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# Analysis of Some Overflow Problems with Queuing

By J. A. MORRISON

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*When calls offered to a primary group of trunks find all of them busy, provisions are often made for these calls to overflow to other groups of trunks. Such traffic overflow systems have been of interest for a long time, but recently overflow systems that allow for some calls to be queued have been of importance. In this paper we analyze a traffic overflow system with queuing. The system consists of two groups, a primary and a secondary. We consider two cases which differ in the treatment of demands waiting in the primary queue. We adopt a novel analytical approach, which considerably reduces the dimensions of the problem and simplifies the calculation of various steady-state quantities of interest. Our theoretical results include expressions for the loss probabilities, the probability of overflow from the primary to the secondary, and the average waiting times in the queues.*

## I. INTRODUCTION

In this paper, a particular overflow system with queuing is analyzed. The system consists of two groups, a primary and a secondary, with $n_k$ servers and $q_k$ waiting spaces, which receive demands from independent Poisson sources $S_k$ with arrival rates $\lambda_k > 0$, $k = 1$ and 2, respectively, as depicted in Fig. 1. The service times of the demands are independent and exponentially distributed with mean service rate $\mu > 0$. If all $n_2$ servers in the secondary are busy when a demand from $S_2$ arrives, the demand is queued if one of the $q_2$ waiting spaces is available; otherwise, it is lost (blocked and cleared from the system). Demands waiting in the secondary queue enter service (in some prescribed order) as servers in the secondary become free.

If all $n_1$ servers in the primary are busy when a demand from $S_1$ arrives and there is a free server in the secondary and no demands waiting in the secondary queue, the demand is served in the secondary.

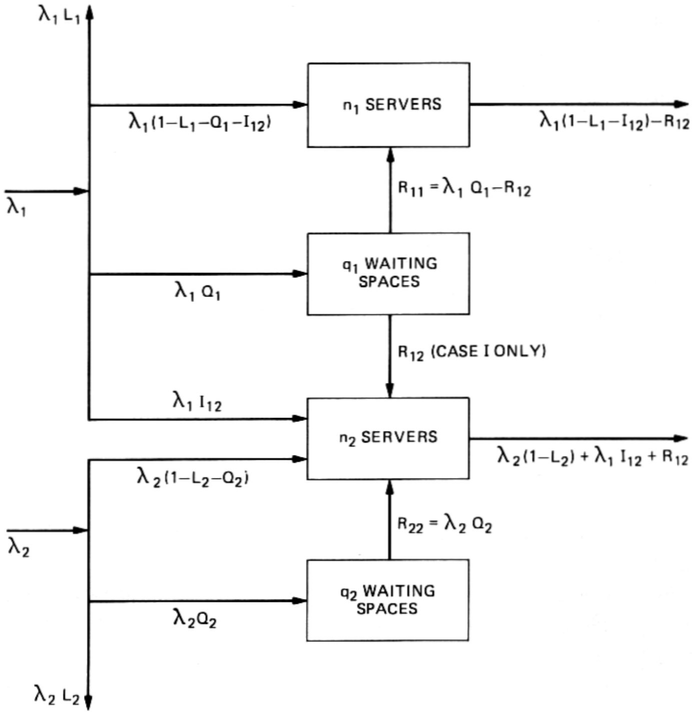


Fig. 1—Mean flow rates for an overflow system with queuing.

If there are no free servers, then the demand is queued in the primary queue, if one of the $q_1$ waiting spaces is available; otherwise, it is lost. Two different cases are considered for the treatment of demands waiting in the primary queue. In case I, a demand waiting in the primary queue may enter service either in the primary, when a server becomes free, or in the secondary, if a server becomes free and no demands are waiting in the secondary queue. In case II, no overflow is permitted from the primary queue, so that a demand in the primary queue must wait for a server in the primary to become free.

The case $q_1 = 0$ and $q_2 = 0$ corresponds to no queuing. This problem has been analyzed by Brockmeyer[1] in the particular case $\lambda_2 = 0$, corresponding to no secondary source. The case $\lambda_2 \neq 0$ is a special case of two groups of servers with a common overflow group. The first group has $n_1$ servers and the second none, while the overflow group has $n_2$ servers. This problem has been analyzed by Kosten,[2,3] but his analysis differs from ours, as is explained in the next section.

Generally, let $p_{ij}$ denote the steady-state probability that there are $i$ demands in the primary and $j$ demands in the secondary. These probabilities satisfy a set of generalized birth-and-death equations, which take the form of partial difference equations connecting neighboring states. Rath[4] has solved these equations numerically in case I, by using a Gauss-Seidel iteration technique. Here we carry out an analysis that reduces the dimensions of the problem, which may be considerable in cases of interest. The basic technique is to separate variables in regions away from certain boundaries of the state space. This leads to various eigenvalue problems for the separation constant. The eigenvalues are roots of polynomial equations. The probabilities $p_{ij}$ are then represented in terms of the corresponding eigenfunctions. The constant coefficients in these representations are determined from the boundary conditions, and the normalization condition that the sum of the probabilities is unity. In general, these constants have to be determined numerically. In case II, additional states are accessible, since in this case demands wait in the primary queue even if a server is free in the secondary. In this additional region, the probabilities $p_{ij}$ are expressed in terms of a fundamental solution of a partial difference equation.

A different procedure for reducing the dimensions of the problem is that of Herzog, Woo, and Chandy.[13] Their procedure, if applied to the problem of this paper, would involve the recursive determination of a number of solutions of a partial difference equation, in order to express the probabilities $p_{ij}$ in terms of their values on certain boundaries. Except in case II, we have to solve only ordinary difference equations in our approach, because of the separation of variables. In case II, as noted above, it is necessary to obtain only a single solution of a partial difference equation in a certain region.

Various steady-state quantities which are of interest may be expressed in terms of the probabilities $p_{ij}$. The quantities include the loss (or blocking) probabilities, the probabilities that a demand is queued, the probability of overflow from the primary to the secondary, and the average waiting times in the queues. These quantities may be expressed directly in terms of the constant coefficients which occur in the representations for the probabilities $p_{ij}$. Thus the steady-state quantities of interest may be calculated directly, once the coefficients have been determined from the boundary and normalization conditions, without having to calculate the probabilities $p_{ij}$. Here again the reduction in the dimensions of the problem is valuable.

Only the theoretical results are presented in this paper. Numerical results will be presented in a forthcoming paper by Kaufman, Seery and Morrison.[5] The reduction in the dimensions of the problem by means of the analysis presented in this paper leads to a dense matrix, rather than the much larger sparse matrix for determining the probabilities $p_{ij}$. Nevertheless, the approach presented here is amenable to numerical calculation for moderately large problems. The largest case run so far using this approach has been for $n_1 = 40 = n_2$ and $q_1 = 10 = q_2$, for case I. Kaufman[6] has used a numerical technique involving matrix separability (block diagonalization) for these problems and, in fact, is led to one of the same eigenvalue problems. In addition, Kaufman has obtained numerical solutions by means of successive over-relaxation techniques. For the example mentioned above, our results for the steady-state quantities of interest agree to 12 significant figures, although some of the probabilities $p_{ij}$, which range over 30 orders of magnitude, agree to fewer significant figures.

An iterative solution of two-dimensional birth-and-death processes has been discussed by Brandwajn.[14] Although his numerical results for some specific problems indicate better execution times than with Gauss-Seidel iteration, the successive over-relaxation technique, with appropriate choice of parameter, would seem to be superior.

The theoretical results are built up in stages, commencing with the analysis of the overflow problem without queuing in Section II. Next, the case in which there is only a secondary queue is investigated in Section III. This analysis pertains to both case I and case II, since the two cases differ only in the treatment of demands queued in the primary. The overflow problem in which there are both a primary and a secondary queue is investigated in Section IV for case I, and the steady-state quantities of interest are calculated in Section V. Another approach for case I, when there is only a primary queue, is presented in Section VI. The analysis of case II, when there is only a primary queue, is given in Section VII. The investigation of case II, when there is both a primary and a secondary queue, is commenced in Section VIII, and the steady-state quantities of interest are calculated in Section IX.

Properties of the eigenfunctions which occur in the representations of the probabilities $p_{ij}$, and which are related to Poisson-Charlier[7,8] and Chebyshev[9] polynomials, are given in Appendix A. Appendix B shows that the corresponding eigenvalues are distinct. Moreover, the eigenvalues, which are roots of polynomial equations, enjoy an interlacing property which facilitates their numerical evaluation.

Another approach to the solution of the overflow problem, which is useful when $q_1$ is large, or even infinite, as is the analysis in Section VI for case I with $q_2 = 0$, will be presented in a separate paper. Also, an alternate approach, which is useful when $q_2$ is large, or even infinite, is presented in another paper.[15]

## II. OVERFLOW WITHOUT QUEUING

We begin by considering an overflow system without queuing. The system consists of two groups, a primary and a secondary, with $n_k \geq 1$ servers, which receive demands from independent Poisson sources

$S_k$ with arrival rates $\lambda_k > 0$, $k = 1$ and 2, respectively. The service times of the demands are independent and exponentially distributed with mean service rate $\mu > 0$. If all $n_2$ servers in the secondary are busy when a demand from $S_2$ arrives, the demand is lost (blocked and cleared from the system). If all $n_1$ servers in the primary are busy when a demand from $S_1$ arrives, then the demand is served in the secondary, if one of the $n_2$ servers is free; otherwise, the demand is lost.

The overflow system corresponding to the case of no secondary source (or $\lambda_2 = 0$) has been analyzed by Brockmeyer.[1] Also, the system described above is a special case of two groups of servers with a common overflow group. The first group has $n_1$ servers and the second none, while the overflow group has $n_2$ servers. This problem has been analyzed by Kosten,[2,3] but his analysis differs from ours, as will be explained.

Let $p_{ij}$ denote the steady-state probability that there are $i$ demands in the primary and $j$ demands in the secondary. These probabilities satisfy a set of generalized birth-and-death equations,[10] which may be derived in a straightforward manner. We define the traffic intensities

$$a_1 = \lambda_1/\mu, \quad a_2 = \lambda_2/\mu, \tag{1}$$

and let $\delta_{ij}$ denote the Kronecker delta; $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. Then it is found that

$$\begin{aligned}
[a_1(1 - \delta_{in_1}\delta_{jn_2}) + a_2(1 - \delta_{jn_2}) + i + j]p_{ij} \\
= a_1(1 - \delta_{i0})p_{i-1,j} + (1 - \delta_{j0})(a_1\delta_{in_1} + a_2)p_{i,j-1} \\
+ (1 - \delta_{in_1})(i + 1)p_{i+1,j} + (1 - \delta_{jn_2})(j + 1)p_{i,j+1},
\end{aligned} \tag{2}$$

for $0 \le i \le n_1$, $0 \le j \le n_2$. The normalization condition is

$$\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} p_{ij} = 1. \tag{3}$$

It is known from the theory of finite Markov chains[11] that there is a unique solution to the system (2) and (3), and $p_{ij} > 0$. Once this solution has been obtained, the steady-state quantities of interest may be calculated. In particular, if $L_k$ is the probability that a demand arriving from $S_k$ is lost, then

$$L_1 = p_{n_1,n_2}, \quad L_2 = \sum_{i=0}^{n_1} p_{i,n_2}. \tag{4}$$

Note that a demand from the primary source is lost only if all the servers in both the primary and the secondary are busy. Such a demand overflows to the secondary if all $n_1$ servers in the primary are busy, but there is a server free in the secondary, so that the probability of overflow is

$$I_{12} = \sum_{j=0}^{n_2-1} p_{n_1,j}. \tag{5}$$

We now proceed to solve the system (2). If $i \neq n_1$, the variables in (2) may be separated, and there are solutions of the form $\alpha_i\beta_j$ where

$$(a_1 + i + \rho)\alpha_i = a_1(1 - \delta_{i0})\alpha_{i-1} + (i + 1)\alpha_{i+1}, \tag{6}$$

for $0 \leq i \leq n_1 - 1$, and

$$[a_2(1 - \delta_{jn_2}) + j - \rho]\beta_j = a_2(1 - \delta_{j0})\beta_{j-1} + (1 - \delta_{jn_2})(j + 1)\beta_{j+1}, \tag{7}$$

for $0 \leq j \leq n_2$, and $\rho$ is a separation constant. The solution of (6) may be expressed in terms of Poisson-Charlier polynomials.[7,8] We here denote the solution of (6) for which $\alpha_0 = 1$ by $s_i(\rho, a_1)$. The properties of $s_i(\rho, a)$ which we will need are given in Appendix A.

