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The Application of Mathematical Programming to Loop Feeder Allocation

By T. R. ELKEN

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Loop feeder allocation is the process of planning for economic use of the spare capacity in a feeder route, which supplies a central office. This paper presents a model of the feeder route for which a good allocation of feeder pairs is derived using a combination of heuristic and mathematical programming techniques. The model of the route reflects the factors relevant to making allocation decisions: subscriber growth, operating costs due to subscriber movement, transmission requirements, relief, and rearrangement costs. The model is flexible enough to handle allocation problems on routes with pair gain systems as well as conventional resistance design cable routes with multiple paths from some customers to the central office. An iterative separable linear programming algorithm is used to find the feasible allocation with the minimum expected operating expense. The need to reserve existing pairs in the feeder route to connect to the pairs placed in future relief jobs requires the solution of a multi-time period model. To save on computation time, a heuristic algorithm is used to solve the dynamic problem approximately.

I. INTRODUCTION

The loop network is divided into two components, distribution and feeder. The distinction is basically economic. Distribution is the portion of the network nearest to the customer. Each distribution cable serves a small, well-defined geographical area, resulting in a low ultimate demand for capacity. Thus, economies of scale imply that cables should be sized according to the ultimate demand. Distribution cables are spliced into larger *feeder* cables that connect them to the central office. Since the cross-sectional demand increase is considerably greater, it becomes economically advantageous to add capacity periodically rather than all at once.

An impending shortage of spare facilities (wire pairs) in some distribution cable is usually due to a shortage of feeder pairs which are spliced to that distribution cable. If similar scarcities of spare feeder pairs appear throughout the geographic area fed by that cable, then more feeder cable must be added to the route. However, if some areas have an excess of spare pairs, splicing operations can sometimes be performed to make those spare pairs available to the distribution cable in need of them.

The economic management of existing facilities and planning for new facilities in the loop network has come to be called feeder administration. This paper is concerned with the management of existing facilities, also referred to as the allocation process.[1] It describes a model of the physical feeder route and the craft activity required to connect subscribers to the central office. This model is solvable by a mathematical programming algorithm which is the basis of a computer program being developed to help outside plant engineers perform the feeder administration function.

The purpose of this algorithm is to suggest an allocation that will move more of the available feeder capacity to those areas which are growing more quickly than anticipated or are exhibiting high operating expenses due to customers connecting to or disconnecting from the network, or "churn." In feeder routes experiencing substantial growth, this movement or rearrangement of capacity has the primary benefit of deferring the next facility (cable or pair gain system) placement. On slowly growing routes for which the next facility placement is not an immediate concern, the primary motivation for rearrangements is to reduce the operating expenses due to churn. To aid in making decisions for both types of routes, it was decided that the theoretical allocation should be one which minimizes route operating expenses subject to some constraints. The constraints should insure that:

(*i*) The allocation satisfies resistance design requirements.

(*ii*) The available feeder capacity is not exceeded.

(*iii*) Demand is routed along the correct path back to the central office (co).

(*iv*) Some facilities remain unallocated initially so that they may be allocated in conjunction with future feeder cable placements.

Once the program has suggested a theoretical allocation, the engineer investigates the feasibility and cost of the rearrangements necessary to effect the proposed allocations. If the rearrangements are feasible, the engineer interactively describes the rearrangements and the costs associated with them to the feeder administration program. Then, using the cost models of Ref. 2, the economic impact of the rearrangements is evaluated as described in Ref. 3. The program was designed to be interactive because factors involved in the cost of rearrangements are very difficult to model and must be studied by the engineer.

A mathematical program with a convex objective function and linear constraints can be solved to provide a theoretical allocation which fulfills the properties mentioned above. The reader who is more interested in mathematical programming than feeder route modeling can get a quick introduction to the model by reading Appendix A and moving directly to Section IV. Section II discusses the constraints and Section III presents the objective function for the model. An adaptation of separable linear programming techniques for the solution of such a convex program will be presented in Section IV. When several facility placements (relief jobs) are anticipated in the next few years, it may be unwise to suggest a rearrangement which will have to be reversed in a short time in order to splice up the relief pairs to the central office. To prevent such short-sighted actions, an estimate must be made of what configuration the pairs in the feeder route will have after the relief job. In Section V, a multi-time-period model is defined which has constraints preventing the temporary allocation of pairs to an allocation area. Section VI describes how to calculate the number of pairs reserved between the central office and future cable placements.

II. THE FEEDER ROUTE PHYSICAL MODEL

2.1 Definitions

To plan for the economic expansion of capacity, the feeder route is divided into feeder sections, which include all the cable and conduit in a cross section between two points along the route. A feeder section is defined so that it has a nearly uniform cross-sectional capacity. The Exchange Feeder Route Analysis Program (EFRAP) is used to determine the economical size and nominal time for cable placements in each section of the route.[4]

An EFRAP schematic is a representation of the feeder route in the form of a graph. Each section has an associated load area which is the geographic area served by lateral cables which enter the feeder route in a particular section. These are sometimes called EFRAP load areas or ELAS. Figure 1 illustrates the EFRAP schematic and an ELA. The EFRAP load area is the unit for forecasting demand in the feeder route. If the loops feeding an ELA follow two or more paths back to the central office, the engineer must predict the future demand which will

