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Saddle-Point Approximation for M-ary Phase-Shift Keying with Adjacent Satellite Interference

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As the geosynchronous orbit for satellite communication becomes increasingly crowded, the effect of adjacent satellite interference is of primary concern. We describe here the saddle-point approximation technique for evaluating the error probability of M-ary PSK systems with adjacent satellite interference. In comparison with previous methods, this technique has the advantages of both computational simplicity and accuracy. Results include a sample calculation to study the effect of fading on the uplink of a 12/14-GHz quadraphase-shift-keying system with 3-degree satellite spacings and 3-m earth stations.

I. INTRODUCTION

The problem of adjacent satellite interference, which is basically one of co-channel interference, has received extensive attention in terms of exact computation methods[1-5] and bounds.[6-9] In spite of this, system designers still often treat the interference as additive Gaussian noise, even though this has been shown to be very pessimistic[2,4] for lower error rates ($<10^{-3}$). The reason for this is that the exact methods are too difficult to use and the simple bounds are not tight enough for large signal-to-noise ratios (i.e., low error rates). As a result, the communication system is overspecified and the geosynchronous orbit, a limited natural resource, is not used efficiently.

In this paper, we describe a saddle-point method[10,11] which is both simple to use and accurate for evaluating adjacent satellite interference. After defining the problem in Section II, we derive the saddle-point approximation from the Fourier inversion integral representation of the error probability in Section III. To illustrate the accuracy of the approximation, we consider the uplink error performance of a QPSK system in the 12/14 GHz band, with 3*m* earth stations and 10 interfering satellites spaced 3 degrees apart.

II. PROBLEM STATEMENT

We consider a M-ary PSK system with coherent detection. The transmitted signal is one of M possible waveforms:

$$s_m(t) = \sqrt{2S}\cos(\omega_0 t + \theta_m), \qquad 0 \le t < T, \tag{1}$$

where $\theta_m = \pi(2m - 1)/M$ is the carrier phase angle corresponding to the mth symbol ($m = 1, 2, \cdots, M$). At the receiver, there are K interfering signals from different PSK transmitters:

$$I(t) = \sum_{j=1}^{K} \sqrt{2I_j}\cos(\omega_j t + \psi_j + \phi_j), \tag{2}$$

where I_j and ω_j are constants, ψ_j is a random variable conveying the information from the jth interferer ($j = 1, 2, \cdots, K$), and ϕ_j is a uniformly distributed random variable in $[0, 2\pi)$. Then for $s_m(t)$ transmitted, the received signal is given by

$$r(t) = s_m(t) + I(t) + n(t), \tag{3}$$

where the receiver noise, $n(t)$, is a stationary, zero-mean Gaussian process with variance σ^2. Rewriting (3) in quadrature components, we obtain

$$r(t) = \left[\sqrt{2S}\cos\theta_m + \sum_{j=1}^{K}\sqrt{2I_j}\cos\lambda_j(t) + n_c(t)\right]\cos\omega_0 t - \left[\sqrt{2S}\sin\theta_m + \sum_{j=1}^{K}\sqrt{2I_j}\sin\lambda_j(t) + n_s(t)\right]\sin\omega_0 t.$$

The receiver samples the envelopes of the in-phase and quadrature terms at $t = t_0$, and chooses one of M symbols as the transmitted one.

Without loss of generality, we assume θ_1 was sent. The envelope samples are given by

$$r_c = \sqrt{2S}\cos\theta_1 + \sum_{j=1}^{K}\sqrt{2I_j}\cos\lambda_j + n_c \tag{4}$$

and

$$r_s = \sqrt{2S}\sin\theta_1 + \sum_{j=1}^{K}\sqrt{2I_j}\sin\lambda_j + n_s, \tag{5}$$

where the λ_j's are independent and uniformly distributed in $[0, 2\pi)$, the n_c and n_s are zero-mean Gaussian with variance σ^2, and the dependence on t_0 has been suppressed. If the phase angle of the vector (r_c, r_s) lies within $[0, 2\pi/M)$, the receiver would decide that θ_1 was transmitted. As shown in Ref. 4, the symbol error probability is bounded by