For $0 \leq j \leq n_2 - 1$, a solution of (7) is $\beta_j = s_j(-\rho, a_2)$, $0 \leq j \leq n_2$. The boundary condition at $j = n_2$ implies that

$$(n_2 - \rho)s_{n_2}(-\rho, a_2) = a_2 s_{n_2-1}(-\rho, a_2). \tag{8}$$

With the help of (169) and (170) in Appendix A, this equation may be written in the form

$$\rho s_{n_2}(1 - \rho, a_2) = 0. \tag{9}$$

But $s_{n_2}(1 - \rho, a_2)$ is a polynomial in $\rho$ of degree $n_2$. It is shown in Appendix B that its zeros are positive and distinct, and we denote them by $\rho_m$, $m = 1, \cdots, n_2$. Then, with $\rho_0 = 0$, we have

$$\rho_m s_{n_2}(1 - \rho_m, a_2) = 0, \quad m = 0, \cdots, n_2. \tag{10}$$

It follows that we may represent the probabilities $p_{ij}$ in the form

$$p_{ij} = \sum_{m=0}^{n_2} c_m s_i(\rho_m, a_1)s_j(-\rho_m, a_2), \quad 0 \leq i \leq n_1, \ 0 \leq j \leq n_2, \tag{11}$$

where the constants $c_m$ are to be determined. It remains to satisfy the boundary conditions corresponding to $i = n_1$ in (2), as well as the normalization condition (3). Hence, for $i = n_1$ and $j \neq n_2$, we must have

$$(a_1 + a_2 + n_1 + j)p_{n_1,j}$$
$$= a_1 p_{n_1-1,j} + (a_1 + a_2)(1 - \delta_{j0})p_{n_1,j-1} + (j + 1)p_{n_1,j+1}, \tag{12}$$

for $0 \leq j \leq n_2 - 1$. If we substitute (11) into (12), it is found, after reduction with the help of the recurrence relations in Appendix A, that

$$\sum_{m=0}^{n_2} c_m[\rho_m s_{n_1}(1+\rho_m, a_1)s_j(-\rho_m, a_2)$$

$$+ a_1 s_{n_1}(\rho_m, a_1)s_j(-1-\rho_m, a_2)] = 0, \quad (13)$$

for $0 \leq j \leq n_2 - 1$.

The boundary condition corresponding to $i = n_1$ and $j = n_2$ in (2), namely,

$$(n_1 + n_2)p_{n_1,n_2} = a_1 p_{n_1-1,n_2} + (a_1 + a_2)p_{n_1,n_2-1}, \quad (14)$$

implies, after reduction and the use of (10), that

$$\sum_{m=0}^{n_2} c_m[\rho_m s_{n_1}(1+\rho_m, a_1)s_{n_2-1}(1-\rho_m, a_2)$$

$$+ a_1 s_{n_1}(\rho_m, a_1)s_{n_2-1}(-\rho_m, a_2)] = 0. \quad (15)$$

As is to be expected, this condition is redundant, as may be verified by summing (13) from $j = 0$ to $n_2 - 1$, and using (171) in Appendix A. Thus the coefficients $c_m$ are determined by (13) only to within a multiplicative constant, which is determined from the normalization condition (3). From (3), (10), (11), and (171), it follows that

$$c_0 s_{n_1}(1, a_1)s_{n_2}(1, a_2) = 1. \quad (16)$$

In general, the constants $c_m$ have to be calculated numerically. In the limiting case $a_2 = 0$, for which $\rho_m = m$, from (10) and (167), we were able to obtain an analytical expression for $c_m$, and to reproduce the results of Brockmeyer,[1] but we omit the details.

We now comment on the approach adopted by Kosten.[2,3] Whereas we obtained an eigenvalue problem by satisfying the boundary conditions at $j = n_2$, but not at $i = n_1$, Kosten obtained an eigenvalue problem by satisfying the boundary conditions at $i = n_1$, but not at $j = n_2$. He obtains the eigenvalue equation $\gamma s_{n_1}(1 + \gamma, a_1) = 0$, with roots $\gamma_0 = 0$ and $\gamma_r$, $r = 1, \cdots, n_1$. However, since the variables cannot be separated in (2) in the region $0 \leq i \leq n_1$, $0 \leq j \leq n_2 - 1$, the corresponding eigenfunctions, which occur in the representation of $p_{ij}$, contain two terms for $r = 1, \cdots, n_1$, rather than one as in (11). Moreover, the eigenfunction corresponding to $\gamma_0 = 0$, which corresponds to the solution for $n_2 = \infty$, is an infinite sum. Hence it would seem that Kosten's approach is preferable if $n_2$ is significantly larger than $n_1$, but that the representation (11) is more suitable if $n_2 \lesssim n_1$.

Once the constants $c_m$ have been determined, the probabilities $p_{ij}$ may be calculated from (11). We note, in particular, that, with the help of (10) and (16), and (167) and (171) in Appendix A, it follows that

$$\sum_{j=0}^{n_2} p_{ij} = \frac{s_i(0, a_1)}{s_{n_1}(1, a_1)} = \frac{a_1^i/i!}{\sum_{r=0}^{n_1} a_1^r/r!}. \tag{17}$$

This is the well-known result for the steady-state probability that there are $i$ demands in the primary. From (4), (5), and (11), with the help of (171), we obtain

$$L_2 = \sum_{m=0}^{n_2} c_m s_{n_1}(1 + \rho_m, a_1) s_{n_2}(-\rho_m, a_2) \tag{18}$$

and

$$I_{12} = \sum_{m=0}^{n_2} c_m s_{n_1}(\rho_m, a_1) s_{n_2-1}(1 - \rho_m, a_2). \tag{19}$$

Also, from (4), (5), and (17), we have

$$L_1 + I_{12} = \sum_{j=0}^{n_2} p_{n_1,j} = \frac{s_{n_1}(0, a_1)}{s_{n_1}(1, a_1)} = \frac{a_1^{n_1}/n_1!}{\sum_{r=0}^{n_1} a_1^r/r!}. \tag{20}$$

We remark that the quantities $L_1$, $L_2$ and $I_{12}$ may be obtained directly, once the constants $c_m$ are known, without calculating the probabilities $p_{ij}$.

## III. SECONDARY QUEUE ONLY

We next consider the overflow problem in which there are $q_2 \geq 1$ waiting spaces in the secondary, but there is no primary queue. Now, if all $n_2$ servers in the secondary are busy when a demand from $S_2$ arrives, the demand is queued if one of the $q_2$ waiting spaces is available; otherwise, it is lost. Demands waiting in the secondary queue enter service (in some prescribed order) as servers in the secondary become free. If all $n_1$ servers in the primary are busy when a demand from $S_1$ arrives, and if there is a free server in the secondary and no demands are waiting in the secondary queue, then the demand is served in the secondary. If there are no free servers, the demand is lost.

Again, we let $p_{ij}$ denote the steady-state probability that there are $i$ demands in the primary and $j$ demands in the secondary, either in service or waiting. It is convenient to introduce the function

$$\chi_\ell = \begin{cases} 1, & \ell \geq 0, \\ 0, & \ell < 0. \end{cases} \tag{21}$$

Then, with $k_2 = n_2 + q_2$, it is found that

$$[a_1(1 - \delta_{in_1}\chi_{j-n_2}) + a_2(1 - \delta_{jk_2}) + i + \min(j, n_2)]p_{ij}$$
$$= a_1(1 - \delta_{i0})p_{i-1,j} + (1 - \delta_{j0})(a_1\delta_{in_1}\chi_{n_2-j} + a_2)p_{i,j-1}$$
$$+ (1 - \delta_{in_1})(i + 1)p_{i+1,j} + (1 - \delta_{jk_2})\min(j + 1, n_2)p_{i,j+1} \quad (22)$$

for $0 \le i \le n_1$, $0 \le j \le k_2$. The normalization condition is

$$\sum_{i=0}^{n_1} \sum_{j=0}^{k_2} p_{ij} = 1. \quad (23)$$

If $i \ne n_1$, the variables in (22) may be separated, and there are solutions of the form $\alpha_i\beta_j$, where $\alpha_i$ satisfies (6) for $0 \le i \le n_1 - 1$ and

$$[a_2(1 - \delta_{jk_2}) + \min(j, n_2) - \rho]\beta_j$$
$$= a_2(1 - \delta_{j0})\beta_{j-1} + (1 - \delta_{jk_2})\min(j + 1, n_2)\beta_{j+1} \quad (24)$$

for $0 \le j \le k_2$. As before, $\alpha_i = s_i(\rho, a_1)$ for $0 \le i \le n_1$. For $0 \le j \le n_2 - 1$, (24) reduces to (7), and so there are solutions $\beta_j$ proportional to $s_j(-\rho, a_2)$ for $0 \le j \le n_2$. Also, for $n_2 \le j \le k_2$,

$$[a_2(1 - \delta_{jk_2}) + n_2 - \rho]\beta_j = a_2\beta_{j-1} + n_2(1 - \delta_{jk_2})\beta_{j+1}. \quad (25)$$

The solution of (25) may be expressed in terms of Chebyshev polynomials of the second kind,[9] $U_\ell(x)$. It is convenient to define

$$\Psi_\ell(\rho) = \left(\frac{n_2}{a_2}\right)^{\ell/2} U_\ell\left(\frac{a_2 + n_2 - \rho}{2\sqrt{a_2n_2}}\right) \quad (26)$$

and

$$\phi_j(\rho) = \Psi_{k_2-j}(\rho) - \Psi_{k_2-j-1}(\rho). \quad (27)$$

The properties of these functions which we will need are given in Appendix A. We note here, however, that $U_0(x) \equiv 1$, $U_{-1}(x) \equiv 0$ and $\phi_{k_2}(\rho) \equiv 1$. It follows from (25) and (177) that $\beta_j$ is proportional to $\phi_j(\rho)$ for $n_2 - 1 \le j \le k_2$.

Consequently, we take

$$\beta_j = \begin{cases} s_j(-\rho, a_2)\phi_{n_2}(\rho), & 0 \le j \le n_2, \\ s_{n_2}(-\rho, a_2)\phi_j(\rho), & n_2 - 1 \le j \le k_2, \end{cases} \quad (28)$$

where

$$s_{n_2-1}(-\rho, a_2)\phi_{n_2}(\rho) = s_{n_2}(-\rho, a_2)\phi_{n_2-1}(\rho). \quad (29)$$

This equation may be written in the form

$$\rho[s_{n_2}(1 - \rho, a_2)\Psi_{q_2}(\rho) - s_{n_2-1}(1 - \rho, a_2)\Psi_{q_2-1}(\rho)] = 0. \quad (30)$$

Note that this equation reduces to (9) if $q_2 = 0$. The expression in the square brackets in (30) is a polynomial in $\rho$ of degree $k_2 = n_2 + q_2$. Appendix B shows that its zeros are positive and distinct, and we denote them by $\rho_m$, $m = 1, \cdots, k_2$. As before, we let $\rho_0 = 0$.

It follows that we may represent the probabilities $p_{ij}$, for $0 \le i \le n_1$, in the form

$$p_{ij} = \begin{cases} \sum_{m=0}^{k_2} c_m s_i(\rho_m, a_1) s_j(-\rho_m, a_2) \phi_{n_2}(\rho_m), \; 0 \le j \le n_2, \\ \sum_{m=0}^{k_2} c_m s_i(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \phi_j(\rho_m), \; n_2 \le j \le k_2, \end{cases} \tag{31}$$

where the coefficients $c_m$ are to be determined. Note that (31) reduces to (11) if $q_2 = 0$, since $\phi_{k_2}(\rho) \equiv 1$. It remains to satisfy the boundary conditions corresponding to $i = n_1$ in (22), as well as the normalization condition (23). Hence, in addition to (12) for $0 \le j \le n_2 - 1$, we must have

$$(a_2 + n_1 + n_2) p_{n_1, n_2} = a_1 p_{n_1 - 1, n_2} + (a_1 + a_2) p_{n_1, n_2 - 1} + n_2 p_{n_1, n_2 + 1} \tag{32}$$

and

$$[a_2(1 - \delta_{jk_2}) + n_1 + n_2] p_{n_1, j} = a_1 p_{n_1 - 1, j} + a_2 p_{n_1, j-1} + n_2 (1 - \delta_{jk_2}) p_{n_1, j+1} \tag{33}$$

for $n_2 + 1 \le j \le k_2$.