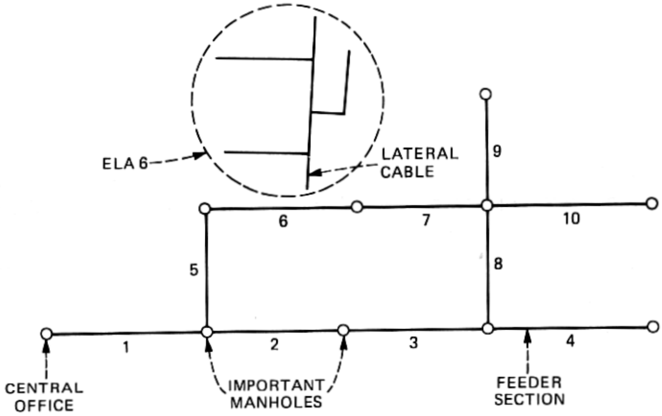

Fig. 1—A feeder route schematic with an outline of an EFRAP load area (ELA).

be placed upon each path from such load areas. More than one path in a route can occur when there are circuits or loops in the feeder route schematic. The demand routing decision is part of the long-range planning process and may be considered fixed for the purposes of allocation. The allocation process is part of short-term planning.

Pair gain systems are electronic devices which use multiplexing, either analog or digital, and switching to allow a large number, N, lines to feed customers from a remote terminal (RT) which uses only n trunks to connect to the central office, where $N > n$. Pair gain systems can be modeled using the multiple-path concept. The availability of this type of capacity for allocation is represented by an alternate path consisting of one section between the RT site and the central office.

The use of physical cable pairs between the RT site and the central office is modeled by creating another ELA at the location of the RT site which demands n pairs for each pair gain system placed. Figure 2 is an example of this model when the *SLC*™-96 systems are in place at an RT site.

The portion of an ELA fed by a single path is called a *computational unit* (CU). In the mathematical model below, a variable x_i, $i = 1, \cdots,$ NCU is defined for the number of pairs allocated to each CU, where NCU is the number of computational units.

Actually, some pairs may have appearances in two different ELAs, so it is difficult to define the current number of pairs available in an ELA, let alone define a future allocation. The allocation area is a geographic area with between 500 and 2000 ultimate available pairs and is defined so that less than 10 percent of its *complements* (groups of pairs, usually in multiples of 25) appear in another allocation area. The allocation area allocations will be represented by the variables, z_i, $i = 1, \cdots,$

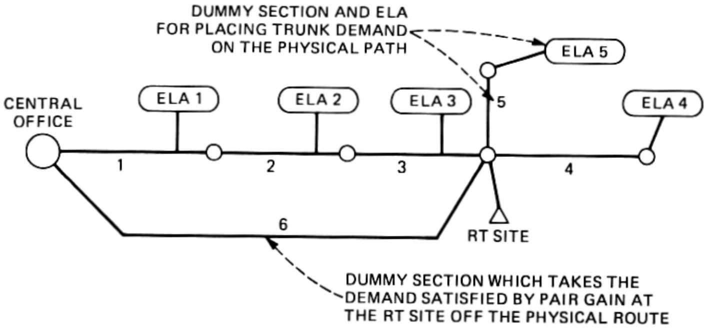

Fig. 2—If two *SLC*™-96 systems (concentrated version) were placed at the RT site, section 6 would have a capacity of 192 pairs and ELA 5 would require 12 pairs in sections 1, 2, and 3.

NAA, where NAA is the number of allocation areas and allocation area is made up of a number of computational units indexed by the set of E_i. Thus the equation

$$z_i = \sum_{j \in E_i} x_j, \qquad i = 1, \cdots, \text{NAA} \tag{1}$$

equates the allocation of an AA to the sum of allocations to the CUs which comprise the AA. Define $z = (z_1, z_2, \cdots, z_{\text{NAA}})$ to be the vector of AA allocations and $x = (x_1, x_2, \cdots, x_{\text{NCU}})$ the vector of CU allocations. Then the NAA equations (1) can be written in matrix form $z = Dx$ where D is the appropriate 0-1 matrix. The CU variables are intended to ensure that the allocation to an AA is feasible and the AA variables are used in cost functions which estimate the influence of the number of pairs allocated to an allocation area upon the operating expenses.

2.2 The section constraints

To ensure that allocation vector $z = (z_1, i = 1, \cdots, \text{NAA})$ is feasible, the cable in the feeder sections must be segregated by gauge (26, 24, 22, or 19 gauge), and the demand must be accumulated by gauge.