$$P_e < 2\Pr[r_s < 0]. \tag{6}$$

III. THE SADDLE-POINT APPROXIMATION

The probability $P_s \triangleq \Pr[r_s < 0]$ can be evaluated via the Fourier inversion integral:[12]

$$P_s = -\frac{1}{2\pi i}\int_C \Phi(u)du/u, \tag{7}$$

where the path of integration C is along the real axis ($-\infty$ to ∞) except for an indentation above the origin, and $\Phi(u)$ is the characteristic function of $r_s/\sqrt{2S}$:

$$\Phi(u) = \exp[iud - u^2/4\rho^2]\prod_{j=1}^{K} J_0(R_j u),$$

with

$$d = \sin(\pi/M),$$

$$\rho^2 = S/\sigma^2, \text{ the carrier-to-noise ratio (CNR)},$$

$$R_j = \sqrt{I_j/S}, \text{ the } j\text{th interference-to-carrier ratio},$$

and

$J_0(x)$, the zeroth order Bessel function of the first kind.

Define $G(u) = \ln[\Phi(u)]$. Then the saddle points of the integrand in (7) (ignoring the effect of u^{-1}) are solutions of $G'(u) = 0$, i.e.,

$$id - u/2\rho^2 - \sum_{j=1}^{K} R_j J_1(R_j u)/J_0(R_j u) = 0. \tag{8}$$

In Appendix A, we show a unique solution of (8) on the positive imaginary axis, i.e., there is only one saddle point. Denoting this by $u_s = iy_s$, we can rewrite the saddle-point equation and define y_s as the solution to

$$H(y) = -d + y/2\rho^2 + \sum_{j=1}^{K} R_j I_1(R_j y)/I_0(R_j y) = 0, \tag{9}$$

where $I_0(x)$ and $I_1(x)$ are the zeroth and first-order modified Bessel functions, respectively. Since it is observed in Appendix A that $H(0)H''(0) > 0$ and $H'(y)$ and $H''(y)$ do not change sign in $[0, y_s]$, Newton's root-finding method (Ref. 13, p. 18):

$$y_{k+1} = y_k - H(y_k)/H'(y_k) \tag{10}$$

would converge monotonically with an initial estimate of $y_0 = 0$.

Since $\Phi(u)$ is analytic in the entire complex u-plane, we can move the contour of integration for (7) away from the pole at the origin, to intersect the imaginary axis at u_s. Then expanding $G(u)$ around the saddle point, we obtain

$$G(u) = G(u_s) + \tfrac{1}{2}(u - u_s)^2 G''(u_s) + \cdots$$
$$\triangleq G_s - \tfrac{1}{2}(u - u_s)^2 \sigma_s^2 + \cdots .$$

Substituting this Taylor series expansion of $G(u)$ into (7), we can integrate term by term to obtain an asymptotic series (as is done in Appendix B), with the leading term given by

$$\hat{P}_s = \tfrac{1}{2} \exp[G_s + y_s^2 \sigma_s^2/2] \mathrm{erfc}[y_s \sigma_s/\sqrt{2}]. \tag{11}$$

where $\mathrm{erfc}(x)$ is the complementary error function (Ref. 13, p. 297). We shall refer to $\hat{P}_s$ as the saddle-point approximation of P_s. It is shown in Appendix B that, as $y_s \to \infty$, which corresponds to either $\rho^2(\text{CNR}) \to \infty$ for an open-eye pattern ($d > D = \sum_{j=1}^{K} R_j$), or $D - d \to 0$ for a closed one ($d < D$), the saddle-point approximation is asymptotically exact. More precisely,

$$P_s = \hat{P}_s + 7K \exp[G_s]/\sqrt{2\pi}\, 8\sigma_s^5 y_s^5 + O(y_s^{-6}). \tag{12}$$

Fortran subroutines for determining the saddle point and evaluating $\hat{P}_s$ are given in Appendix C.

