normalization condition (71). From (70), these boundary conditions are

$$(a_1 + a_2 + n_1 + j)p_{n_1,j} = a_1 p_{n_1-1,j} + (a_1 + a_2)(1 - \delta_{j0})p_{n_1,j-1} + n_1 p_{n_1+1,j} + (j+1)p_{n_1,j+1}, \quad (78)$$

for  $0 \leq j \leq n_2 - 1$  and

$$[a_1(1 - \delta_{ik_1}) + a_2 + n_1 + n_2]p_{i,n_2} = a_1 p_{i-1,n_2} + (a_1 + a_2)p_{i,n_2-1} + n_1(1 - \delta_{ik_1})p_{i+1,n_2} + n_2 p_{i,n_2+1}, \quad (79)$$

for  $n_1 \leq i \leq k_1$ .

From (27), (76), and (78), we find, after reduction, that

$$\begin{aligned} & \sum_{r=0}^{k_1} b_r \theta_{n_1}(\kappa_r) [\kappa_r s_{n_1}(1 + \kappa_r, a_1) s_j(-\kappa_r, a_2) + a_1 s_{n_1}(\kappa_r, a_1) s_j(-1 - \kappa_r, a_2)] \\ & + \sum_{\ell=1}^{n_2} c_\ell [\eta_\ell s_{n_1}(1 + \eta_\ell, a_1) s_j(-\eta_\ell, a_2) + a_1 s_{n_1}(\eta_\ell, a_1) s_j(-1 - \eta_\ell, a_2)] \\ & - n_1 \sum_{m=n_1+1}^{k_1} p_{m0} \Pi_{m-n_1-1,j} = 0, \quad (80) \end{aligned}$$

for  $0 \leq j \leq n_2 - 1$ . Also, from (23), (76), and (79), it is found that

$$a_2 \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \theta_i(\kappa_r) \frac{\phi_{n_2-1}(\kappa_r)}{\phi_{n_2}(\kappa_r)} - (a_1 + a_2) \sum_{m=i}^{k_1} p_{m0} \Pi_{m-i, n_2-1} = 0, \quad (81)$$

for  $n_1 + 1 \leq i \leq k_1$ .

From (23) and (27), the boundary condition corresponding to  $i = n_1$  in (79) reduces to

$$a_2 \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \theta_{n_1}(\kappa_r) \frac{\phi_{n_2-1}(\kappa_r)}{\phi_{n_2}(\kappa_r)} - (a_1 + a_2) \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2-1}(-\kappa_r, a_2) \theta_{n_1}(\kappa_r) - (a_1 + a_2) \sum_{\ell=1}^{n_2} c_\ell s_{n_1}(\eta_\ell, a_1) s_{n_2-1}(-\eta_\ell, a_2) = 0. \quad (82)$$

This condition may be shown to be redundant, by summing (80) from  $j = 0$  to  $n_2 - 1$ , (34) from  $i = 0$  to  $n_1 - 1$ , and (81) from  $i = n_1 + 1$  to  $k_1$ , and adding.

The constants  $b_r$ ,  $r = 0, \dots, k_1$  and  $c_\ell$ ,  $\ell = 1, \dots, n_2$ , and the probabilities  $p_{m0}$ ,  $m = n_1 + 1, \dots, k_1$ , are determined by (34), (77), (80), and (81) only to within a multiplicative constant, which is determined by the normalization condition (71). But, from (70), if we sum on  $j$  from 0 to  $k_2$ , and on  $i$  from  $\ell$  to  $k_1$ , we obtain

$$n_1 \sum_{j=0}^{k_2} p_{\ell j} = a_1 \sum_{j=n_2}^{k_2} p_{\ell-1, j}, \quad n_1 + 1 \leq \ell \leq k_1. \quad (83)$$

Hence, from (23), (27), and (83), with the help of (11), (15), and (98), the normalization condition (71) implies that

$$\sum_{r=0}^{k_1} b_r s_{n_1}(1 + \kappa_r, a_1) \theta_{n_1}(\kappa_r) \left[ s_{n_2}(-\kappa_r, a_2) \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)} + s_{n_2-1}(1 - \kappa_r, a_2) \right] + \sum_{\ell=1}^{n_2} c_\ell s_{n_1}(1 + \eta_\ell, a_1) s_{n_2-1}(1 - \eta_\ell, a_2) + \frac{a_1}{n_1} \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) [\Omega_{q_1}(\kappa_r) - 1] \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)} = 1. \quad (84)$$