If we substitute (31) into (12), we find, after reduction, that

$$\sum_{m=0}^{k_2} c_m [\rho_m s_{n_1}(1 + \rho_m, a_1) s_j(-\rho_m, a_2) + a_1 s_{n_1}(\rho_m, a_1) s_j(-1 - \rho_m, a_2)] \phi_{n_2}(\rho_m) = 0, \tag{34}$$

for $0 \le j \le n_2 - 1$. Also, from (33) we obtain

$$\sum_{m=0}^{k_2} c_m \rho_m s_{n_1}(1 + \rho_m, a_1) s_{n_2}(-\rho_m, a_2) \phi_j(\rho_m) = 0 \tag{35}$$

for $n_2 + 1 \le j \le k_2$, and from (32) we find that

$$\sum_{m=0}^{k_2} c_m [\rho_m s_{n_1}(1 + \rho_m, a_1) s_{n_2}(-\rho_m, a_2) - a_1 s_{n_1}(\rho_m, a_1) s_{n_2 - 1}(-\rho_m, a_2)] \phi_{n_2}(\rho_m) = 0. \tag{36}$$

This last condition is redundant, as may be verified by summing (34) from $j = 0$ to $n_2 - 1$ and (35) from $j = n_2 + 1$ to $k_2$, and using the fact that $\rho_m$ satisfies (30). Hence, the coefficients $c_m$ are determined by (34) and (35) only to within a multiplicative constant, which is determined from the normalization condition (23). From (23) and (31), with the help of (27), (30), and (170) and (171) of Appendix A, it is found that

$$c_0 s_{n_1}(1, a_1)[s_{n_2}(1, a_2)\Psi_{q_2}(0) - s_{n_2-1}(1, a_2)\Psi_{q_2-1}(0)] = 1. \tag{37}$$

We note, from (167), (168), and (176), that

$$s_n(1, a) = \sum_{r=0}^{n} \frac{a^r}{r!}, \quad \Psi_\ell(0) = \sum_{r=0}^{\ell} \left(\frac{n_2}{a_2}\right)^r. \tag{38}$$

Once the constants $c_m$ have been determined, the probabilities $p_{ij}$ may be calculated from (31). In particular, with the help of (27), (30), (37), (167), and (170), it is found that

$$\sum_{j=0}^{k_2} p_{ij} = \frac{s_i(0, a_1)}{s_{n_1}(1, a_1)} = \frac{a_1^i/i!}{\sum_{r=0}^{n_1} a_1^r/r!}, \tag{39}$$

as is to be expected. Moreover, the steady-state quantities of interest may be calculated. If $L_k$ is the probability that a demand from $S_k$ is lost, then

$$L_1 = \sum_{j=n_2}^{k_2} p_{n_1,j}, \quad L_2 = \sum_{i=0}^{n_1} p_{i,\,k_2}. \tag{40}$$

Also, the probability of overflow $I_{12}$ is as given by (5), and so, from (39),

$$L_1 + I_{12} = \sum_{j=0}^{k_2} p_{n_1,j} = \frac{s_{n_1}(0, a_1)}{s_{n_1}(1, a_1)} = \frac{a_1^{n_1}/n_1!}{\sum_{r=0}^{n_1} a_1^r/r!}. \tag{41}$$

From (31), it is found that

$$L_2 = \sum_{m=0}^{k_2} c_m s_{n_1}(1 + \rho_m, a_1) s_{n_2}(-\rho_m, a_2), \tag{42}$$

since $\phi_{k_2}(\rho) \equiv 1$, from (27), and

$$I_{12} = \sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, a_1) s_{n_2-1}(1 - \rho_m, a_2)\, \phi_{n_2}(\rho_m). \tag{43}$$

Other quantities of interest are the probability that a demand from the secondary source is queued on arrival, namely,

$$Q_2 = \sum_{i=0}^{n_1} \sum_{j=n_2}^{k_2-1} p_{ij}, \tag{44}$$

and the average number of demands in the secondary queue,

$$V_2 = \sum_{i=0}^{n_1} \sum_{j=n_2+1}^{k_2} (j - n_2)p_{ij}. \tag{45}$$

But, if we let

$$r_j = \sum_{i=0}^{n_1} p_{ij}, \tag{46}$$

and sum on $i$ in (22), we obtain

$$[a_2(1 - \delta_{jk_2}) + n_2]r_j = a_2 r_{j-1} + n_2(1 - \delta_{jk_2})r_{j+1}, \tag{47}$$

for $n_2 + 1 \le j \le k_2$. It follows that

$$n_2 r_j = a_2 r_{j-1}, \quad n_2 + 1 \le j \le k_2. \tag{48}$$

Hence, since $L_2 = r_{k_2}$, from (40) and (46),

$$r_j = \left(\frac{n_2}{a_2}\right)^{k_2-j} L_2, \quad n_2 \le j \le k_2. \tag{49}$$

From (45), (46), and (49), we obtain

$$V_2 = \Delta_{q_2}\left(\frac{n_2}{a_2}\right) L_2, \tag{50}$$

where

$$\Delta_q(\xi) = \sum_{\ell=1}^{q} \ell\xi^{q-\ell} = \begin{cases} [q - (q+1)\xi + \xi^{q+1}]/(1-\xi)^2, \ \xi \ne 1, \\ \tfrac{1}{2}\, q(q+1), \ \xi = 1. \end{cases} \tag{51}$$

Similarly, from (44),

$$Q_2 = \Gamma_{q_2}\left(\frac{n_2}{a_2}\right) L_2, \tag{52}$$

where

$$\Gamma_q(\xi) = \sum_{\ell=0}^{q-1} \xi^{q-\ell} = q + (\xi - 1)\Delta_q(\xi). \tag{53}$$

The departure rate from the secondary queue is

$$R_{22} = n_2\mu \sum_{i=0}^{n_1} \sum_{j=n_2+1}^{k_2} p_{ij} = n_2\mu \sum_{j=n_2+1}^{k_2} r_j, \tag{54}$$

from (46). It may be verified from (49) that $R_{22} = \lambda_2 Q_2$. This result holds since, in the steady state, the departure rate from the secondary queue is equal to the arrival rate to it.

Now, according to Little's theorem,[10] the average number of demands in a queuing system is equal to the average rate of arrival of demands to that system times the average time spent in that system. If we apply this result to the secondary queue, we find that the average waiting time of the demands which are queued in the secondary is

$$W_2 = \frac{V_2}{\lambda_2 Q_2}, \tag{55}$$

independently of the order of service within the queue. Now the average number of demands being served by the secondary group is

$$X_2 = \sum_{i=0}^{n_1} \sum_{j=0}^{k_2} \min(j, n_2) p_{ij}. \tag{56}$$

Hence, if we apply Little's theorem to the secondary group of servers, we obtain

$$a_2(1 - L_2) + a_1 I_{12} = X_2, \tag{57}$$

from (1), since the mean service rate is $\mu$. From (31) and (56), with the help of (23) and (171) and (172) of Appendix A, it is found that

$$X_2 = n_2 - \sum_{m=0}^{k_2} c_m s_{n_1}(1 + \rho_m, a_1) s_{n_2-1}(2 - \rho_m, a_2) \phi_{n_2}(\rho_m). \tag{58}$$

The relationship (57) may be verified from (42), (43), and (58), with the help of the boundary conditions (34) and (35), although the verification is difficult. Consequently, the relationship (57) provides a useful check on the numerical calculation of the constants $c_m$.

## IV. PRIMARY AND SECONDARY QUEUES: CASE I

We now begin the investigation of the overflow problem in which there is a primary queue, with $q_1 \geq 1$ waiting spaces, as well as a secondary queue, with $q_2$ waiting spaces. As before, if all $n_2$ servers in the secondary are busy when a demand from $S_2$ arrives, the demand is queued if one of the $q_2$ waiting spaces is available; otherwise, it is lost. Demands waiting in the secondary queue enter service as servers in the secondary become free. If all $n_1$ servers in the primary are busy when a demand from $S_1$ arrives, and if there is a free server in the secondary and no demands are waiting in the secondary queue, then the demand is served in the secondary. If there are no free servers, then the demand is queued in the primary queue, if one of the $q_1$ waiting spaces is available; otherwise, it is lost.

We will consider two different cases for the treatment of demands that are waiting in the primary queue. In the first case, to be considered now, a demand waiting in the primary queue may enter service either in the primary, when a server becomes free, or in the secondary, if a server becomes free and there are no demands waiting in the secondary queue. Thus overflow from the primary queue is permitted in the first case. In the second case, to be considered later, no overflow is permitted from the primary queue, so that a demand in the primary queue must wait for a server in the primary to become free.

We let

$$k_1 = n_1 + q_1, \quad k_2 = n_2 + q_2. \tag{59}$$

Then, in the first case, it is found that the steady-state probabilities $p_{ij}$ satisfy the equations

$$\begin{aligned}[a_1(1 - \delta_{ik_1}\chi_{j-n_2}) + a_2(1 - \delta_{jk_2}) + \min(i, n_1) + \min(j, n_2)]p_{ij} \\ = (1 - \chi_{i-1-n_1}\chi_{n_2-1-j})[a_1(1 - \delta_{i0})p_{i-1,j} \\ + (1 - \delta_{jk_2})\min(j+1, n_2)p_{i,j+1}] \\ + (1 - \delta_{j0})[\mathrm{a}_1\delta_{in_1}\chi_{n_2-j} + \mathrm{a}_2(1 - \chi_{i-1-n_1}\chi_{n_2-j})]\mathrm{p}_{i,j-1} \\ + (1 - \delta_{ik_1})[(1 - \chi_{i-n_1}\chi_{n_2-1-j})\min(i+1, n_1) + n_2\chi_{i-n_1}\delta_{jn_2}]p_{i+1,j},\end{aligned} \tag{60}$$

for $0 \le i \le k_1$, $0 \le j \le k_2$. These equations were constructed so as to imply that

$$p_{ij} = 0, \quad n_1 + 1 \le i \le k_1, \quad 0 \le j \le n_2 - 1, \tag{61}$$

since it is impossible for demands to be waiting in the primary queue when there is a server free in the secondary. The normalization condition is

$$\sum_{i=0}^{n_1}\sum_{j=0}^{k_2} p_{ij} + \sum_{i=n_1+1}^{k_1}\sum_{j=n_2}^{k_2} p_{ij} = 1. \tag{62}$$

We assume for the time being that $q_2 \ge 1$. For $0 \le i \le n_1 - 1$ and $0 \le j \le k_2$, (60) is equivalent to (22), and hence there are solutions of the form $\alpha_i\beta_j$, where $\alpha_i = s_i(\rho, a_1)$ for $0 \le i \le n_1$, $\beta_j$ is given by (28), and $\rho$ is a root of equation (30). For $n_1 + 1 \le i \le k_1$ and $n_2 + 1 \le j \le k_2$, the variables in (60) may be separated, and there are solutions of the form $\gamma_i\delta_j$, where

$$\begin{aligned}[a_1(1 - \delta_{ik_1}) + n_1 + \nu]\gamma_i \\ = a_1\,\gamma_{i-1} + n_1(1 - \delta_{ik_1})\gamma_{i+1}, \; n_1 + 1 \le i \le k_1\end{aligned} \tag{63}$$

and

$$\begin{aligned}[a_2(1 - \delta_{jk_2}) + n_2 - \nu]\delta_j \\ = a_2\delta_{j-1} + n_2(1 - \delta_{jk_2})\delta_{j+1}, \; n_2 + 1 \le j \le k_2,\end{aligned} \tag{64}$$

and $\nu$ is a separation constant. Corresponding to (26) and (27), we define

$$\Omega_\ell(\nu) = \left(\frac{n_1}{a_1}\right)^{\ell/2} U_\ell\left(\frac{a_1 + n_1 + \nu}{2\sqrt{a_1 n_1}}\right) \tag{65}$$

and

$$\theta_i(\nu) = \Omega_{k_1-i}(\nu) - \Omega_{k_1-i-1}(\nu). \tag{66}$$

Then $\gamma_i$ is proportional to $\theta_i(\nu)$ for $n_1 \le i \le k_1$ and $\delta_j$ is proportional to $\phi_j(\nu)$ for $n_2 \le j \le k_2$. We note that $\theta_{k_1}(\nu) \equiv 1$.