The minimum two-gauge resistance design (an overview by Long[5] defines this and other loop terminology) for each CU determines which gauge wire it requires in the feeder sections between it and the central office. Overgauging is also allowed so that coarser gauges than the required gauge can be used to serve a CU. Consider a feeder section which contains only 26- and 24-gauge facilities. In general terms, the constraints for section can be represented as

$$D_{24}(t) \leqslant s_{24}$$

$$D_{24}(t) + D_{26}(t) \leqslant s_{24} + s_{26}$$

$$0 \leqslant t \leqslant T,$$

where $D_g(t)$ is the demand for gauge g facilities in the section at time t, s_g is the number of gauge g pairs available during the interval $[0, T]$ and T is the time until the next capacity expansion. To define these inequalities in terms of the allocations to CU's, let G_{24} be the set of indices i such that x_i requires 24-gauge facilities. In the section under consideration, G_{26} is similarly defined. Also, let $\underline{x}_i$ be the maximum number of pairs demanded in CU i during the interval $[0, T]$. Then the constraints for the section in question are

$$\sum_{i \in G_{24}} x_i \leqslant s_{24} \tag{2}$$

$$\sum_{i \in G_{24}} x_i + \sum_{i \in G_{26}} x_i \leqslant s_{24} + s_{26} \tag{3}$$

$$x_i \geqslant \underline{x}_i, \qquad i \in G_{24} \quad \text{or} \quad G_{26}. \tag{4}$$

The form of the constraints is analogous when three or four gauges are present in a section. Appendix A gives a simple feeder route and the resulting system of constraints when only one gauge of cable is present.

For simplicity of notation, we now refer to cable gauges as 1, 2, 3, 4 rather than 19, 22, 24, and 26. Define $t_{gl} \equiv \sum_{j=1}^{g} s_{jl}$, the pairs available in section l of gauge g or coarser. If constraints similar to (2), (3), and (4) are constructed for each section, the total set of inequalities will define the set of feasible allocations for the feeder route. Often sections will not have every gauge of capacity or demand, so a smaller number of constraints may define the feasible set. Let S be the set of pairs (g, l) which correspond to an inequality in the model and $t \equiv (t_{gl} \mid (g, l) \in S)$ be the vector of capacities. Then if the coefficients of the variables x_i, $i = 1, \cdots,$ NCU are in the form of a 0-1 matrix, C, the constraint set can be written as

$$Cx \leqslant t \tag{5}$$

$$x \geqslant \underline{x}. \tag{6}$$

Even though C is a 0-1 matrix, it is not totally unimodular.* The nature of the multipath routes makes it possible to construct examples with nonunimodular submatrices. An example of a feeder route with this property is in Appendix B. This means that it is not possible to transform the problem into a network flow problem.[6]

III. THE ALLOCATION AREA OPERATING COST MODEL

The modeling of operating costs in the loop network has been discussed by Koontz.[2] Those models were quite detailed because they aim to predict the economic effect of a particular sequence of network

* A matrix is totally unimodular if and only if each of its square submatrices has a determinant with value 0 or ±1.

changes (rearrangements, conversion to the Serving Area Concept or SAC, etc.) as accurately as possible. The purpose of this allocation model is to choose the allocation for the route which has the lowest total operating cost. Since these cost functions must be evaluated quite a few times during the course of the algorithm below, a simplification of the models developed in Ref. 2 is used. The sum of these operating cost functions serve as the objective function for the mathematical program defined below.

The operating expenses are divided among those incurred in the Multiple Outside Plant (MOP) and those incurred in the SAC areas. To predict the expenses in the MOP portion of the AA when pairs are available, the following exponential approximation model is used:[2]

$$C_{\text{MOP}}(\omega) = (1 - \pi) \int_0^{T_1} \lambda_{IN} \cdot C_{BLK} \cdot F(\omega, t)^{\alpha} e^{-rt}\, dt,$$

where

π = the fraction of the AA which is interfaced,
λ_{IN} = the number of inward movements (arrivals) per year for this AA,
C_{BLK} = the cost of clearing a blockage in this AA,
$F(\omega, t)$ = fill in the AA at time t given that ω pairs are available
= (assigned pairs at time t)/ω,
α = the average serving terminal size in the AA, and
r = the cost of money.

The parameters for this cost model can be obtained from the LATIS (Loop Activity Tracking Information System)[7] program. The parameters are defined so that the initial operating expenses, at least, will correspond closely to the recent history.

The expenses in the portion of the AA served by SAC are the present worth of the expenses due to BCTs (break connect-through) and RTCs (reterminated connections). For definitions of these operations, see Ref. 2.

$$C_{\text{SAC}}(\omega) = \pi_0 \int_0^{T_1} \lambda_{IN}(C_{\text{BCT}} + C_{\text{RTC}}/(1 + r\tau_v)) \cdot F(\omega, t)^{\text{KSAC}} e^{-rt}\, dt,$$

where

C_{BCT} = cost of a BCT
C_{RTC} = cost of an RTC
τ_v = the mean vacancy time for premises in the AA
KSAC = a positive parameter chosen so that $F(\omega, t)^{\text{KSAC}}$ approximates the probability of a BCT.

Since several of the parameters above are indexed by allocation area, we can write the expected present worth of operating costs for AA i as

$$f_i(\omega) = C_{\text{MOP}}(\omega) + C_{\text{SAC}}(\omega).$$

It is possible to factor out the terms involving ω and express the rest of the terms as a constant and write

$$f_i(\omega) = \beta_i \omega^{-\alpha_i} + \gamma_i \omega^{-\text{KSAC}}.$$

Since both α_i and KSAC are positive, it is clear that f_i is a convex decreasing function. This cost function will result in very high operating costs when the maximum number of pairs demanded gets close to ω, the number of pairs allocated, because $F(\omega, t)^{\alpha}$ will be close to 1 implying that nearly every inward order results in a blockage. A high churn area will have a large value for λ_{IN}, the inward movement rate, and, hence, may have high operating costs even if the growth rate is low. Thus a search for the allocation which minimizes the sum of all the operating cost functions, $f_i(\omega)$, results in an equitable allocation of pairs to both growing and nongrowing areas. This feature is important for many urban feeder routes in the Bell System that have little net growth but high operating expenses due to churn.