The deformation of the path of integration to pass through the saddle point provides not only an accurate approximation, but also a very efficient method of evaluating the inversion integral numerically.[14] Let $u = x + iy_s$. Then eq. (7) becomes

$$P_s = \exp[G_s] \int_{-\infty}^{\infty} \exp[-x^2/4\rho^2 + ix(d - y_s/2\rho^2)]$$
$$\cdot \prod_{j=1}^{K} \{J_0[R_j(x + iy_s)]/I_0[R_j y_s]\}\, dx/2\pi(y_s - ix). \tag{13}$$

Using the inequality

$$|J_0(x + iy)| \le I_0(y), \tag{14}$$

we immediately obtain a simple upper bound in terms of y_s:

$$P_s \le \exp[G_s]\rho/y_s\sqrt{\pi} \triangleq P_u, \tag{15}$$

which is tighter than the Chernoff bound[15] for $y_s > \rho/\sqrt{\pi}$. To truncate the range of integration in (13) to $(-X, X)$, we first bound the remainder:

$$|R_x| \le \exp[G_s] \int_{X}^{\infty} \exp[-x^2/4\rho^2] dx/\pi y_s = P_u \mathrm{erfc}[X/2\rho]$$
$$\le P_u \exp[-X^2/4\rho^2]. \tag{16}$$

Then the truncated integral is given by

$$P_s - R_x = \exp[G_s] \int_0^X \exp[-x^2/4\rho^2 + E_r(x)] \cdot \{ y_s \cos E_i(x) - x \sin E_i(x) \} \, dx/\pi(x^2 + y_s^2), \quad (17)$$

where

$$\exp[E_r(x) + iE_i(x)] = \exp[ix(d - y_s/2\rho^2)] \prod_{j=1}^{K} \{J_0[R_j(x + iy_s)]/I_0[R_j y_s]\}.$$

Therefore, with (16) and (17), P_s can be determined to any degree of accuracy desired, and (17) is easier to compute numerically than (7) because the latter has a singularity at the origin.[14]

IV. EXAMPLE

To illustrate the accuracy of the saddle-point approximation, we consider the case of a geostationary communication satellite, flanked on each side by $K/2$ interfering satellites equally spaced by $\Delta\theta$ degrees. According to the model in Ref. 4, if all satellites transmit at the same power level, the jth interference-to-carrier ratio at the earth station is given by

$$\hat{R}_j = (k\Delta\theta)^{-1.25}\, 18\lambda/D_a, \quad \text{where} \quad k = \begin{cases} j, j \le K/2 \\ -j, j > K/2 \end{cases}. \quad (18)$$

D_a/λ is the ratio of diameter to wavelength of the earth station antenna, and is approximately 100 for $3m$ stations in the 12/14 GHz band.

If $K + 1$ earth stations are transmitting to their respective satellites with the same CCIR antenna sidelobe characteristic,[4] the interference-to-carrier ratios at the desired satellite are also given by (18) (see Fig. 3). We assume that the earth stations are separated far apart geographically to experience different fading conditions due to precipitation, but close enough so that the angle subtended by the maximum separation at the satellite is small as compared to $\Delta\theta$. Then the worst case condition is when the desired uplink signal is subject to fading while the interfering signals are not, and (18) is modified as

$$R_j = \phi \hat{R}_j = \phi (k\Delta\theta)^{-1.25}\, 18\lambda/D_a, \quad (19)$$

with the fading parameter, $1 \le \phi < \infty$.

Figure 1 shows the uplink performance of a QPSK ($M = 4$) system with interfering satellites at $\Delta\theta = 3°$ spacings and $3m$ earth stations, for 0-, 6-, and 12-dB fading. The two parameters of interest are the