Now $\theta_{n_1}(\nu)$ is a polynomial of degree $q_1$ in $\nu$. It is shown in Appendix B that its zeros are negative and distinct, and we denote them by $\nu_\ell$, $\ell = 1, \cdots, q_1$. Then

$$\theta_{n_1}(\nu_\ell) = 0, \quad \ell = 1, \cdots, q_1. \tag{67}$$

Since the roots $\rho_m$, $m = 0, \cdots, k_2$ of (30) are nonnegative and distinct, we have a joint set of distinct roots. In view of the above results, we may represent the probabilities $p_{ij}$ in the form

$$p_{ij} = \begin{cases} \sum_{m=0}^{k_2} c_m s_i(\rho_m, a_1) s_j(-\rho_m, a_2) \theta_{n_1}(\rho_m) \phi_{n_2}(\rho_m), \ 0 \le j \le n_2, \\ \\ \sum_{m=0}^{k_2} c_m s_i(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \theta_{n_1}(\rho_m) \phi_j(\rho_m), \ n_2 \le j \le k_2, \end{cases} \tag{68}$$

for $0 \le i \le n_1$, and

$$p_{ij} = \sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \theta_i(\rho_m) \phi_j(\rho_m)$$
$$+ \sum_{\ell=1}^{q_1} b_\ell \theta_i(\nu_\ell) \phi_j(\nu_\ell), \quad n_1 \le i \le k_1, \ n_2 \le j \le k_2, \tag{69}$$

where the constants $c_m$ and $b_\ell$ are to be determined. Note that the representations in (68) and (69) agree for $i = n_1$, in view of (67). We also note that (68) reduces to (31) if $q_1 = 0$, since $\theta_{k_1}(\nu) \equiv 1$.

It remains to satisfy the boundary conditions at $i = n_1$, $0 \le j \le k_2$ and at $j = n_2$, $n_1 + 1 \le i \le k_1$, as well as the normalization condition (62). Hence, in addition to (12) for $0 \le j \le n_2 - 1$, we must have

$$(a_1 + a_2 + n_1 + n_2) p_{n_1, n_2}$$
$$= a_1 p_{n_1 - 1, n_2} + (a_1 + a_2) p_{n_1, n_2 - 1} + (n_1 + n_2) p_{n_1 + 1, n_2} + n_2 p_{n_1, n_2 + 1} \tag{70}$$

and

$$[a_1 + a_2(1 - \delta_{jk_2}) + n_1 + n_2] p_{n_1, j}$$
$$= a_1 p_{n_1 - 1, j} + a_2 p_{n_1, j-1} + n_1 p_{n_1 + 1, j} + n_2 (1 - \delta_{jk_2}) p_{n_1, j+1} \tag{71}$$

for $n_2 + 1 \le j \le k_2$. Also,

$$[a_1(1 - \delta_{ik_1}) + a_2 + n_1 + n_2] p_{i, n_2}$$
$$= a_1 p_{i-1, n_2} + (n_1 + n_2)(1 - \delta_{ik_1}) p_{i+1, n_2} + n_2 p_{i, n_2 + 1} \tag{72}$$

for $n_1 + 1 \le i \le k_1$.

If we substitute (68) into (12), we find, after reduction, that

$$\sum_{m=0}^{k_2} c_m[\rho_m s_{n_1}(1+\rho_m, a_1)s_j(-\rho_m, a_2) + a_1 s_{n_1}(\rho_m, a_1)s_j(-1-\rho_m, a_2)]\theta_{n_1}(\rho_m)\phi_{n_2}(\rho_m) = 0, \quad (73)$$

for $0 \le j \le n_2 - 1$. Next, from (68), (69), and (71), with the help of (184), we obtain

$$\sum_{m=0}^{k_2} c_m \rho_m s_{n_2}(-\rho_m, a_2)[s_{n_1}(1+\rho_m, a_1)\theta_{n_1}(\rho_m) + s_{n_1}(\rho_m, a_1)\Omega_{q_1-1}(\rho_m)]\phi_j(\rho_m) = n_1 \sum_{\ell=1}^{q_1} b_\ell \theta_{n_1+1}(\nu_\ell)\phi_j(\nu_\ell), \quad (74)$$

for $n_2 + 1 \le j \le k_2$. Also, from (72), with the help of the recurrence relation (63) satisfied by $\theta_i(\nu)$, it is found that

$$\sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, a_1)s_{n_2}(-\rho_m, a_2)[a_2\theta_i(\rho_m)\phi_{n_2-1}(\rho_m) - n_2(1-\delta_{ik_1})\theta_{i+1}(\rho_m)\phi_{n_2}(\rho_m)] + \sum_{\ell=1}^{q_1} b_\ell[a_2\theta_i(\nu_\ell)\phi_{n_2-1}(\nu_\ell) - n_2(1-\delta_{ik_1})\theta_{i+1}(\nu_\ell)\phi_{n_2}(\nu_\ell)] = 0, \quad (75)$$

for $n_1 + 1 \le i \le k_1$. It may be verified that the boundary condition (70) is redundant.

The coefficients $c_m$ and $b_\ell$ are determined by (73) to (75) only to within a multiplicative constant, which is determined from the normalization condition (62). But, from (68), with the help of (27), (30), and (170) and (171) of Appendix A, it is found that

$$\sum_{j=0}^{k_2} p_{ij} = c_0 s_i(0, a_1)\theta_{n_1}(0)[s_{n_2}(1, a_2)\Psi_{q_2}(0) - s_{n_2-1}(1, a_2)\Psi_{q_2-1}(0)] \quad (76)$$

for $0 \le i \le n_1$. Then, from (62) and (69), with the help of (66) and (171), it follows that

$$c_0 s_{n_1}(1, a_1)\theta_{n_1}(0)[s_{n_2}(1, a_2)\Psi_{q_2}(0) - s_{n_2-1}(1, a_2)\Psi_{q_2-1}(0)] + \sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, a_1)s_{n_2}(-\rho_m, a_2)\Omega_{q_1-1}(\rho_m)\Psi_{q_2}(\rho_m) + \sum_{\ell=1}^{q_1} b_\ell \Omega_{q_1-1}(\nu_\ell)\Psi_{q_2}(\nu_\ell) = 1. \quad (77)$$

Note that (77) reduces to (37) if $q_1 = 0$, since $\Omega_{-1}(\nu) \equiv 0$ and $\theta_{k_1}(\nu) \equiv 1$. Once the constants $c_m$ and $b_\ell$ have been determined, the steady-state probabilities $p_{ij}$ may be calculated from (68) and (69). We remark that the number of constants to be determined is only $k_2 + 1 + q_1$, whereas the number of probabilities $p_{ij}$ is $(n_1 + 1)(k_2 + 1) + q_1(q_2 + 1)$, which in general is considerably larger.

## V. SOME STEADY-STATE QUANTITIES: CASE I

Other steady-state quantities of interest are depicted in Fig. 1, which indicates the mean flow rates. The loss probabilities $L_1$ and $L_2$ are given by

$$L_1 = \sum_{j=n_2}^{k_2} p_{k_1,j}, \quad L_2 = \sum_{i=0}^{k_1} p_{i,k_2}, \tag{78}$$

and the probabilities that a demand from the primary, or secondary, source is queued on arrival are

$$Q_1 = \sum_{i=n_1}^{k_1-1} \sum_{j=n_2}^{k_2} p_{ij}, \quad Q_2 = \sum_{i=0}^{k_1} \sum_{j=n_2}^{k_2-1} p_{ij}. \tag{79}$$

The probability that a demand arriving from the primary source overflows immediately is $I_{12}$, as given by (5).

Since the mean service rate is $\mu$, the mean departure rate from the primary queue to the primary servers is

$$R_{11} = n_1\mu \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} p_{ij}, \tag{80}$$

while the mean rate of overflow from the primary queue to the secondary servers is

$$R_{12} = n_2\mu \sum_{i=n_1+1}^{k_1} p_{i,n_2}. \tag{81}$$

The mean departure rate from the secondary queue is

$$R_{22} = n_2\mu \sum_{i=0}^{k_1} \sum_{j=n_2+1}^{k_2} p_{ij}. \tag{82}$$

It may be verified from (60) that

$$R_{11} + R_{12} = \lambda_1 Q_1, \quad R_{22} = \lambda_2 Q_2. \tag{83}$$

The average number of demands in the primary and secondary queues are

$$V_1 = \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} (i - n_1)p_{ij}, \quad V_2 = \sum_{i=0}^{k_1} \sum_{j=n_2+1}^{k_2} (j - n_2)p_{ij}. \tag{84}$$

Also, the average number of demands in service in the two groups are

$$X_1 = \sum_{i=0}^{n_1} \sum_{j=0}^{k_2} ip_{ij} + n_1 \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} p_{ij} \tag{85}$$

and

$$X_2 = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2-1} j p_{ij} + n_2 \sum_{i=0}^{k_1} \sum_{j=n_2}^{k_2} p_{ij}. \tag{86}$$

If we apply Little's theorem to the primary and secondary queues, we find that the average waiting times of the demands which are queued in the primary or in the secondary are given by

$$W_1 = \frac{V_1}{\lambda_1 Q_1}, \quad W_2 = \frac{V_2}{\lambda_2 Q_2}, \tag{87}$$

respectively, independently of the order of service within each queue. Also, if we apply Little's theorem to the primary and secondary groups of servers, we obtain

$$\lambda_1(1 - L_1 - I_{12}) - R_{12} = \mu X_1, \quad \lambda_2(1 - L_2) + \lambda_1 I_{12} + R_{12} = \mu X_2, \tag{88}$$

since the mean service rate is $\mu$.

The steady-state quantities of interest may be expressed in terms of the constants $c_m$ and $b_\ell$ by use of the representations in (68) and (69). From (78), with the help of (27), since $\theta_{k_1}(\nu) \equiv 1$, it is found that

$$L_1 = \sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \Psi_{q_2}(\rho_m) + \sum_{\ell=1}^{q_1} b_\ell \Psi_{q_2}(\nu_\ell), \tag{89}$$

and, with the help of (66) and (171), since $\phi_{k_2}(\rho) \equiv 1$, it is found that

$$L_2 = \sum_{m=0}^{k_2} c_m s_{n_2}(-\rho_m, a_2)[s_{n_1}(1 + \rho_m, a_1)\theta_{n_1}(\rho_m) + s_{n_1}(\rho_m, a_1)\Omega_{q_1-1}(\rho_m)] + \sum_{\ell=1}^{q_1} b_\ell \Omega_{q_1-1}(\nu_\ell). \tag{90}$$

Similarly, it follows from (79) that

$$Q_1 = \sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2)[\Omega_{q_1}(\rho_m) - 1]\Psi_{q_2}(\rho_m) + \sum_{\ell=1}^{q_1} b_\ell[\Omega_{q_1}(\nu_\ell) - 1]\Psi_{q_2}(\nu_\ell). \tag{91}$$

Next, from (5), with the help of (171), it is found that

$$I_{12} = \sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, a_1) s_{n_2-1}(1 - \rho_m, a_2)\theta_{n_1}(\rho_m)\phi_{n_2}(\rho_m). \tag{92}$$

Also, from (80) and (81), it follows that

$$R_{11} = n_1\mu\left[\sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2)\Omega_{q_1-1}(\rho_m)\Psi_{q_2}(\rho_m) + \sum_{\ell=1}^{q_1} b_\ell \Omega_{q_1-1}(\nu_\ell)\Psi_{q_2}(\nu_\ell)\right], \tag{93}$$

and

$$R_{12} = n_2\mu\left[\sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m,\, a_1)s_{n_2}(-\rho_m,\, a_2)\Omega_{q_1-1}(\rho_m)\phi_{n_2}(\rho_m) + \sum_{\ell=1}^{q_1} b_\ell \Omega_{q_1-1}(\nu_\ell)\phi_{n_2}(\nu_\ell)\right]. \quad (94)$$

The first relationship in (83) provides a useful numerical check, in view of the expressions in (91), (93), and (94).