IV. THE OPTIMIZATION MODEL

In this section, we assume that the interval $[0, T]$ during which no capacity expansions occur is long enough that the planning engineer is not really concerned with the effect that the allocation during the interval has upon the rearrangement costs after T. This is often the case in low growth routes in which the next cable placement is not expected in the next five years or so. The next section will discuss the case in which one or more cable placements are being planned shortly and some pairs should be held in the feeder route so that the planned relief cable can be spliced to the central office without undue expense.

The optimization problem for the single period model is

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{\text{NAA}} f_i(z_i) \\ \text{subject to} \quad & Cx \leqslant t \\ & Dx - z = 0 \\ & \underline{x} \leqslant x. \end{aligned} \tag{7}$$

This problem has a nonlinear, convex objective function and linear constraints. A number of algorithms have been developed that solve problems with this structure; the generalized reduced gradient method of Lasdon et al[8] and the gradient projection algorithm of Rosen[9] are two well-known examples. However, since the objective is also separable (it is a sum of single variable functions), the techniques of separable linear programming can be employed. These techniques convert the nonlinear objective function into a linear one by constructing piecewise linear approximations to the single variable functions. Computational experiments by Collins et al[10] indicate that the separable programming approach is efficient and robust when compared to some of the leading nonlinear programming algorithms. Linear programming codes are also more readily available than nonlinear codes. For these reasons, the separable linear programming approach was selected for these allocation problems.

There are two methods for representing convex functions as piecewise linear functions for the purposes of separable linear programming. The technique of lambda-separable programming (see Dantzig,[11] p. 483) was used in a preliminary version of the allocation algorithm. However, to eliminate NAA equations from the LP (linear programming) model, the method of delta-separable programming can be used (Ref. 10, p. 485). With this technique, the slopes for the linear segments are the cost coefficients, and the variables are the number of pairs which are used at the various constant slope intervals. Figure 3 has an example:

$$f_i(z_i) \doteq d_0 + d_{i1}v_{i1} + d_{i2}v_{i2} + d_{i3}v_{i3} + d_{i4}v_{i4},$$

where

$$z_i = L_i + v_{i1} + v_{i2} + v_{i3} + v_{i4}$$

$$0 \leqslant v_{ij} \leqslant \bar{v}_{ij}, \qquad j = 1, 2, 3, 4.$$

L_i is defined as the lowest possible allocation for AA i; hence, it is equal to $\sum_{j \epsilon E_i} \underline{x}_j$. The upper bounds $\bar{v}_{ij}$, $j = 1, \cdots, 4$ are defined so that $L_i + \sum_{j=1}^{4} \bar{v}_{ij} = U_i$, the upper bound on pairs which can be allocated to AA i. Since the interval of possible allocations $[L_i, U_i]$ must be finite, we set $U_i \equiv 2 \cdot L_i$ because this upper bound should rarely be attained in practice.

The problem of choosing the correct number of intervals to define the piecewise linear approximation is a difficult one. To get a reasonably accurate approximation, one would like to define many intervals, but since this strategy increases both storage and computational requirements, it is not recommended. Instead, a dynamic approach is chosen in which only four intervals of equal widths are chosen initially. But after the optimal solution is found with the coarse approximation, the two middle intervals with length TOL are placed on each side of $\hat{z}_i$, the optimal solution (typical values for TOL are 10 or 25 pairs). The method is specified in more detail below. First, we write the linear program which is to be solved.

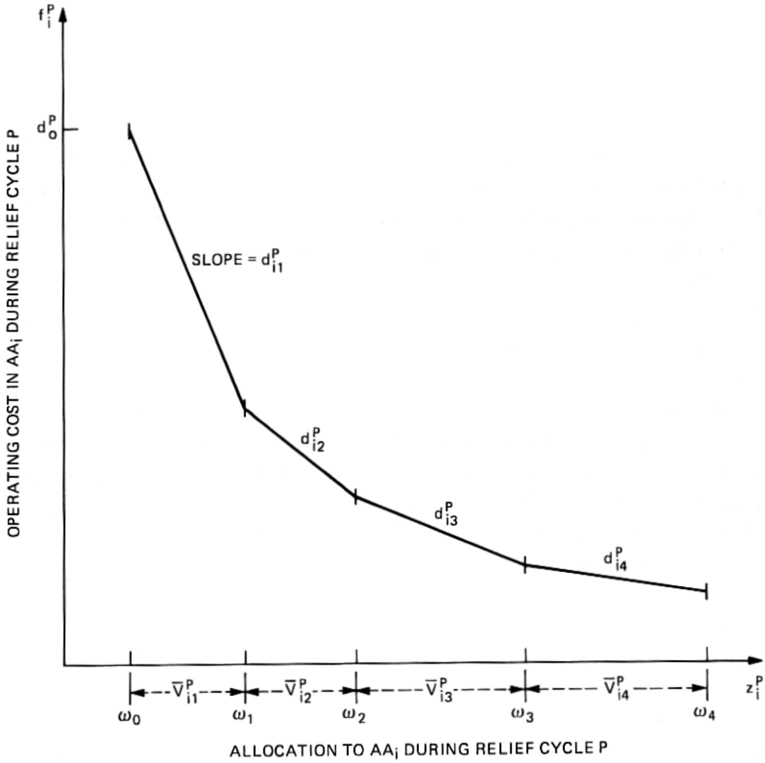

Fig. 3—Example of delta-separable programming.