## VI. SOME STEADY-STATE QUANTITIES: CASE II

We now consider the calculation of various steady-state quantities of interest. Since no overflow is permitted from the primary queue,  $R_{12} = 0$  in Fig. 1. The loss probabilities  $L_1$  and  $L_2$  are given by (41), and

the probabilities  $Q_1$  and  $Q_2$  that demands from the primary and secondary sources are queued on arrival are given by (42). Hence, from (23) and (27), it follows that (53) to (56) still hold. We emphasize, however, that the constants  $b_r$ , as well as  $c_r$ , differ for the two cases. Similarly,  $R_{22}$  is given by (46) and  $V_2$  is given by (48), so that  $R_{22} = \lambda_2 Q_2$ , as before, and (62) holds. Also,

$$X_1 = \sum_{i=0}^{k_1} \sum_{j=0}^{k_2} \min(i, n_1) p_{ij} = n_1 - \sum_{i=0}^{n_1} \sum_{j=0}^{k_2} (n_1 - i) p_{ij}, \quad (85)$$

and it follows, from (23) and (27), that (64) holds.

The probability that a demand arriving from the primary source overflows (immediately) is

$$I_{12} = \sum_{i=n_1}^{k_1} \sum_{j=0}^{n_2-1} p_{ij} = \sum_{j=0}^{n_2-1} \left( p_{n_1, j} + \sum_{i=n_1+1}^{k_1} p_{ij} \right). \quad (86)$$

Hence, with the help of (83), we obtain

$$I_{12} = \sum_{j=0}^{n_2-1} p_{n_1, j} + \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} \left( \frac{\alpha_1}{n_1} p_{i-1, j} - p_{ij} \right). \quad (87)$$

From (23) and (27), with the help of (11), (15), and (98), it is found that

$$\begin{aligned} I_{12} = & \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, \alpha_1) s_{n_2-1}(1 - \kappa_r, \alpha_2) \theta_{n_1}(\kappa_r) \\ & + \sum_{\ell=1}^{n_2} c_\ell s_{n_1}(\eta_\ell, \alpha_1) s_{n_2-1}(1 - \eta_\ell, \alpha_2) + \sum_{r=0}^{k_1} b_r s_{n_1}(\kappa_r, \alpha_1) s_{n_2}(-\kappa_r, \alpha_2) \\ & \cdot \left\{ \frac{\alpha_1}{n_1} [\Omega_{q_1}(\kappa_r) - 1] - \Omega_{q_1-1}(\kappa_r) \right\} \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)}. \quad (88) \end{aligned}$$

Since  $R_{12} = 0$  and  $\lambda_1 = \alpha_1 \mu$ , the first relationship in (52) becomes

$$\alpha_1 (1 - L_1 - I_{12}) = X_1. \quad (89)$$

In view of (53), (64), and (88), this relationship provides a useful numerical check.

The mean departure rate from the primary queue to the primary servers is

$$R_{11} = n_1 \mu \sum_{i=n_1+1}^{k_1} \sum_{j=0}^{k_2} p_{ij}. \quad (90)$$

From (42) and (83), it follows that  $R_{11} = \lambda_1 Q_1$ , as expected. Finally, the average number of demands in the primary queue is

$$V_1 = \sum_{i=n_1+1}^{k_1} \sum_{j=0}^{k_2} (i - n_1) p_{ij}. \quad (91)$$

Hence, with the help of (11), (15), (23), (83), (113), and (115), it is found that

$$V_1 = b_0 s_{n_1}(0, a_1) s_{n_2}(0, a_2) \left( \frac{n_1}{a_1} \right)^{q_1} \Lambda_{q_1} \left( \frac{a_1}{n_1} \right) \frac{\Psi_{q_2}(0)}{\phi_{n_2}(0)} + \frac{a_1}{n_1} \sum_{r=1}^{k_1} b_r s_{n_1}(\kappa_r, a_1) s_{n_2}(-\kappa_r, a_2) \frac{\Psi_{q_2}(\kappa_r)}{\phi_{n_2}(\kappa_r)} \cdot \left\{ \Omega_{q_1}(\kappa_r) - (q_1 + 1) + \frac{a_1}{\kappa_r} \left[ \Omega_{q_1}(\kappa_r) - 1 - \frac{n_1}{a_1} \Omega_{q_1-1}(\kappa_r) \right] \right\}, \quad (92)$$

where  $\Lambda_q(\xi)$  is as defined in (61).