From (84), with the help of (27), (182), and (183), it is found that

$$V_1 = c_0 s_{n_1}(0,\, a_1)s_{n_2}(0,\, a_2)\Psi_{q_2}(0)\Delta_{q_1}\left(\frac{n_1}{a_1}\right) - a_1\sum_{m=1}^{k_2}\frac{c_m}{\rho_m}s_{n_1}(\rho_m,\, a_1)s_{n_2}(-\rho_m,\, a_2)\Psi_{q_2}(\rho_m) \cdot\left[1 - \Omega_{q_1}(\rho_m) + \frac{n_1}{a_1}\Omega_{q_1-1}(\rho_m)\right] - a_1\sum_{\ell=1}^{q_1}\frac{b_\ell}{\nu_\ell}\Psi_{q_2}(\nu_\ell)\left[1 - \Omega_{q_1}(\nu_\ell) + \frac{n_1}{a_1}\Omega_{q_1-1}(\nu_\ell)\right], \quad (95)$$

where $\Delta_q(\xi)$ is as defined in (51). From (85), with the help of (62), (76), and (172), it follows that

$$X_1 = n_1 - c_0 s_{n_1-1}(2,\, a_1)\theta_{n_1}(0) \cdot[s_{n_2}(1,\, a_2)\Psi_{q_2}(0) - s_{n_2-1}(1,\, a_2)\Psi_{q_2-1}(0)]. \quad (96)$$

Similarly, from (86), with the help of (62), (68), (171), and (172),

$$X_2 = n_2 - \sum_{m=0}^{k_2} c_m s_{n_1}(1 + \rho_m,\, a_1)s_{n_2-1}(2 - \rho_m,\, a_2)\theta_{n_1}(\rho_m)\phi_{n_2}(\rho_m). \quad (97)$$

In view of (89), (90), (92), (94), (96), and (97), the relationships in (88) provide a useful numerical check.

If we now define

$$r_j = \sum_{i=0}^{k_1} p_{ij}, \quad (98)$$

and sum on $i$ in (60), for $n_2 + 1 \le j \le k_2$, we obtain (47), and hence (48). Also, $L_2 = r_{k_2}$, from (78) and (98), and it follows that (49) holds and, from (79) and (84), that (50) and (52) hold. Moreover, from (82) and (98),

$$R_{22} = n_2\mu\sum_{j=n_2+1}^{k_2} r_j, \quad (99)$$

and it may be verified, as before, that $R_{22} = \lambda_2 Q_2$. This completes the calculation of expressions for the steady-state quantities of interest.

We have assumed that $q_2 \geq 1$. We remark, however, that the representations in (68) and (69) remain valid in the case $q_2 = 0$, so that $k_2 = n_2$ and $\phi_{n_2}(\rho) \equiv 1$. We have verified that (73) and (75) to (77) hold for $q_2 = 0$, and note that $\Psi_0(\rho) \equiv 1$ and $\Psi_{-1}(\rho) \equiv 0$. We have also verified that (89) to (97) hold for $q_2 = 0$. Moreover, $V_2 = 0$ and $Q_2 = 0$, which is consistent with (50) and (52), as is seen from (51) and (53). In the next section, we investigate the case $q_2 = 0$ in a different manner.

## VI. ANOTHER APPROACH: CASE I

We now give special consideration to the case $q_1 \geq 1$, $q_2 = 0$, so that $k_2 = n_2$. Then (60) for $0 \leq i \leq n_1 - 1$ and $0 \leq j \leq n_2$ leads to the representation

$$p_{ij} = \sum_{m=0}^{n_2} d_m s_i(\rho_m, a_1) s_j(-\rho_m, a_2), \quad 0 \leq i \leq n_1,\ 0 \leq j \leq n_2, \tag{100}$$

corresponding to (11), where $\rho_m$, $m = 0, \cdots, n_2$ are the roots of (10), with $\rho_0 = 0$. The coefficients $d_m$ are to be determined. The only other nonzero steady-state probabilities are $p_{i,n_2}$ for $n_1 + 1 \leq i \leq k_1$. But, from (60), with $k_2 = n_2$,

$$[a_1(1 - \delta_{ik_1}) + n_1 + n_2] p_{i,n_2} = a_1 p_{i-1,n_2} + (n_1 + n_2)(1 - \delta_{ik_1}) p_{i+1,n_2} \tag{101}$$

for $n_1 + 1 \leq i \leq k_1$. It follows that

$$(n_1 + n_2) p_{i,n_2} = a_1 p_{i-1,n_2}, \quad n_1 + 1 \leq i \leq k_1. \tag{102}$$

Hence,

$$p_{i,n_2} = \left(\frac{a_1}{n_1 + n_2}\right)^{i-n_1} p_{n_1,n_2}, \quad n_1 \leq i \leq k_1, \tag{103}$$

and $p_{n_1,n_2}$ is given by (100). We remark that (103) holds if $q_1 = \infty$, provided that the stability condition $a_1 < n_1 + n_2$ is satisfied.

It remains to satisfy the boundary conditions at $i = n_1$, $0 \leq j \leq n_2$, as well as the normalization condition (62) with $k_2 = n_2$. But (60) implies that (12) must hold for $0 \leq j \leq n_2 - 1$, and from (100) it follows that, corresponding to (13),

$$\sum_{m=0}^{n_2} d_m [\rho_m s_{n_1}(1 + \rho_m, a_1) s_j(-\rho_m, a_2) + a_1 s_{n_1}(\rho_m, a_1) s_j(-1 - \rho_m, a_2)] = 0, \tag{104}$$

for $0 \leq j \leq n_2 - 1$. It may be verified that the boundary condition corresponding to $i = n_1$ and $j = n_2$, namely,

$$(a_1 + n_1 + n_2) p_{n_1,n_2} = a_1 p_{n_1-1,n_2} + (a_1 + a_2) p_{n_1,n_2-1} + (n_1 + n_2) p_{n_1+1,n_2}, \tag{105}$$

is redundant. In fact, (105) reduces to (14), which was shown to be redundant, since

$$(n_1 + n_2)p_{n_1+1,n_2} = a_1 p_{n_1,n_2} \tag{106}$$

from (103).

The coefficients $d_m$ are determined by (104) only to within a multiplicative constant, which is determined by the normalization condition. From (62), with $k_2 = n_2$, it is found, with the help of (10), (53), (100), (103), and (171), that

$$d_0 s_{n_1}(1, a_1)s_{n_2}(1, a_2) + \left(\frac{a_1}{n_1 + n_2}\right)^{q_1+1} \Gamma_{q_1}\left(\frac{n_1 + n_2}{a_1}\right) \cdot \sum_{m=0}^{n_2} d_m s_{n_1}(\rho_m, a_1)s_{n_2}(-\rho_m, a_2) = 1. \tag{107}$$

We note that, for $a_1 < n_1 + n_2$,

$$\lim_{q_1\to\infty} \left(\frac{a_1}{n_1 + n_2}\right)^{q_1+1} \Gamma_{q_1}\left(\frac{n_1 + n_2}{a_1}\right) = \frac{a_1}{(n_1 + n_2 - a_1)}. \tag{108}$$

Once the coefficients $d_m$ have been determined from (104) and (107), the steady-state quantities of interest may be calculated with the help of (100) and (103). Since the calculation of these quantities is a straightforward extension of previous calculations, in view of the simplicity of the result in (103), we do not present the results here. We remark that the approach adopted here in the case $q_2 = 0$ requires the determination of only $n_2 + 1$ constants $d_m$, $m = 0, \cdots, n_2$, whereas the approach adopted in the two previous sections requires the determination of the $n_2 + 1$ constants $c_m$, $m = 0, \cdots, n_2$ and the $q_1$ constants $b_\ell$, $\ell = 1, \cdots, q_1$. This raises the question of an analogous approach for $q_2 \geq 1$. We have carried out the corresponding analysis and will present the results in a separate paper. The results are useful, in particular, when $q_1$ is large, or even infinite.

## VII. PRIMARY QUEUE ONLY: CASE II

We now begin the investigation of the second case, in which no overflow is permitted from the primary queue, so that a demand in the primary queue must wait for a server in the primary to become free. Hence, $R_{12} \equiv 0$ in the diagram in Fig. 1, and this is the only aspect in which the second case differs from the first. In this section, we consider the case in which there is a primary queue only, with $q_1 \geq 1$ waiting spaces. The steady-state probabilities $p_{ij}$ now satisfy the equations

$$\begin{aligned}
&[a_1(1 - \delta_{ik_1}\delta_{jn_2}) + a_2(1 - \delta_{jn_2}) + \min(i, n_1) + j]p_{ij} \\
&\quad = a_1(1 - \delta_{i0})[1 - \chi_{i-n_1-1}(1 - \delta_{jn_2})]p_{i-1,j} \\
&\qquad + (1 - \delta_{j0})(a_1\chi_{i-n_1} + a_2)p_{i,j-1} \\
&\qquad + (1 - \delta_{ik_1})\min(i + 1, n_1)p_{i+1,j} + (1 - \delta_{jn_2})(j + 1)p_{i,j+1},
\end{aligned} \tag{109}$$

for $0 \le i \le k_1$, $0 \le j \le n_2$, where $k_1 = n_1 + q_1$. The normalization condition is

$$\sum_{i=0}^{k_1} \sum_{j=0}^{n_2} p_{ij} = 1. \tag{110}$$

But (109) for $0 \le i \le n_1 - 1$ and $0 \le j \le n_2$ leads, analogously to case I, to the representation

$$p_{ij} = \sum_{m=0}^{n_2} D_m s_i(\rho_m, a_1) s_j(-\rho_m, a_2), \quad 0 \le i \le n_1, \ 0 \le j \le n_2, \tag{111}$$

where $\rho_m$, $m = 0, \cdots, n_2$ are the roots of (10), with $\rho_0 = 0$. The coefficients $D_m$ are to be determined. Also, for $n_1 + 1 \le i \le k_1$ and $0 \le j \le n_2 - 1$,

$$\begin{aligned}(a_1 + a_2 + n_1 + j)p_{ij} \\ = (a_1 + a_2)(1 - \delta_{j0})p_{i,j-1} + n_1(1 - \delta_{ik_1})p_{i+1,j} + (j+1)p_{i,j+1}.\end{aligned} \tag{112}$$

We define the quantities $\Pi_{rj}$, for $r, j = 0, 1, \cdots$, as the solutions of the equations

$$\begin{aligned}(a_1 + a_2 + n_1 + j)\Pi_{rj} \\ = (a_1 + a_2)(1 - \delta_{j0})\Pi_{r,j-1} + n_1(1 - \delta_{r0})\Pi_{r-1,j} + (j+1)\Pi_{r,j+1},\end{aligned} \tag{113}$$

which satisfy the initial conditions

$$\Pi_{r0} = \delta_{r0}, \quad r = 0, 1, \cdots . \tag{114}$$

It follows from (163) that

$$\sum_{r=0}^{\infty} \Pi_{rj} v^r = s_j(n_1(1 - v), a_1 + a_2), \tag{115}$$

and, in particular,

$$\Pi_{0j} = s_j(n_1, a_1 + a_2). \tag{116}$$

Also, (115) implies that

$$\Pi_{rj} = 0, \quad r > j. \tag{117}$$

It may be verified, from (112), (113), and (117), that

$$p_{ij} = \sum_{r=i}^{k_1} p_{r0} \Pi_{r-i,j}, \quad n_1 + 1 \le i \le k_1, \ 0 \le j \le n_2. \tag{118}$$

The quantities $\Pi_{rj}$ may be calculated sequentially from the recurrence relations (113), with the help of (117) and the initial condition $\Pi_{00} = 1$.

It remains to satisfy the boundary conditions at $i = n_1$, $0 \le j \le n_2$ and at $j = n_2$, $n_1 + 1 \le i \le k_1$, as well as the normalization condition

(110). Hence, from (109) we must have

$$[a_1 + a_2(1 - \delta_{jn_2}) + n_1 + j]p_{n_1,j} = a_1 p_{n_1-1,j} + (a_1 + a_2)(1 - \delta_{j0})p_{n_1,j-1} + n_1 p_{n_1+1,j} + (1 - \delta_{jn_2})(j + 1)p_{n_1,j+1} \quad (119)$$

for $0 \le j \le n_2$ and

$$[a_1(1 - \delta_{ik_1}) + n_1 + n_2]p_{i,n_2} = a_1 p_{i-1,n_2} + (a_1 + a_2)p_{i,n_2-1} + n_1(1 - \delta_{ik_1})p_{i+1,n_2} \quad (120)$$

for $n_1 + 1 \le i \le k_1$.