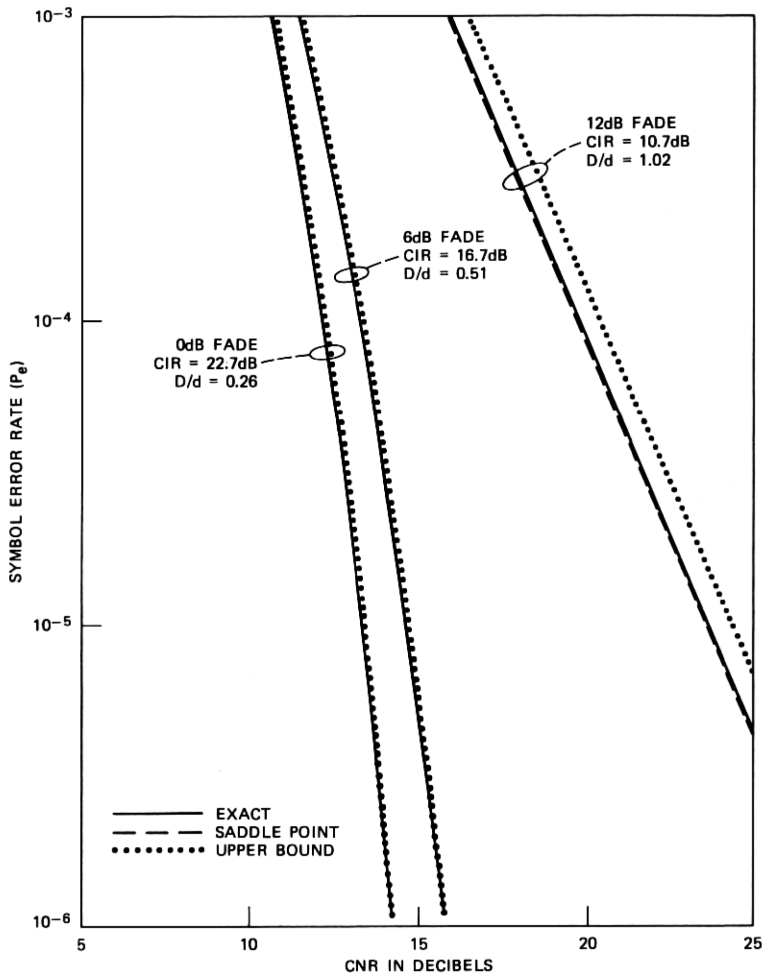

Fig. 1—Performance of a QPSK system subject to fading, with 10 interfering satellites at 3-degree spacings. All curves are upper bounds on the actual symbol error rate [eq. (6)]. The earth station antenna has diameter $3m$ ($D_a \cong 100\lambda$). CIR is the total carrier-to-interference ratio and D/d is the peak distortion.

total carrier-to-interference ratio (CIR):

$$\rho_I^2 = 1 \Big/ \sum_{j=1}^{K} R_j^2 = (\Delta\theta)^{2.5}(D_a/\lambda 18\phi)^2 \Big/ \sum_{k=1}^{K/2} 2k^{-2.5}, \tag{20}$$

and the peak distortion ratio

$$\frac{D}{d} = \frac{1}{d}\sum_{j=1}^{K} R_j.$$

We plot the upper bound (6) on the symbol error probability using $\hat{P}_s$ and P_u from (11) and (15), respectively, and the exact P_s using (16) and (17). For fading conditions of 0 and 6 dB, the exact and the approximate curves are indistinguishable. In the 12-dB case, we note that $D/d > 1$, which means the eye pattern is closed and the saddle point location does not approach infinity for large CNR. However, even though the saddle-point approximation is not exact asymptotically, it is still quite accurate, deviating from the exact result only in the second significant figure.

To determine the fade margin against interference at the satellite, we plot CNR vs CIR for different error rates in Fig. 2. For example, if