## APPENDIX

We define  $s_i(\lambda, a)$  by the recurrence relation

$$(a + i + \lambda) s_i(\lambda, a) = a(1 - \delta_{i0}) s_{i-1}(\lambda, a) + (i + 1) s_{i+1}(\lambda, a);$$

$$s_0(\lambda, a) = 1, \quad (93)$$

for  $i = 0, 1, \dots$ . Thus  $s_n(\lambda, a)$  is a polynomial of degree  $n$  in both  $\lambda$  and  $a$ , and it may be related to a Poisson-Charlier polynomial.<sup>4,5</sup> However, we will give here the properties of  $s_n(\lambda, a)$  which we will need. An explicit formula is<sup>1</sup>

$$s_i(\lambda, a) = \sum_{r=0}^i \frac{(\lambda)_r a^{i-r}}{r!(i-r)!}, \quad (94)$$

where

$$(\lambda)_0 = 1, \quad (\lambda)_r = \lambda(\lambda + 1) \dots (\lambda + r - 1), \quad r = 1, 2, \dots \quad (95)$$

It was also shown<sup>1</sup> that

$$(i + 1) s_{i+1}(\lambda, a) = a s_i(\lambda, a) + \lambda s_i(\lambda + 1, a) \quad (96)$$

and

$$s_i(\lambda, a) = s_i(\lambda + 1, a) - (1 - \delta_{i0}) s_{i-1}(\lambda + 1, a). \quad (97)$$

From (97), it follows that

$$\sum_{i=0}^n s_i(\lambda, a) = s_n(\lambda + 1, a), \quad (98)$$

and, from (96) and (98), we deduce that

$$\sum_{i=0}^n (n - i) s_i(\lambda, a) = (1 - \delta_{n0}) s_{n-1}(\lambda + 2, a). \quad (99)$$

We now turn our attention to the Chebyshev polynomials of the second kind,<sup>3</sup>  $U_\ell(x)$ . They may be defined by the recurrence relation

$$2x U_\ell(x) = U_{\ell+1}(x) + U_{\ell-1}(x); \quad U_{-1}(x) \equiv 0, \quad U_0(x) \equiv 1, \quad (100)$$

for  $\ell = 0, 1, \dots$ . From (10) and (100), it follows that

$$\begin{aligned} (a_2 + n_2 - \kappa)\Psi_\ell(\kappa) &= a_2\Psi_{\ell+1}(\kappa) + n_2\Psi_{\ell-1}(\kappa); \\ \Psi_{-1}(\kappa) &\equiv 0, \quad \Psi_0(\kappa) \equiv 1. \end{aligned} \quad (101)$$

From (11) and (101), we deduce that

$$[a_2(1 - \delta_{jk_2}) + n_2 - \kappa]\phi_j(\kappa) = a_2\phi_{j-1}(\kappa) + n_2(1 - \delta_{jk_2})\phi_{j+1}(\kappa), \quad (102)$$

for  $j \leq k_2$ . Since<sup>3</sup>

$$U_\ell \left[ \frac{1}{2} \left( \xi + \frac{1}{\xi} \right) \right] = \sum_{r=0}^{\ell} \xi^{2r-\ell}, \quad (103)$$

it follows that

$$\Psi_\ell(0) = \left( \frac{n_2}{a_2} \right)^{\ell/2} U_\ell \left( \frac{a_2 + n_2}{2\sqrt{a_2 n_2}} \right) = \sum_{r=0}^{\ell} \left( \frac{n_2}{a_2} \right)^r. \quad (104)$$

Hence, from (11), we have

$$\phi_j(0) = \left( \frac{n_2}{a_2} \right)^{k_2-j}. \quad (105)$$

Now<sup>3</sup>

$$U_\ell(x) = \frac{[(x + \sqrt{x^2 - 1})^{\ell+1} - (x - \sqrt{x^2 - 1})^{\ell+1}]}{2\sqrt{x^2 - 1}}, \quad x \neq 1. \quad (106)$$