From (111), (118), and (119), it is found, after reduction, that

$$\sum_{m=0}^{n_2} D_m[\rho_m s_{n_1}(1 + \rho_m, a_1)s_j(-\rho_m, a_2) + a_1 s_{n_1}(\rho_m, a_1)s_j(-1 - \rho_m, a_2)] - n_1 \sum_{r=n_1+1}^{k_1} p_{r0}\Pi_{r-n_1-1,j} = 0 \quad (121)$$

for $0 \le j \le n_2 - 1$. Also, with the help of (113), the boundary conditions (120) may be written in the form

$$\sum_{r=n_1+1}^{k_1} p_{r0}[(n_2 + 1)\Pi_{r-n_1-1,n_2+1} - (a_1\delta_{n_1+1,k_1} + a_2)\Pi_{r-n_1-1,n_2}] - a_1 \sum_{m=0}^{n_2} D_m s_{n_1}(\rho_m, a_1)s_{n_2}(-\rho_m, a_2) = 0, \quad (122)$$

and, for $q_1 \ge 2$,

$$\sum_{r=i}^{k_1} p_{r0}[(n_2 + 1)\Pi_{r-i,n_2+1} - (a_1\delta_{ik_1} + a_2)\Pi_{r-i,n_2}] - a_1 \sum_{r=i-1}^{k_1} p_{r0}\Pi_{r-i+1,n_2} = 0, \quad (123)$$

for $n_1 + 2 \le i \le k_1$.

The boundary condition corresponding to $j = n_2$ in (119) may be reduced, with the help of (10), to that in (121) for $j = n_2$. However, this last condition may be shown to be redundant, and it suffices to use (121) for $0 \le j \le n_2 - 1$. These conditions, together with those in (122) and (123), determine the constants $D_m$, $m = 0, \cdots, n_2$ and the probabilities $p_{r0}$, $r = n_1 + 1, \cdots, k_1$, only to within a multiplicative constant, which is determined from the normalization condition (110). But, from (111), with the help of (10) and (171), it follows that

$$\sum_{j=0}^{n_2} p_{ij} = D_0 s_i(0, a_1)s_{n_2}(1, a_2), \quad 0 \le i \le n_1. \quad (124)$$

Also, from (109), if we sum on $j$ from 0 to $n_2$, and on $i$ from $\ell$ to $k_1$, we obtain

$$n_1 \sum_{j=0}^{n_2} p_{\ell j} = a_1 p_{\ell-1,n_2}, \quad n_1 + 1 \le \ell \le k_1. \tag{125}$$

Hence, from (124) and (125), with the help of (111), (118), and (171), the normalization condition (110) implies that

$$D_0 s_{n_1}(1, a_1) s_{n_2}(1, a_2) + \frac{a_1}{n_1} \sum_{m=0}^{n_2} D_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) + \frac{a_1}{n_1}\left( \sum_{r=n_1+1}^{k_1} p_{r0} \sum_{\ell=0}^{r-n_1-1} \Pi_{\ell,n_2} - p_{k_1,0}\Pi_{0,n_2}\right) = 1. \tag{126}$$

Once the constants $D_m$, $m = 0, \cdots, n_2$ and the probabilities $p_{r0}$, $r = n_1 + 1, \cdots, k_1$ have been determined, the remaining probabilities may be calculated from (111) and (118). Also, other steady-state quantities of interest may be calculated directly. In particular, the loss probabilities are

$$L_1 = p_{k_1,n_2} = p_{k_1,0}\Pi_{0,n_2}, \tag{127}$$

and, with the help of (171),

$$L_2 = \sum_{i=0}^{k_1} p_{i,n_2} = \sum_{m=0}^{n_2} D_m s_{n_1}(1 + \rho_m, a_1) s_{n_2}(-\rho_m, a_2) + \sum_{r=n_1+1}^{k_1} p_{r0} \sum_{\ell=0}^{r-n_1-1} \Pi_{\ell,n_2}. \tag{128}$$

The probability that a demand from the primary source is queued on arrival is

$$Q_1 = \sum_{i=n_1}^{k_1-1} p_{i,n_2} = \sum_{m=0}^{n_2} D_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) + \sum_{r=n_1+1}^{k_1} p_{r0} \sum_{\ell=0}^{r-n_1-1} \Pi_{\ell,n_2} - p_{k_1,0}\Pi_{0,n_2}, \tag{129}$$

which may be simplified in view of (126).

The probability that a demand arriving from the primary source overflows (immediately) is

$$I_{12} = \sum_{i=n_1}^{k_1} \sum_{j=0}^{n_2-1} p_{ij} = \sum_{j=0}^{n_2-1} \left( p_{n_1,j} + \sum_{i=n_1+1}^{k_1} p_{ij} \right). \tag{130}$$

With the help of (125) and (171), it is found that

$$I_{12} = \sum_{m=0}^{n_2} D_m s_{n_1}(\rho_m, a_1)\left[ s_{n_2-1}(1 - \rho_m, a_2) + \frac{a_1}{n_1} s_{n_2}(-\rho_m, a_2)\right] + \left(\frac{a_1}{n_1} - 1\right) \sum_{r=n_1+1}^{k_1} p_{r0} \sum_{\ell=0}^{r-n_1-1} \Pi_{\ell,n_2} - \frac{a_1}{n_1} p_{k_1,0}\Pi_{0,n_2}. \tag{131}$$

The mean departure rate from the primary queue to the primary servers is

$$R_{11} = n_1\mu \sum_{i=n_1+1}^{k_1} \sum_{j=0}^{n_2} p_{ij}. \tag{132}$$

It follows from (1) and (125) that $R_{11} = \lambda_1 Q_1$, as is to be expected.

The average number of demands in the primary queue is

$$V_1 = \sum_{i=n_1+1}^{k_1} \sum_{j=0}^{n_2} (i - n_1)p_{ij}. \tag{133}$$

Again with the help of (125), it is found that

$$V_1 = \frac{a_1}{n_1}\left[\sum_{m=0}^{n_2} D_m s_{n_1}(\rho_m, a_1)s_{n_2}(-\rho_m, a_2) - (q_1 + 1)p_{k_1,0}\Pi_{0,n_2} + \sum_{r=n_1+1}^{k_1} p_{r0} \sum_{\ell=0}^{r-n_1-1} (r + 1 - \ell - n_1)\Pi_{\ell,n_2}\right]. \tag{134}$$

The average number of demands in service in the primary is

$$X_1 = \sum_{i=0}^{k_1} \sum_{j=0}^{n_2} \min(i, n_1)p_{ij}. \tag{135}$$

With the help of (110), (124) and (172), we obtain

$$X_1 = n_1 - D_0 s_{n_1-1}(2, a_1)s_{n_2}(1, a_2). \tag{136}$$

The average waiting time of the demands which are queued in the primary is $W_1$, as given by (87). Also, since $R_{12} = 0$ and $\lambda_1 = a_1\mu$, the first relationship in (88) becomes

$$a_1(1 - L_1 - I_{12}) = X_1. \tag{137}$$

In view of (127), (131), and (136), this relationship provides a useful numerical check. Finally, we comment that the only $\Pi_{rj}$ which occur in (127) to (129), (131), and (134) are $\Pi_{\ell,n_2}$, for $\ell = 0, \cdots, q_1 - 1$.

## VIII. PRIMARY AND SECONDARY QUEUES: CASE II

We now consider the second case when there is a secondary queue, with $q_2 \geq 1$ waiting spaces, as well as a primary queue, with $q_1 \geq 1$ waiting spaces. The steady-state probabilities satisfy the equations

$$\begin{aligned}
&[a_1(1 - \delta_{ik_1}\chi_{j-n_2}) + a_2(1 - \delta_{jk_2}) + \min(i, n_1) + \min(j, n_2)]p_{ij} \\
&= a_1(1 - \delta_{i0})(1 - \chi_{i-n_1-1}\chi_{n_2-1-j})p_{i-1,j} \\
&\quad + (1 - \delta_{j0})(a_1\chi_{i-n_1}\chi_{n_2-j} + a_2)p_{i,j-1} + (1 - \delta_{ik_1}) \min(i + 1, n_1)p_{i+1,j} \\
&\quad + (1 - \delta_{jk_2}) \min(j + 1, n_2)p_{i,j+1},
\end{aligned} \tag{138}$$

for $0 \leq i \leq k_1$, $0 \leq j \leq k_2$, where $k_1$ and $k_2$ are as defined in (59). The normalization condition is

$$\sum_{i=0}^{k_1} \sum_{j=0}^{k_2} p_{ij} = 1. \tag{139}$$

Now (138) for $0 \le i \le n_1 - 1$ and $0 \le j \le k_2$ leads, analogously to case I, to the representation

$$p_{ij} = \begin{cases} \sum_{m=0}^{k_2} C_m s_i(\rho_m, a_1) s_j(-\rho_m, a_2) \theta_{n_1}(\rho_m) \phi_{n_2}(\rho_m), & 0 \le j \le n_2, \\ \sum_{m=0}^{k_2} C_m s_i(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \theta_{n_1}(\rho_m) \phi_j(\rho_m), & n_2 \le j \le k_2, \end{cases} \tag{140}$$

for $0 \le i \le n_1$, where $\rho_m$, $m = 0, \cdots, k_2$ are the roots of (30), with $\rho_0 = 0$, and the constants $C_m$ are to be determined. Similarly, (138) for $n_1 + 1 \le i \le k_1$ and $n_2 + 1 \le j \le k_2$, analogously to case I, leads to the representation

$$p_{ij} = \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \theta_i(\rho_m) \phi_j(\rho_m) + \sum_{\ell=1}^{q_1} B_\ell \theta_i(\nu_\ell) \phi_j(\nu_\ell), \quad n_1 \le i \le k_1,\ n_2 \le j \le k_2, \tag{141}$$

where $\nu_\ell$, $\ell = 1, \cdots, q_1$, are the roots of $\theta_{n_1}(\nu) = 0$, and the constants $B_\ell$ are to be determined. As before, the representations in (140) and (141) agree for $i = n_1$.

Finally, (138) for $n_1 + 1 \le i \le k_1$ and $0 \le j \le n_2 - 1$ leads, as in the previous section, to the representation

$$p_{ij} = \sum_{r=i}^{k_1} p_{r0} \Pi_{r-i,j}, \quad n_1 + 1 \le i \le k_1, \quad 0 \le j \le n_2. \tag{142}$$

In order that the representations in (141) and (142) agree for $j = n_2$, we must have

$$\sum_{r=i}^{k_1} p_{r0} \Pi_{r-i,n_2} = \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \theta_i(\rho_m) \phi_{n_2}(\rho_m) + \sum_{\ell=1}^{q_1} B_\ell \theta_i(\nu_\ell) \phi_{n_2}(\nu_\ell), \quad n_1 + 1 \le i \le k_1. \tag{143}$$

Since, from (116) and (167), $\Pi_{0,n_2} = s_{n_2}(n_1, a_1 + a_2) > 0$, these equations may be used to solve successively for $p_{k_1,0}, \cdots, p_{n_1+1,0}$ in terms of $C_m$, $m = 0, \cdots, k_2$ and $B_\ell$, $\ell = 1, \cdots, q_1$.

It remains to satisfy the boundary conditions at $i = n_1$, $0 \le j \le k_2$ and at $j = n_2$, $n_1 + 1 \le i \le k_1$, as well as the normalization condition (139). Hence, from (138), in addition to (119) for $0 \le j \le n_2 - 1$ and

(71) for $n_2 + 1 \le j \le k_2$, we must have

$$(a_1 + a_2 + n_1 + n_2)p_{n_1,n_2} = a_1 p_{n_1-1,n_2} + (a_1 + a_2)p_{n_1,n_2-1} + n_1 p_{n_1+1,n_2} + n_2 p_{n_1,n_2+1}, \quad (144)$$

and

$$[a_1(1 - \delta_{ik_1}) + a_2 + n_1 + n_2]p_{i,n_2} = a_1 p_{i-1,n_2} + (a_1 + a_2)p_{i,n_2-1} + n_1(1 - \delta_{ik_1})p_{i+1,n_2} + n_2 p_{i,n_2+1}, \quad (145)$$

for $n_1 + 1 \le i \le k_1$.