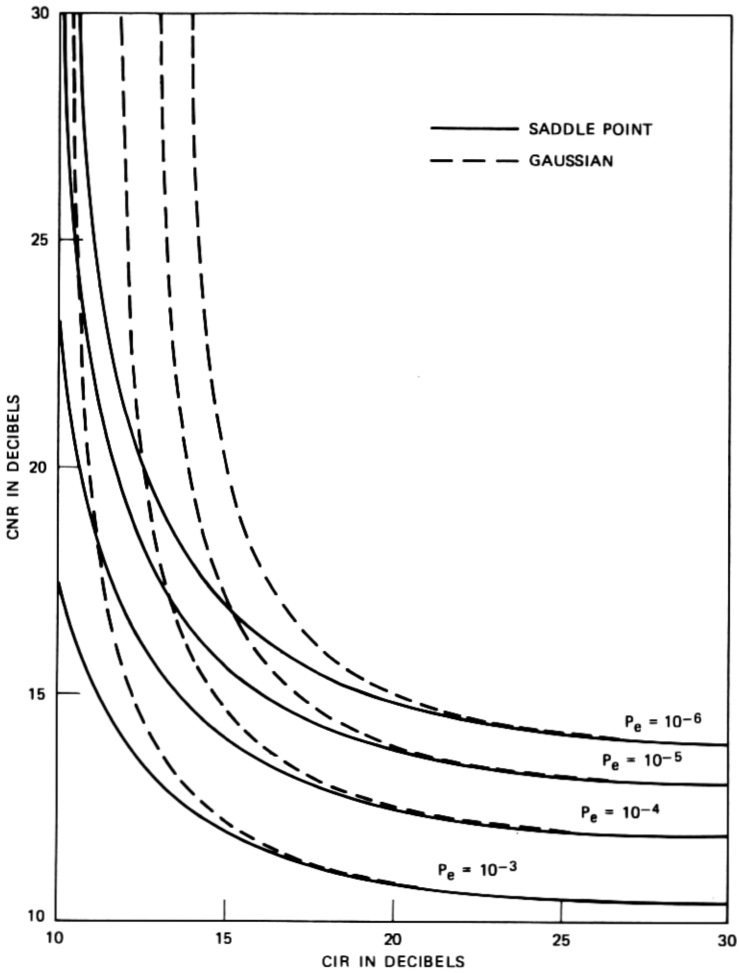

Fig. 2—Effect of variations in CIR on the required CNR to maintain constant error rate. The results are applicable to the uplink QPSK system with 10 interfering satellites equally spaced on both sides of the desired one, regardless of orbital spacing and earth station antenna size.

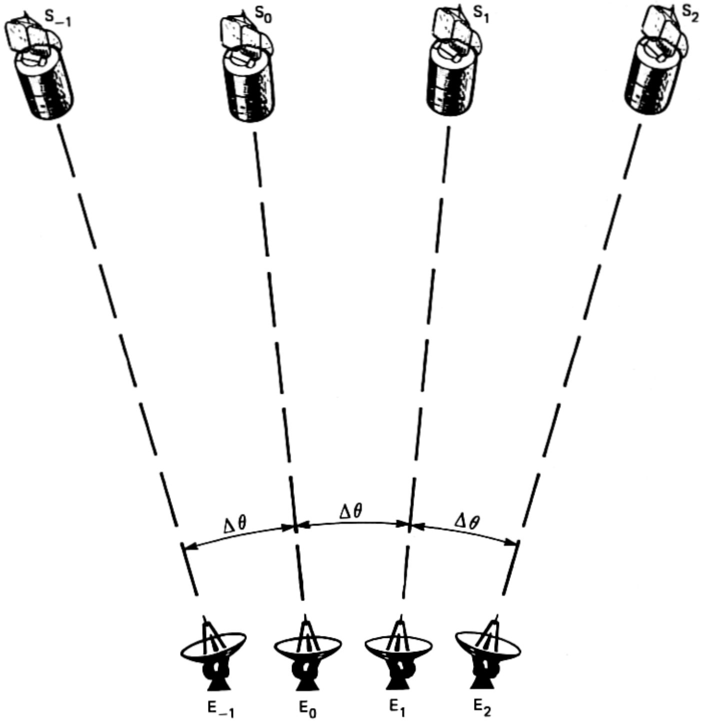

Fig. 3—Equally spaced satellites and their earth stations. Earth station E_k ($k \neq 0$) is pointed at satellite S_k, but its signal would interfere through its sidelobe with the transmission from E_0 to S_0.

the system is designed for $P_e = 10^{-6}$ and a nominal CIR of 23 dB, a 6-dB fade would require an additional increase of 1.7 dB in CNR, which means a total fade margin of 7.7 dB. For comparison, we also show the result from the Gaussian approximation in Fig. 2, using

$$P_s \cong \tfrac{1}{2}\,\mathrm{erfc}(d\rho_e),$$

with the equivalent CNR given by $\rho_e^{-2} = \rho^{-2} + \rho_I^{-2}$. The required fade margin in this case is 8.4 dB, which is too high by 0.7 dB; for lower nominal CIR's, the discrepancy is even larger. Moreover, the Gaussian approximation would overestimate the minimum CIR for $P_e = 10^{-6}$ by at least 3.5 dB.