Hence, from (10), we obtain

$$\Psi_\ell(\kappa_r) = \left( \frac{n_2}{a_2} \right)^{\ell/2} \frac{[(\epsilon_r + \sqrt{\epsilon_r^2 - 1})^{\ell+1} - (\epsilon_r - \sqrt{\epsilon_r^2 - 1})^{\ell+1}]}{2\sqrt{\epsilon_r^2 - 1}}, \quad (107)$$

for  $r = 1, \dots, k_1$ , where  $\epsilon_r$  is given by (20). We note that  $\epsilon_r > 1$  for  $r = 1, \dots, k_1$ , since  $\kappa_r < 0$  for  $r = 1, \dots, k_1$ . We take the positive square root of  $\epsilon_r^2 - 1$  in (107). From (11), it follows that

$$\begin{aligned} \phi_j(\kappa_r) &= \frac{(\epsilon_r + \sqrt{\epsilon_r^2 - 1} - \sqrt{a_2/n_2})}{2\sqrt{\epsilon_r^2 - 1}} \left[ \sqrt{\frac{n_2}{a_2}} (\epsilon_r + \sqrt{\epsilon_r^2 - 1}) \right]^{k_2-j} \\ &\quad - \frac{(\epsilon_r - \sqrt{\epsilon_r^2 - 1} - \sqrt{a_2/n_2})}{2\sqrt{\epsilon_r^2 - 1}} \left[ \sqrt{\frac{n_2}{a_2}} (\epsilon_r - \sqrt{\epsilon_r^2 - 1}) \right]^{k_2-j}, \end{aligned} \quad (108)$$

for  $r = 1, \dots, k_1$ . Since  $\kappa_r < 0$  for  $r = 1, \dots, k_1$ , we deduce from (20) that

$$\epsilon_r + \sqrt{\epsilon_r^2 - 1} > \epsilon_0 + \sqrt{\epsilon_0^2 - 1} = \max \left( \sqrt{\frac{a_2}{n_2}}, \sqrt{\frac{n_2}{a_2}} \right) \geq \sqrt{\frac{a_2}{n_2}} \quad (109)$$

for  $r = 1, \dots, k_1$ .

Next, from (14) and (100), it follows that

$$(a_1 + n_1 + \kappa)\Omega_\ell(\kappa) = a_1\Omega_{\ell+1}(\kappa) + n_1\Omega_{\ell-1}(\kappa);$$

$$\Omega_{-1}(\kappa) \equiv 0, \quad \Omega_0(\kappa) \equiv 1. \quad (110)$$

From (15) and (110), we deduce that

$$[a_1(1 - \delta_{ik_1}) + n_1 + \kappa]\theta_i(\kappa) = a_1\theta_{i-1}(\kappa) + n_1(1 - \delta_{ik_1})\theta_{i+1}(\kappa), \quad (111)$$

for  $i \leq k_1$ . Also, from (14) and (103), we obtain

$$\Omega_\ell(0) = \left(\frac{n_1}{a_1}\right)^{\ell/2} U_\ell\left(\frac{a_1 + n_1}{2\sqrt{a_1 n_1}}\right) = \sum_{r=0}^{\ell} \binom{n_1}{a_1}^r. \quad (112)$$

Hence, from (15), we have

$$\theta_i(0) = \left(\frac{n_1}{a_1}\right)^{k_1-i}. \quad (113)$$

Since  $k_1 = n_1 + q_1$ , it follows that

$$\sum_{i=n_1}^{k_1} (i - n_1)\theta_i(0) = \left(\frac{n_1}{a_1}\right)^{q_1} \Lambda_{q_1}\left(\frac{a_1}{n_1}\right), \quad (114)$$

where  $\Lambda_q(\xi)$  is as defined in (61). Finally, if we multiply (111) by  $(i - n_1)$  and sum on  $i$ , we obtain, with the help of (15),

$$\kappa \sum_{i=n_1}^{k_1} (i - n_1)\theta_i(\kappa) = a_1[\Omega_{q_1}(\kappa) - 1] - n_1\Omega_{q_1-1}(\kappa). \quad (115)$$

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