$$\text{Minimize} \quad \sum_{i=1}^{\text{NELAC}} \sum_{j=1}^{4} d_{ij} v_{ij}$$

$$\text{subject to} \quad Cx \leqslant t$$

$$\sum_{k \epsilon E_i} x_k = L_i + \sum_{j=1}^{4} v_{ij}, \qquad i = 1, \cdots, \text{NAA}$$

$$\underline{x}_i \leqslant x_i, \qquad i = 1, \cdots, \text{NCU}, \qquad 0 \leqslant v_{ij} \leqslant \bar{v}_{ij}, \qquad \begin{matrix} i = 1, \cdots, \text{NAA} \\ j = 1, \cdots, 4. \end{matrix} \tag{8}$$

Note that the variable z_i has been omitted, but its two representations are equated: $\sum_{k \epsilon E_i} x_k$ and $L_i + \sum_{j=1}^{4} v_{ij}$. The constraint $z_i \leqslant U_i$ is maintained because the choice of $\bar{v}_{ij}, j = 1, \cdots, 4$ implies that $U_i \equiv L_i + \sum_{j=1}^{4} \bar{v}_{ij}$. The following alogrithm solves the convex program (7).

Algorithm 1

0. $\omega_0 := L_i$.
   Repeat for $j = 1, 4$:

$$\omega_j := L_i + j/4(U_i - L_i)$$

$$\bar{v}_{ij} := \omega_j - \omega_{j-1}$$

$$d_{ij} := (f_i(\omega_j) - f_i(\omega_{j-1}))/\bar{v}_{ij}.$$

1. Solve the linear program (8).
   Determine the "optimal" AA allocations $\hat{z}_i$, $i = 1, \cdots,$ NAA.
2. Optimality test—Check to see if there is a breakpoint $(\omega_0, \cdots, \omega_4)$ in the interval $[\hat{z}_i - \text{TOL}, \hat{z}_i)$ and the interval $(\hat{z}_i, \hat{z}_i + \text{TOL}]$.
   If this is true for $i = 1, \cdots,$ NAA, STOP.
   Otherwise, for each i such that the check fails, alter the breakpoints ω_1, ω_2, and ω_3 so that

$$\omega_2 := \hat{z}_i$$

$$\omega_1 := \max(\hat{z}_i - \text{TOL}, (\omega_0 + \omega_2)/2)$$

$$\omega_3 := \min(\hat{z}_i + \text{TOL}, (\omega_2 + \omega_4)/2).$$

   Repeat for $j = 1, 4$;

$$\bar{v}_{ij} := \omega_j - \omega_{j-1}$$

$$d_{ij} := (f_i(\omega_j) - f_i(\omega_{j-1}))/\bar{v}_{ij}.$$

   Return to Step 1.

The optimality test is a check to see whether the approximation to f_i is good in the neighborhood of $\hat{z}_i$. If the test fails, the fixup insures that the test will not fail on the next iteration if the new optimal solution $\bar{z}_i$ is in the open interval $(\hat{z}_i - \text{TOL}, \hat{z}_i + \text{TOL})$.

R. R. Meyer[12] provides a proof that this algorithm for solving (7) by dynamically changing the breakpoints for the linear approximations does result in an "optimal" (actually, each value z_i is within $\pm$TOL of being optimal) solution to (7). The time necessary to solve the linear programs following the first iteration should be quite small because the LP-optimal solution $(\hat{x}, \hat{z})$ can be used as a starting point for the next problem.

V. THE DYNAMIC MODEL

The static model described above is quite adequate when the next relief job is far away or the next relief job reinforces the central office section. But when the next cable placement needs to be connected to the central office by pairs in intervening sections, a judgment must be made as to whether any spare pairs in those sections should be held in the feeder or spliced to an allocation area to reduce operating expenses. Figure 4 depicts the situation under consideration. Should the unallocated pairs in section 1101 be committed to AA 1101 now and cut and

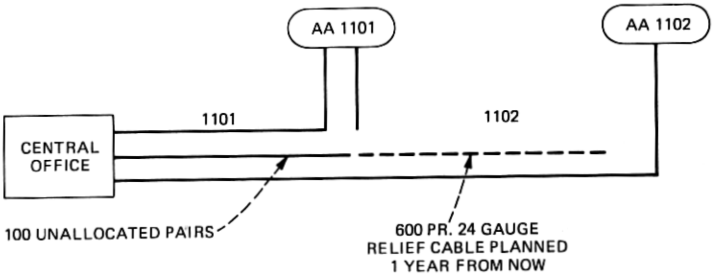

Fig. 4—Should the unallocated pairs be committed to AA 1101 now and cut and spliced to the relief cable one year from now?

spliced to the relief cable a year from now? In general, the answer is "No."