As seen from (20), given a total CIR, the allocation of interference power among the 10 interferers is independent of satellite spacing ($\Delta\theta$) and the size of the earth station antenna (D_a/λ), so that Fig. 2 is applicable to the model in Ref. 4 for all satellite spacings and antenna sizes.

V. CONCLUSION

In this paper we have described the saddle-point approximation for analyzing M-ary PSK systems with co-channel interference and demonstrated its usage and accuracy in an example. Along with the results for intersymbol interference,[11] this method can be extended to analyze the combined effect of filtering, co-channel and adjacent channel interference.[16–18] It should be noted that the inequality (14) is a fundamental property of characteristic functions and that the upper bound (15) in terms of the saddle point is applicable to all types of additive interference, as long as the receiver noise is Gaussian.

VI. ACKNOWLEDGMENT

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APPENDIX A

Uniqueness of the Imaginary-Axis Saddle Point

The saddle-point equation (9) is repeated here for easy reference:

$$H(y) = -d + y/2\rho^2 + \sum_{j=1}^{K} R_j I_1(R_j y)/I_0(R_j y) = 0. \qquad (21)$$

Define $B(x) = I_1(x)/I_0(x)$. Then the derivative of $H(y)$ is given by

$$H'(y) = \frac{1}{2\rho^2} + \sum_{j=1}^{K} R_j^2 B'(R_j y).$$

Since $H(0) = -d$ and $H(\infty) = \infty$ for $\rho < \infty$, at least one solution to (21) exists in $0 < y < \infty$. It can be shown that $B'(x) > 0$ for $0 < x < \infty$, so that $H'(y) > 0$ in $0 < y < \infty$, which means that only one saddle point is on the positive imaginary axis.

Figure 4 shows $B(x)$ and its first two derivatives. We note that $B''(x)$, and therefore $H''(y)$, is strictly negative for positive arguments. Using the asymptotic expansion of $I_0(x)$ and $I_1(x)$, we obtain for large x,