From (119) for $0 \le j \le n_2 - 1$, (140) and (142), it is found, analogously to (121), that

$$\sum_{m=0}^{k_2} C_m[\rho_m s_{n_1}(1 + \rho_m, a_1)s_j(-\rho_m, a_2) + a_1 s_{n_1}(\rho_m, a_1)s_j(-1 - \rho_m, a_2)]\theta_{n_1}(\rho_m)\phi_{n_2}(\rho_m) - n_1 \sum_{r=n_1+1}^{k_1} p_{r0}\Pi_{r-n_1-1,j} = 0, \quad (146)$$

for $0 \le j \le n_2 - 1$. Also, from (71), (140), and (141), it is found, analogously to (74), that

$$\sum_{m=0}^{k_2} C_m \rho_m s_{n_2}(-\rho_m, a_2)[s_{n_1}(1 + \rho_m, a_1)\theta_{n_1}(\rho_m) + s_{n_1}(\rho_m, a_1)\Omega_{q_1-1}(\rho_m)]\phi_j(\rho_m) = n_1 \sum_{\ell=1}^{q_1} B_\ell \theta_{n_1+1}(\nu_\ell)\phi_j(\nu_\ell), \quad (147)$$

for $n_2 + 1 \le j \le k_2$. From (141), (142), and (145), with the help of (177) and (180), it follows that

$$a_2 \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1)s_{n_2}(-\rho_m, a_2)\theta_i(\rho_m)\phi_{n_2-1}(\rho_m) + a_2 \sum_{\ell=1}^{q_1} B_\ell \theta_i(\nu_\ell)\phi_{n_2-1}(\nu_\ell) - (a_1 + a_2) \sum_{r=i}^{k_1} p_{r0}\Pi_{r-i,n_2-1} = 0, \quad (148)$$

for $n_1 + 1 \le i \le k_1$.

The boundary condition (144), which may be reduced with the help of (30) to that in (146) for $j = n_2$, can be shown to be redundant. The conditions in (143) and (146) to (148) determine the constants $C_m$, $m = 0, \cdots, k_2$ and $B_\ell$, $\ell = 1, \cdots, q_1$, and the probabilities $p_{r0}$, $r = n_1 + 1, \cdots, k_1$, only to within a multiplicative constant, which is determined from the normalization condition (139). But, from (140), with the help of (27), (30), and (171), it follows that

$$\sum_{j=0}^{k_2} p_{ij} = C_0 s_i(0, a_1)\theta_{n_1}(0)[s_{n_2}(1, a_2)\Psi_{q_2}(0) - s_{n_2-1}(1, a_2)\Psi_{q_2-1}(0)], \quad (149)$$

for $0 \leq i \leq n_1$. Also, from (138), if we sum on $j$ from 0 to $k_2$, and on $i$ from $\ell$ to $k_1$, we obtain

$$n_1 \sum_{j=0}^{k_2} p_{\ell j} = a_1 \sum_{j=n_2}^{k_2} p_{\ell-1,j}, \quad n_1 + 1 \leq \ell \leq k_1. \quad (150)$$

Hence, from (149) and (150), with the help of (27), (66), (141), and (171), the normalization condition (139) implies that

$$C_0 s_{n_1}(1, a_1)\theta_{n_1}(0)[s_{n_2}(1, a_2)\Psi_{q_2}(0) - s_{n_2-1}(1, a_2)\Psi_{q_2-1}(0)]$$
$$+ \frac{a_1}{n_1} \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2)[\Omega_{q_1}(\rho_m) - 1]\Psi_{q_2}(\rho_m)$$
$$+ \frac{a_1}{n_1} \sum_{\ell=1}^{q_1} B_\ell[\Omega_{q_1}(\nu_\ell) - 1]\Psi_{q_2}(\nu_\ell) = 1. \quad (151)$$

Once the constants $C_m$, $m = 0, \cdots, k_2$ and $B_\ell$, $\ell = 1, \cdots, q_1$, and the probabilities $p_{r0}$, $r = n_1 + 1, \cdots, k_1$ have been determined, the remaining probabilities may be calculated from (140) to (142).

## IX. SOME STEADY-STATE QUANTITIES: CASE II

Other steady-state quantities of interest are depicted in the diagram of Fig. 1. The loss probabilities $L_1$ and $L_2$ are given by (78), and the queuing probabilities $Q_1$ and $Q_2$ are given by (79). Also, the mean departure rate from the secondary queue, $R_{22}$ is given by (82). The average number of demands in the secondary queue, $V_2$, is as given by (84), but the average number of demands in the primary queue is

$$V_1 = \sum_{i=n_1+1}^{k_1} \sum_{j=0}^{k_2} (i - n_1) p_{ij}. \quad (152)$$

The average waiting times of the demands which are queued in the primary or in the secondary, $W_1$ and $W_2$, are given by (87). The (immediate) overflow probability $I_{12}$ is given by (130). The mean departure rate from the primary queue to the primary servers is

$$R_{11} = n_1\mu \sum_{i=n_1+1}^{k_1} \sum_{j=0}^{k_2} p_{ij}, \quad (153)$$

and the average number of demands in service in the primary is

$$X_1 = \sum_{i=0}^{k_1} \sum_{j=0}^{k_2} \min(i, n_1) p_{ij}. \quad (154)$$

The above steady-state quantities may be expressed in terms of the constants $C_m$ and $B_\ell$, with the help of the representations in (140) and (141) and the relationship (150). Because of (150), we do not have to use the representation in (142), and hence we are able to eliminate the quantities $\Pi_{\ell j}$. From (78), analogously to case I, and corresponding to (89) and (90),

$$L_1 = \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \Psi_{q_2}(\rho_m) + \sum_{\ell=1}^{q_1} B_\ell \Psi_{q_2}(\nu_\ell) \quad (155)$$

and

$$L_2 = \sum_{m=0}^{k_2} C_m s_{n_2}(-\rho_m, a_2)[s_{n_1}(1 + \rho_m, a_1)\theta_{n_1}(\rho_m) + s_{n_1}(\rho_m, a_1)\Omega_{q_1-1}(\rho_m)] + \sum_{\ell=1}^{q_1} B_\ell \Omega_{q_1-1}(\nu_\ell). \quad (156)$$

Similarly, from (79) we obtain, corresponding to (91),

$$Q_1 = \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2)[\Omega_{q_1}(\rho_m) - 1]\Psi_{q_2}(\rho_m) + \sum_{\ell=1}^{q_1} B_\ell [\Omega_{q_1}(\nu_\ell) - 1]\Psi_{q_2}(\nu_\ell), \quad (157)$$

which may be simplified in view of (151).

From (130), with the help of (27), (66), (150), and (171) it is found that

$$I_{12} = \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2-1}(1 - \rho_m, a_2)\theta_{n_1}(\rho_m)\phi_{n_2}(\rho_m) + \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \cdot \left\{ \frac{a_1}{n_1}[\Omega_{q_1}(\rho_m) - 1] - \Omega_{q_1-1}(\rho_m) \right\} \Psi_{q_2}(\rho_m) + \sum_{\ell=1}^{q_1} B_\ell \left\{ \frac{a_1}{n_1}[\Omega_{q_1}(\nu_\ell) - 1] - \Omega_{q_1-1}(\nu_\ell) \right\} \Psi_{q_2}(\nu_\ell). \quad (158)$$

From (1), (150), and (153), it follows directly that $R_{11} = \lambda_1 Q_1$, as is to be expected. From (154), with the help of (139), (149), and (172), it follows that

$$X_1 = n_1 - C_0 s_{n_1-1}(2, a_1)\theta_{n_1}(0)[s_{n_2}(1, a_2)\Psi_{q_2}(0) - s_{n_2-1}(1, a_2)\Psi_{q_2-1}(0)]. \quad (159)$$

In view of (155), (158), and (159), the relationship in (137) provides a useful numerical check. From (152), with the help of (27), (51), (66), (150), (182), and (183), it is found that

$$V_1 = C_0 s_{n_1}(0, a_1) s_{n_2}(0, a_2) \Psi_{q_2}(0) \Delta_{q_1}\left(\frac{n_1}{a_1}\right)$$

$$+ \frac{a_1}{n_1} \sum_{m=1}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \Psi_{q_2}(\rho_m)$$

$$\cdot \left\{ \Omega_{q_1}(\rho_m) - (q_1 + 1) - \frac{a_1}{\rho_m}[1 - \Omega_{q_1}(\rho_m) + \frac{n_1}{a_1}\Omega_{q_1-1}(\rho_m)] \right\}$$

$$+ \frac{a_1}{n_1} \sum_{\ell=1}^{q_1} B_\ell \Psi_{q_2}(\nu_\ell) \left\{ \Omega_{q_1}(\nu_\ell) - (q_1 + 1) \right.$$

$$\left. - \frac{a_1}{\nu_\ell}[1 - \Omega_{q_1}(\nu_\ell) + \frac{n_1}{a_1}\Omega_{q_1-1}(\nu_\ell)] \right\}. \quad (160)$$

Now (60) and (138) are identical for $0 \le i \le k_1$, $n_2 + 1 \le j \le k_2$. Hence, if we define $r_j$ as in (98), and proceed as in case I, we deduce that (50) and (52) hold, and $R_{22} = \lambda_2 Q_2$, as is to be expected.

Although we assumed here that $q_2 \ge 1$, we remark that the expressions in (155) to (160) remain valid for $q_2 = 0$. This may be verified from (127) to (129), (131), (134), and (136), by substituting $D_m = C_m \theta_{n_1}(\rho_m)$ and using (143), noting that $\phi_{k_2}(\rho) \equiv 1$, and $k_2 = n_2$ for $q_2 = 0$. We point out that

$$\sum_{r=n_1+1}^{k_1} p_{r0} \sum_{\ell=0}^{r-n_1-1} \Pi_{\ell,n_2} = \sum_{i=n_1+1}^{k_1} \sum_{r=i}^{k_1} p_{r0} \Pi_{r-i,n_2}, \quad (161)$$

and

$$\sum_{r=n_1+1}^{k_1} p_{r0} \sum_{\ell=0}^{r-n_1-1} (r + 1 - \ell - n_1) \Pi_{\ell,n_2}$$

$$= \sum_{i=n_1+1}^{k_1} (i + 1 - n_1) \sum_{r=i}^{k_1} p_{r0} \Pi_{r-i,n_2}. \quad (162)$$

Similarly, the normalization condition (151) holds for $q_2 = 0$, as may be verified from (126). Also, the boundary condition (146) reduces to (121) for $q_2 = 0$. Finally, it may be verified, with the help of (113) and (143), that the boundary conditions (122) and (123) are equivalent to (148) for $q_2 = 0$.

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## APPENDIX A

We define $s_i(\lambda, a)$ by the recurrence relation

$$(a + i + \lambda)s_i(\lambda, a) = a(1 - \delta_{i0})s_{i-1}(\lambda, a) + (i + 1)s_{i+1}(\lambda, a);$$
$$s_0(\lambda, a) = 1, \quad (163)$$

for $i = 0, 1, \cdots$. Thus $s_n(\lambda, a)$ is a polynomial of degree $n$ in both $\lambda$ and $a$, and it may be related to a Poisson-Charlier polynomial.[7,8] However, we give here the properties of $s_n(\lambda, a)$ which we will need. It follows from (163) that the generating function

$$g(u) = \sum_{i=0}^{\infty} s_i(\lambda, a)u^i \quad (164)$$

satisfies the equation

$$(1 - u)\frac{dg}{du} = [a(1 - u) + \lambda]g; \quad g(0) = 1. \quad (165)$$

Hence,

$$\sum_{i=0}^{\infty} s_i(\lambda, a)u^i = e^{au}(1 - u)^{-\lambda}. \quad (166)$$

This leads to the explicit formula

$$s_i(\lambda, a) = \sum_{r=0}^{i} \frac{(\lambda)_r a^{i-r}}{r!(i - r)!}, \quad (167)$$

where

$$(\lambda)_0 = 1, \quad (\lambda)_r = \lambda(\lambda + 1) \cdots (\lambda + r - 1), \quad r = 1, 2, \cdots. \quad (168)$$

If we differentiate (166) with respect to $u$, we obtain

$$(i + 1)s_{i+1}(\lambda, a) = as_i(\lambda, a) + \lambda\, s_i(\lambda + 1, a). \quad (169)$$

Also, if we write $(1 - u)^{-\lambda} = (1 - u)(1 - u)^{-\lambda-1}$, it follows from (166) that

$$s_i(\lambda, a) = s_i(\lambda + 1, a) - (1 - \delta_{i0})s_{i-1}(\lambda + 1, a). \quad (170)$$

From (170), we obtain

$$\sum_{i=0}^{n} s_i(\lambda, a) = s_n(\lambda + 1, a). \quad (171)$$

From (169) and (171), we deduce that

$$\sum_{i=0}^{n} (n - i)s_i(\lambda, a) = (1 - \delta_{n0})s_{n-1}(\lambda + 2, a). \quad (172)$$

We now turn our attention to the Chebyshev polynomials of the second kind,[9] $U_\ell(x)$. They may be defined by the recurrence relation

$$2x\, U_\ell(x) = U_{\ell+1}(x) + U_{\ell-1}(x); \quad U_{-1}(x) \equiv 0,\ U_0(x) \equiv 1, \tag{173}$$

for $\ell = 0, 1, \cdots$. From (26) and (173), it follows that

$$(a_2 + n_2 - \rho)\Psi_\ell(\rho) = a_2\, \Psi_{\ell+1}(\rho) + n_2\, \Psi_{\ell-1}(\rho);$$

$$\Psi_{-1}(\rho) \equiv 0,\ \Psi_0(\rho) \equiv 1. \tag{174}$$

Since[9]

$$U_\ell\left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right] = \sum_{r=0}^{\ell} z^{2r-\ell}, \tag{175}$$

it also follows that

$$\Psi_\ell(0) = \left(\frac{n_2}{a_2}\right)^{\ell/2} U_\ell\left(\frac{a_2 + n_2}{2\sqrt{a_2 n_2}}\right) = \sum_{r=0}^{\ell} \left(\frac{n_2}{a_2}\right)^r. \tag{176}$$

From (27) and (174), we deduce that

$$[a_2(1 - \delta_{jk_2}) + n_2 - \rho]\phi_j(\rho) = a_2\phi_{j-1}(\rho) + n_2(1 - \delta_{jk_2})\phi_{j+1}(\rho), \tag{177}$$

for $j \le k_2$.