Although the primary objective is to reduce operating expenses and to defer relief, this must not be done at the expense of generating too many recommitments. In order to consider the effects that recommitments performed before the first facility placement have upon later recommitments, the study period is extended to the time of the fourth facility placement or the first facility placement which reinforces the central office section, whichever happens first. The allocation module assumes that rearrangements (changes in the AA allocations) are done at the current time and the time of each facility placement. To reduce the number of planned recommitments, we continue to make the assumption, implicit in the load balance worksheet of Ref. 10, that no commitments will be made to an AA which are planned for another AA at a later date. Deviations from the current forecast may require recommitments later, but the above assumption should reduce the number of recommitments. This assumption implies that the allocations to AAs will be monotonically nondecreasing over time and, hence, sufficient pairs will remain unallocated in the sections on the central office side of future facility placements.

Thus, the model will be concerned with $M(\leq 3)$ facility placements at times $T_1, \cdots, T_M$ which separate $M + 1$ time intervals called relief cycles. The end of relief cycle $M + 1$, T_{M+1} is the time of facility exhaust following the placement at time T_M. Since we must compute an allocation for each relief cycle, let z^p be the allocation to the AAs during relief cycle p, $p = 1, \cdots, M + 1$. Similarly, x^p, $\underline{x}^p$, L_i^p, U_i^p, d_{ij}^p, v_{ij}^p, and $\bar{v}_{ij}^p$ are defined for relief cycle p. The vector t^1 is the initial facilities available in the feeder route. The user inputs subsequent facility additions so that one can construct the capacity vectors $t^2, \cdots, t^{M+1}$ to reflect those additions.

To represent the assumption that the allocations to AAs are nondecreasing over time, the inequalities

$$z^p \leq z^{p+1}, \qquad p = 1, \cdots, M \tag{9}$$

are added to the model. After the solution of the dynamic model described below, certain differences $z_i^{p+1} - z_i^p$ are accumulated to calculate the number of pairs which should remain unallocated. This calculation is described in the following section.

The dynamic model is just a repetition of the constraints (7) for each relief cycle with the appropriate parameters for each relief cycle. Only the inequalities (9) link one relief cycle to the next.

The dynamic model is

$$\text{Minimize} \quad \sum_{p=1}^{M+1} \sum_{i=1}^{\text{NAA}} f_i^p(z_i^p)$$

$$\begin{aligned} \text{subject to} \quad & Cx^p \leqslant t^p, \\ & \underline{x}^p \leqslant x^p, \\ & Dx^p = z^p, \qquad p = 1, \cdots, M+1, \quad \text{and} \\ & z^p \leqslant z^{p+1}, \qquad p = 1, \cdots, M. \end{aligned} \tag{10}$$

In considering algorithms to solve such a problem, the maximum size problem which can be expected must be estimated. Experience with EFRAP indicates that the following should be the maximum dimensions that the algorithm will need to solve:

Number of sections = 200.
Number of allocation areas = 50.
Number of computational units = 600.
Number of relief cycles = 4.
Number of section constraints = 500.

Thus the number of variables in the problem could be $4 \cdot 600 + 4 \cdot 50 = 2600$ and the number of constraints could be $4(500 + 50) + 3 \cdot 50 = 2350$. Lower and upper bounds on variables are not counted as constraints because they do not add to computation time. Such a problem could be solved by mathematical programming techniques, but it would be quite expensive. The problem's constraint matrix has a staircase structure (see Fig. 5) which is typical of dynamic planning problems. Specialized linear programming techniques have been developed to handle these problems,[13, 14] but the time and storage to solve such a problem would be unacceptable for an engineering planning program. For this reason, an algorithm is presented which results in a feasible, but not a necessarily optimal, solution to problem (10). The algorithm decomposes the problem into $M + 1$ smaller problems to be solved rather than one large one. For this algorithm to work, we assume that $L^p \leqslant L^{p+1}$. In effect, we are assuming that there is no negative growth in an allocation area.

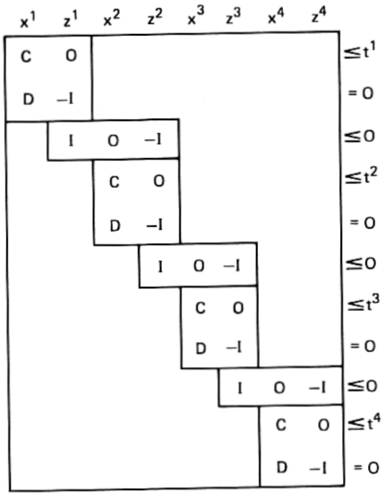

Fig. 5—The staircase mathematical programming structure exhibited by the dynamic allocation model.

5.1 Algorithm 2

0. Set $p = M + 1$, let U^{M+1} be an upper bound vector for z^{M+1}, $U^{M+1} := 2 \cdot L^{M+1}$.
1. Solve

$$\text{Minimize} \quad \sum_{i=1}^{\text{NAA}} f_i^p(z_i^p)$$

$$\text{subject to} \quad Cx^p \leqslant t^p$$

$$Dx^p = z^p$$

$$\underline{x}^p \leqslant x^p, \qquad L^p \leqslant z^p \leqslant U^p,$$

   using Algorithm 1.
   Let the resulting optimal solution be $(\hat{x}^p, \hat{z}^p)$.
2. If $p = 1$, STOP.
   Otherwise, let $U^{p-1} := \hat{z}^p$ and $p := p - 1$.
   Return to Step 1.