$$B(x) \sim 1 - x^{-1}/2 + \cdots,$$

$$B'(x) \sim x^{-2}/2 + x^{-3}/4 + \cdots,$$

and

$$B''(x) \sim -x^{-3} - 3x^{-4}/4 + \cdots.$$

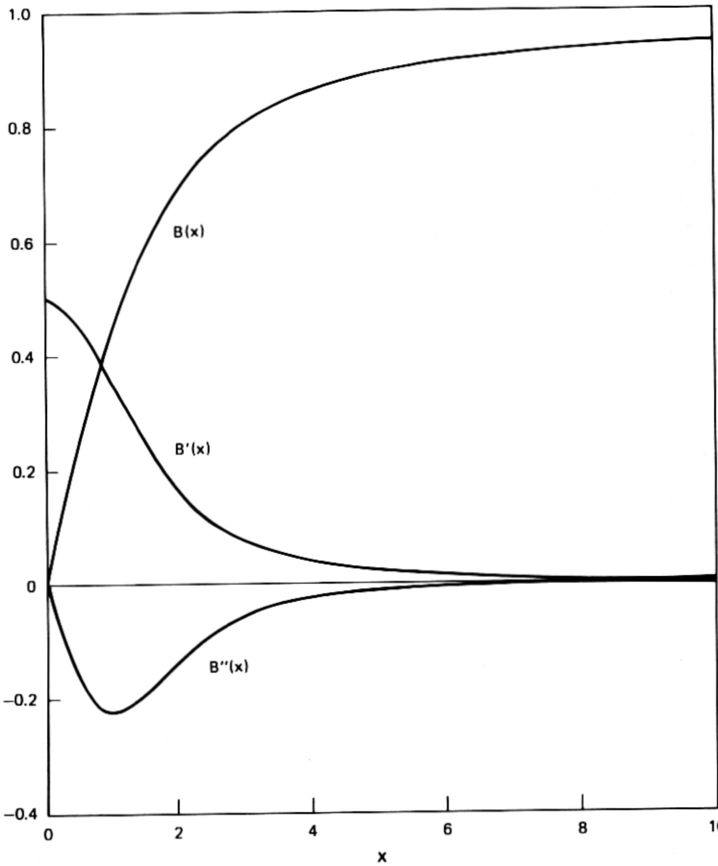

Fig. 4—Behavior of $B(x) = I_1(x)/I_0(x)$ and its derivatives.

APPENDIX B

Asymptotic Expansion of P_s

Along the path of integration which is parallel to the real axis, passing through the saddle point $u_s = iy_s$, $G(u)$ can be written as

$$G(x + iy_s) = F(y_s - ix)$$

$$= F(y_s) + (-ix)F'(y_s) + \frac{(-ix)^2}{2!} F''(y_s) + \sum_{k=3}^{\infty} \frac{(-ix)^k}{k!} F^{(k)}(y_s)$$

$$\triangleq G_s - \frac{x^2}{2} \sigma_s^2 + F_r(x) + iF_i(x),$$

with

$$F_r(x) = \sum_{k=2}^{\infty} (-)^k \frac{x^{2k}}{(2k)!} F^{(2k)}(y_s)$$

and

$$F_i(x) = \sum_{k=2}^{\infty} (-)^k \frac{x^{2k-1}}{(2k-1)!} F^{(2k-1)}(y_s). \tag{22}$$

Substituting (22) into the inversion integral (7), we obtain

$$P_s = \exp[G_s] \int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{2}\sigma_s^2 + F_r(x)\right] \exp[iF_i(x)]dx/(y_s - ix)2\pi$$

$$= \exp[G_s]y_s \int_0^{\infty} \exp\left[-\frac{x^2}{2}\sigma_s^2 + F_r(x)\right]$$

$$\cdot\left\{\cos F_i(x) - \frac{x}{y_s}\sin F_i(x)\right\} dx/(x^2 + y_s^2)\pi$$

$$\triangleq \exp[G_s]y_s \int_0^{\infty} \exp\left[-\frac{x^2}{2}\sigma_s^2\right]\left\{1 + \sum_{k=2}^{\infty} C_k x^{2k}\right\} dx/(x^2 + y_s^2)\pi, \tag{23}$$

with the first two C_k's given by

$$C_2 = F^{(4)}(y_s)/4! - F^{(3)}(y_s)/y_s 3!$$

and

$$C_3 = F^{(5)}(y_s)/y_s 5! - F^{(6)}(y_s)/6! - (F^{(3)}/3!)^2/2.$$

Then, with $t = x^2\sigma_s^2/2$ and $z = \sigma_s^2 y_s^2/2$, term-by-term integration of (23) gives

$$P_s = \sqrt{z}\exp[G_s] \int_0^{\infty} e^{-t}\left[1 + \sum_{k=2}^{\infty} C_k(2/\sigma_s^2)^k t^k\right] t^{-1/2}dt/(t + z)2\pi$$

$$= \frac{1}{2}\exp[G_s + z]\left\{\operatorname{erfc}(\sqrt{z}) + \frac{1}{\pi}\sum_{k=2}^{\infty} C_k(2/\sigma_s^2)^k\right.$$

$$\left.\cdot\Gamma\left(k + \frac{1}{2}\right)z^k\Gamma\left(\frac{1}{2} - k, z\right)\right\}, \tag{24}$$

where $\Gamma(\alpha, z)$ is the incomplete gamma function (Ref. 13, p. 260).