Similarly, from (65) and (173), it follows that

$$(a_1 + n_1 + \nu)\Omega_\ell(\nu) = a_1\Omega_{\ell+1}(\nu) + n_1\Omega_{\ell-1}(\nu);$$

$$\Omega_{-1}(\nu) \equiv 0,\ \Omega_0(\nu) \equiv 1. \tag{178}$$

Also, from (175), we obtain

$$\Omega_\ell(0) = \left(\frac{n_1}{a_1}\right)^{\ell/2} U_\ell\left(\frac{a_1 + n_1}{2\sqrt{a_1 n_1}}\right) = \sum_{r=0}^{\ell} \left(\frac{n_1}{a_1}\right)^r. \tag{179}$$

From (66) and (178), we deduce that

$$[a_1(1 - \delta_{ik_1}) + n_1 + \nu]\theta_i(\nu) = a_1\theta_{i-1}(\nu) + n_1(1 - \delta_{ik_1})\theta_{i+1}(\nu), \tag{180}$$

for $i \le k_1$. Also, from (179),

$$\theta_i(0) = \left(\frac{n_1}{a_1}\right)^{k_1 - i}. \tag{181}$$

Hence, since $k_1 = n_1 + q_1$,

$$\sum_{i=n_1+1}^{k_1} (i - n_1)\theta_i(0) = \Delta_{q_1}\left(\frac{n_1}{a_1}\right), \tag{182}$$

where $\Delta_q(\xi)$ is as defined in (51). If we multiply (180) by $(i - n_1)$ and sum on $i$, we obtain

$$\nu \sum_{i=n_1+1}^{k_1} (i - n_1)\theta_i(\nu) = a_1 \sum_{i=n_1}^{k_1-1} \theta_i(\nu) - n_1 \sum_{i=n_1+1}^{k_1} \theta_i(\nu)$$

$$= a_1[\Omega_{q_1}(\nu) - 1] - n_1\Omega_{q_1-1}(\nu), \quad (183)$$

from (66). We also note, from (178), that

$$a_1\theta_{n_1}(\nu) - n_1\theta_{n_1+1}(\nu) = \nu\Omega_{q_1-1}(\nu). \quad (184)$$

## APPENDIX B

We first consider the roots of $s_n(1 - \rho, a) = 0$, where $a > 0$. We relate this to an eigenvalue problem for a real symmetric matrix, so that we may apply a result of Slepian and Landau[12] to prove that the roots are distinct. For $j = 0, 1, \cdots$, we let

$$y_j = \sqrt{\frac{j!}{a^j}}\, s_j(1 - \rho, a). \quad (185)$$

Then, from (163) it follows that

$$\sqrt{j}\, y_{j-1} + \frac{1}{\sqrt{a}}(\rho - 1 - a - j)\, y_j + \sqrt{j+1}\, y_{j+1} = 0. \quad (186)$$

If $s_n(1 - \rho, a) = 0$, then $y_n = 0$ and

$$\left[\mathbf{A}_n - \frac{1}{\sqrt{a}}(1 + a - \rho)\mathbf{I}_n\right](y_0, \cdots, y_{n-1})^T = 0, \quad (187)$$

where $T$ denotes transpose, $\mathbf{I}_n$ is the $n \times n$ unit matrix, and

$$\mathbf{A}_1 = [0], \quad \mathbf{A}_n = \begin{bmatrix} 0 & 1 & & \\ 1 & \frac{-1}{\sqrt{a}} & \sqrt{2} & \\ \hdashline & & & \\ & & \sqrt{n-1} & -\frac{(n-1)}{\sqrt{a}} \end{bmatrix}, \; n \geq 2. \quad (188)$$

Hence the equation $s_n(1 - \rho, a) = 0$ is equivalent to

$$\det\left[\mathbf{A}_n - \frac{1}{\sqrt{a}}(1 + a - \rho)\mathbf{I}_n\right] = 0. \quad (189)$$

We remark that Kaufman,[6] who has analyzed the same overflow problem using matrix techniques, arrived at the same eigenvalue problem by block diagonalization.

Now $\mathbf{A}_n$ is a real symmetric matrix, so that its eigenvalues are real.

Moreover, $\mathbf{A}_n$ may be obtained from $\mathbf{A}_{n+1}$ by deleting the last row and column. Hence we may apply the results of Slepian and Landau.[12] In order to establish that the eigenvalues of $\mathbf{A}_n$ separate those of $\mathbf{A}_{n+1}$ for $n = 1, 2, \cdots$, we have to verify that certain quantities (see (6) of Ref. 12) are different from zero. Specifically, let $\boldsymbol{\alpha}$ be the $n$-dimensional column vector obtained from the last column of $\mathbf{A}_{n+1}$ by omitting the diagonal element, so that

$$\boldsymbol{\alpha} = \begin{cases} (1), \ n = 1, \\ (0, \cdots, 0, \sqrt{n})^T, \quad n \geq 2. \end{cases} \tag{190}$$

Hence

$$(\boldsymbol{\alpha}, (y_0, \cdots, y_{n-1})^T) = \sqrt{n}\, y_{n-1} \neq 0, \tag{191}$$

since $y_{n-1} \neq 0$ for any eigenvector corresponding to $y_n = 0$. This establishes the desired result, and we may conclude that the zeros of $s_n(1 - \rho, a)$ separate those of $s_{n+1}(1 - \rho, a)$, for $n = 1, 2, \cdots$. In particular, the zeros of $s_n(1 - \rho, a)$ are real and distinct. Since, from (167), $s_n(\lambda, a) > 0$ for $\lambda \geq 0$, it follows that the zeros of $s_n(1 - \rho, a)$ are larger than 1, and hence positive.

We next consider the zeros of

$$P_{n,q}(\rho) \equiv s_n(1 - \rho, a)\psi_q(\rho) - s_{n-1}(1 - \rho, a)\psi_{q-1}(\rho), \tag{192}$$

where

$$\psi_\ell(\rho) = \left(\frac{n}{a}\right)^{\ell/2} U_\ell\left(\frac{a + n - \rho}{2\sqrt{an}}\right). \tag{193}$$

We let

$$z_j = \begin{cases} y_j, \quad j = 0, \cdots, n, \\ z_{n+q-1}U_{n+q-j-1}\left(\dfrac{a + n - \rho}{2\sqrt{an}}\right), \quad j = n - 1, \cdots, n + q, \end{cases} \tag{194}$$

where $y_j$ is as in (185). We note that $z_{n+q} \equiv 0$, since $U_{-1}(x) \equiv 0$. Also, from (185), (192), and (193), the consistency of the definitions of $z_{n-1}$, and of $z_n$, in (194) implies that $P_{n,q}(\rho) = 0$. From (186), it follows that

$$\sqrt{j}\, z_{j-1} + \frac{1}{\sqrt{a}}(\rho - 1 - a - j)z_j + \sqrt{j+1}\, z_{j+1} = 0, \qquad j = 0, \cdots, n - 1, \tag{195}$$

and from (173) we obtain

$$\sqrt{n}\, z_{j-1} + \frac{1}{\sqrt{a}}(\rho - a - n)z_j + \sqrt{n}\, z_{j+1} = 0, \qquad j = n, \cdots, n + q - 1. \tag{196}$$

But $P_{n,0}(\rho) = s_n(1-\rho, a)$ and we have shown that its zeros are real and distinct. Proceeding as before, it follows that the zeros of $P_{n,q}(\rho)$ separate those of $P_{n,q+1}(\rho)$, for $q = 0, 1, \cdots$. In particular, the zeros of $P_{n,q}(\rho)$ are real and distinct.

We now consider $\rho \le 0$, and will show that $P_{n,q}(\rho) \ne 0$, for $n = 1, 2, \cdots$ and $q = 1, 2, \cdots$. Since, from (170),

$$s_n(1-\rho, a) = s_n(-\rho, a) + s_{n-1}(1-\rho, a), \tag{197}$$

and, from (167), $s_n(\lambda, a) > 0$ for $\lambda \ge 0$, it follows that $s_n(1-\rho, a) > s_{n-1}(1-\rho, a) > 0$ for $\rho \le 0$. Hence

$$\frac{s_n(1-\rho, a)}{s_{n-1}(1-\rho, a)} > 1, \quad \rho \le 0. \tag{198}$$

Also, for $\rho \le 0$,

$$\frac{a+n-\rho}{2\sqrt{an}} = \frac{1}{2}\left(\eta + \frac{1}{\eta}\right), \quad \eta \ge \max\left(\sqrt{\frac{n}{a}}, \sqrt{\frac{a}{n}}\right). \tag{199}$$

But, from (175),

$$\frac{\eta U_{q-1}[(1/2)(\eta + (1/\eta))]}{U_q[(1/2)(\eta + (1/\eta))]} = \frac{\sum_{r=1}^{q} \eta^{2r-q}}{\sum_{r=0}^{q} \eta^{2r-q}} < 1. \tag{200}$$

It follows from (193), (199), and (200), that

$$\frac{\psi_{q-1}(\rho)}{\psi_q(\rho)} < \frac{1}{\eta}\sqrt{\frac{a}{n}} \le 1, \quad \rho \le 0. \tag{201}$$

From (192), (198), and (201), we deduce that $P_{n,q}(\rho) \ne 0$ for $\rho \le 0$. Thus the zeros of $P_{n,q}(\rho)$ are positive.

Finally we consider the zeros of $\theta_{n_1}(\nu)$. For $\ell = 0, 1, 2, \cdots$, we define

$$w_\ell(\nu) \equiv U_\ell\left(\frac{a_1 + n_1 + \nu}{2\sqrt{a_1 n_1}}\right) - \sqrt{\frac{a_1}{n_1}}\, U_{\ell-1}\left(\frac{a_1 + n_1 + \nu}{2\sqrt{a_1 n_1}}\right). \tag{202}$$

Then, from (59), (65), and (66), $\theta_{n_1}(\nu) = 0$ is equivalent to $w_{q_1}(\nu) = 0$. From (173) we deduce that

$$(n_1 + \nu) w_0 = \sqrt{a_1 n_1}\, w_1, \tag{203}$$

and

$$\sqrt{a_1 n_1}\, w_{\ell-1} - (a_1 + n_1 + \nu) w_\ell + \sqrt{a_1 n_1}\, w_{\ell+1} = 0, \tag{204}$$

for $\ell = 1, 2, \cdots$. Proceeding as before, it follows that the zeros of $w_q(\nu)$ separate those of $w_{q+1}(\nu)$, for $q = 1, 2, \cdots$. In particular, the zeros of $w_q(\nu)$ are real and distinct. But, from (193), (201), and (202), it follows that $w_q(\nu) \ne 0$ for $\nu \ge 0$. Hence, the zeros of $w_q(\nu)$ are negative.

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