This algorithm solves the $M + 1$ subproblems beginning with the one farthest away in time. The only interaction between subproblems is the fact that $\hat{z}^{p+1}$, the optimal ELAC allocation for problem $p + 1$ becomes an upper bound for z^p on the next iteration. This algorithm does not necessarily provide an optimal solution to problem (10), but we have reason to believe that it is nearly optimal. For AAs i such that $\hat{z}_i^{p+1} = \hat{z}_i^p$, i.e., z_i^p achieves its upper bound, it may be economic to increase the allocation to both $\hat{z}_i^{p+1}$ and $\hat{z}_i^p$. This is true because when z_i^{p+1} was being set at the optimal level for relief cycle $p + 1$, no recognition was made that z_i^{p+1} was also influencing the allocation during relief cycle p (and perhaps earlier relief cycles as well). The reason that the decomposition of the problem in Algorithm 2 probably does not hurt the objective function too much is that, if $\hat{z}_i^{p+1}$ is an optimal allocation during relief cycle $p + 1$, it is probably a good allocation for cycles $1, \cdots, p$ as well. This is true because the assumption that the AA growth rates are nonnegative implies that the fill during these cycles will be less than or equal to the fill during relief cycle $p + 1$.

Algorithm 2 begins by solving the subproblems in reverse order, starting with relief cycle $M + 1$. An intuitive reason for this procedure is that feeder administration attempts to transform the feeder network to its future configuration as economically as possible. A first step in that process is to estimate the future allocations so that a minimal number of recommitments need be performed to attain the ultimate configuration.

The practical reason for solving the subproblems in reverse chronological order is that it works that way. It is easy to show that, if sufficient capacity has been added so that each subproblem is feasible when solved without upper bounds on z_i^p, then the subproblems to be solved during Step 1 of Algorithm 2 are also feasible.

Suppose the most obvious algorithm for solving the subproblems in their natural order were constructed. This algorithm must maintain the constraints $z^p \leq z^{p+1}$, $p = 1, \cdots, M$, and $L^p \leq z^p$, $p = 1, \cdots, M + 1$. The following algorithm satisfies those requirements.

5.2 Algorithm 2′

0′. Let $p := 1$ and $U^1 := 2 \cdot L^1$.
1′. Same as in Algorithm 2.
2′. If $p = M + 1$, STOP.
Otherwise, let $L_i^{p+1} := \max(\hat{z}_i^p, L_i^{p+1})$, $i = 1, \cdots,$ NAA, $U^{p+1} := 2 \cdot L^{p+1}$ and $p := p + 1$.
Return to Step 1.

The following simple problem shows that Algorithm 2′ can result in infeasible subproblems even when the original subproblems are feasible. In this example, AAs are identical to CUs.

$$\text{Subproblem 1: Minimize} \quad (z_1^1)^{-1} + (z_2^1)^{-1}$$
$$\text{subject to} \quad z_1^1 + z_2^1 \leq 500$$
$$z_1^1 \leq 150$$
$$L_1^1 = 200 \leq z_1^1$$
$$L_2^1 = 100 \leq z_2^1.$$

$$\text{Subproblem 2: Minimize} \quad (z_1^2)^{-1} + (z_2^2)^{-1}$$

$$\text{subject to} \quad z_1^2 + z_2^2 \leq 500$$

$$z_2^2 \leq 350$$

$$L_1^2 = 275 \leq z_1^2$$

$$L_2^2 = 175 \leq z_2^2.$$

Following Algorithm 2′, the optimal solution to Subproblem 1 is $\hat{z}_1^1 = 350$ and $\hat{z}_2^1 = 150$. The new lower bounds for Subproblem 2 are

$$L_1^2 := \max(\hat{z}_1^1, L_1^2) = \max(350, 275) = 350$$

$$L_2^2 := \max(\hat{z}_2^1, L_2^2) = \max(150, 175) = 175.$$

The resulting subproblem to be solved in Step 1 is infeasible because the inequalities

$$z_1^2 + z_2^2 \leq 500$$

$$z_1^2 \geq 350$$

$$z_2^2 \geq 175$$

have no solution.

VI. CALCULATION OF PAIRS ALLOCATED TO SECTIONS

After the allocations have been determined for relief cycles 1, $\cdots$, $M + 1$, it is quite easy to calculate the number of pairs that should be allocated to sections on the central office side of the sections which are receiving additional facilities. Suppose we are considering the pairs to be held for facility placement $p \leq M$. First, the routing of demand from CUs to the CO determines which section is on the CO side of a particular relief job. Call this the reserve section. Then each computational unit which is fed by the reserve section is examined to determine the number and type of additional pairs it receives as a result of the facility placement. By "type of pairs" we mean the gauge and break section and path back to the CO required of a particular CU. The break section is the section between the reserve section and the CO in which a two-gauge loop changes from the coarser to the finer gauge wire; the fine gauge portion is usually closer to the CO. The total number of pairs by "type" which are held unallocated in the reserve section is calculated and will be displayed on an output report of the feeder administration program. Figures 6 and 7 depict an example of this process of analyzing the pairs to be allocated to sections.