To illustrate the asymptotic behavior of P_s, for large y_s, we have

$$F'''(y) = -K/y^3 + O(y^{-4}),$$

$$F^{(4)}(y) = 3K/y^4 + O(y^{-5}),$$

and

$$F^{(k)}(y) = O(y^{-k}), \quad \text{for} \quad k \geq 5,$$

so that

$$C_2 = 7K/y_s^4 4! + O(y_s^{-5})$$

and

$$C_k = O(y_s^{-2k}), \quad \text{for} \quad k \geq 3.$$

Then the discrepancy between P_s and $\hat{P}_s$ is given by

$$P_s - \hat{P}_s = \frac{\exp[G_s]}{2\sqrt{\pi z}} \frac{7K}{32} z^{-2} + O(z^{-3}).$$

However, it should be noted that y_s may not approach infinity as $\rho^2 \to \infty$. For large y_s, the saddle-point equation (9) becomes

$$H(y_s) \sim y_s/2\rho^2 + D - K/2y_s - d = 0,$$

where $D = \sum_{j=1}^{K} R_j$, and we have two asymptotic solutions for y_s:

$$y_s \sim 2\rho^2(d - D) + K/2(d - D), \quad \text{for} \quad d > D$$

and

$$y_s \sim K/2(D - d), \quad \text{for} \quad d < D.$$

Therefore y_s would approach infinity either as ρ^2 becomes large in the open eye-pattern case ($d > D$), or as $D - d$ tends to zero; if the eye pattern is closed ($d < D$), y_s would approach a finite limit.

APPENDIX C

Fortran Subroutines

The saddle-point approximation is computed to a desired accuracy using two subroutines. The number of interferers K, the individual interference-to-carrier ratios $\{R(j)\}$, and $d = \sin(\pi/M)$ are specified via the common statement. Subroutine "ggg" is used to calculate

$$g0 \triangleq G(iy) = -yd + y^2/4\rho^2 + \sum_{j=1}^{K} \ln[I_0(R_j y)],$$

$$g1 \triangleq -iG'(iy) = H(y) = -d + y/2\rho^2 + \sum_{j=1}^{K} R_j I_1(R_j y)/I_0(R_j y),$$

and

$$g2 \triangleq -G''(iy) = H'(y) = \frac{1}{2}\rho^2 + \sum_{j=1}^{K} R_j^2 \{I_1'(R_j y) I_0(R_j y) - I_1^2(R_j y)\}/I_0^2(R_j y).$$

The modified Bessel functions $I_0(y)$ and $I_1(y)$ are evaluated by calling BESRI, and $I_1'(y) = I_0(y) - I_1(y)/y$.

Subroutine "sadpnt" implements the saddle-point search algorithm as defined in (10), terminating when the latest relative change in the saddle-point estimate is less than eps. It then computes $\hat{P}_s$ according to (11).

```
          subroutine sadpnt(ps, eps)
          common/ggg/y, d, ro2, K, R(2Ø)
c   ps – saddle point approximation
c eps – relative accuracy of saddle point estimate
c All other parameters are transmitted via the labelled common.
          y = Ø.Ø                        !initial guess
5         call ggg(gØ, g1, g2)
7         yinc = -g1/g2
          y = y + yinc
          if(abs(yinc/y) > eps) go to 5
          call ggg(gØ, g1, g2)
          ysig = sqrt(Ø.5 * g2) * y
          ps = Ø.5 * exp(gØ) * erfw(ysig) !erfw(z) = exp(z*z) * erfc(z)
          return
          end


          subroutine ggg(gØ, g1, g2)
c Constant amplitude sinusoidal interferers
          common/ggg/y, d, ro2, K, R(2Ø)
          dimension b(2)
          sig2 = .5/ro2
          gØ = Ø.5 * y * y * sig2 - y * d
          g1 = y * sig2 - d
          g2 = sig2
          do 1Ø j = 1, K
          rj = R(j)
          ry = rj * y
          nb = 2
          call besri(ry, nb, b)        !Modified Bessel fcns: Ik, k < nb
          bØ = b(1)
          b1 = b(2)
          b1p = Ø.5
          if(ry.ne.Ø.) b1p = b1/ry
          b1p = bØ - b1p
          gØ = gØ + alog(bØ)
          g1 = g1 + rj * b1/bØ
          g2 = g2 + rj * rj * (b1p * bØ - b1 * b1)/(bØ * bØ)
1Ø        continue
          return
          end
```

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