The example route has nine sections and two paths. Relief cable is to be placed in sections 1205, 1208, 1209. The route must be examined to determine what additional allocations were made possible by the

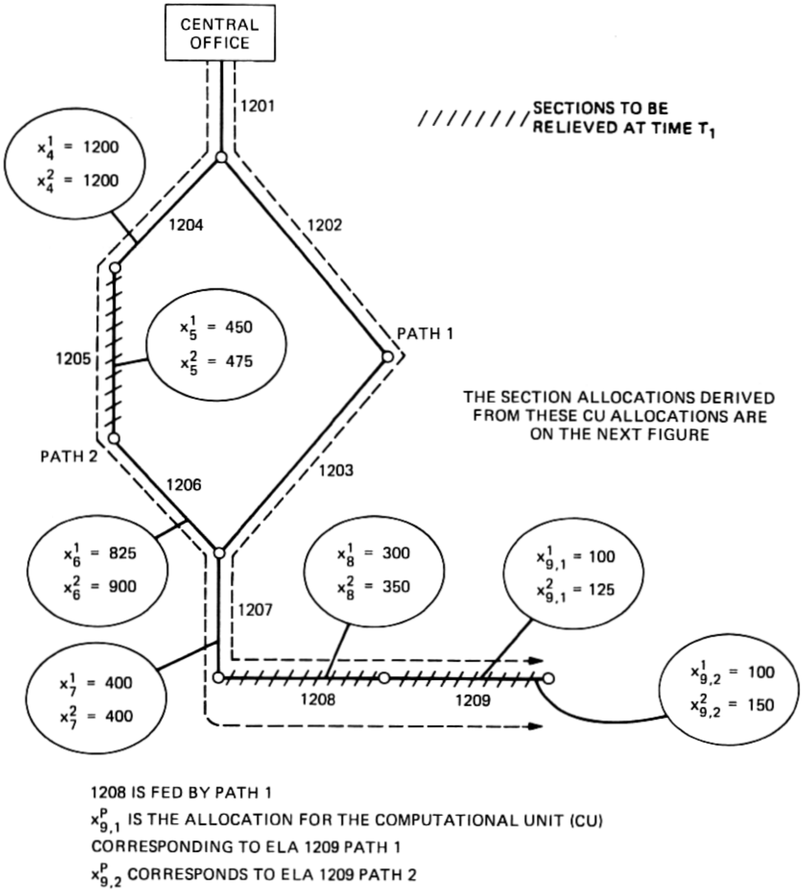

Fig. 6—Example of analyzing pairs to be allocated to sections.

capacity expansions. For example, the relief job in 1205 allows the allocation to ELA 1205 to increase by 25 pairs: $x_5^2 - x_5^1 = 475 - 450$. Since section 1205 is in path 2, only increased allocations to computational units which are fed by path 2 are added to the section allocation for 1204. Thus, the increase of 50 pairs for the portion of ELA 1209 fed by path 2 is credited to 1204 while the increase of 25 pairs for ELA 1209 path 1 is not. Figure 7 shows the results of the calculation.

VII. CONCLUSION

An algorithm for computing the allocation of pair groups to allocation areas for the reduction of feeder route operating expenses has been described. The route model is quite general. It allows two-gauge

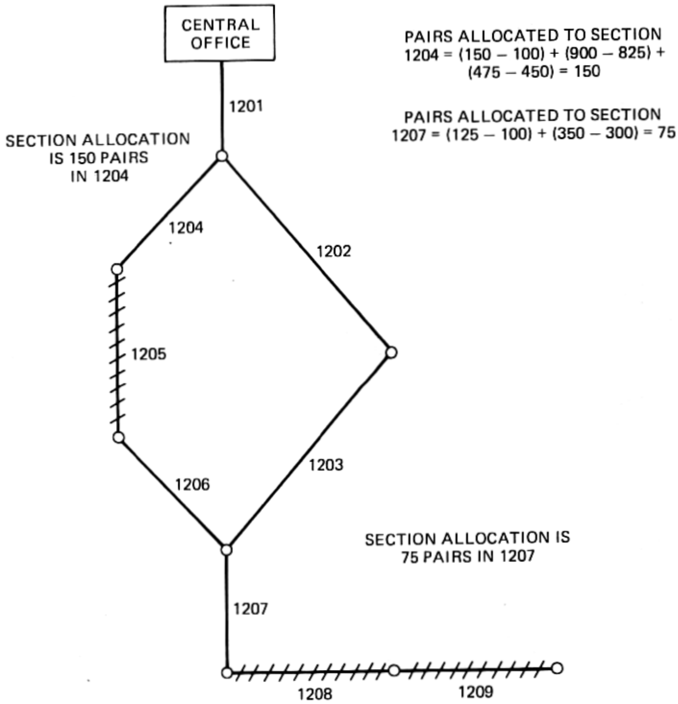

Fig. 7—Pairs reserved in the section(s) on the central office side of the cable placement to energize the relief pairs.

resistance design with overgauging. Since the model allows feeder routes with multiple paths, routes with pair gain systems can be modeled by creating an alternate path for each remote terminal site.

The driving force of the allocation algorithm is to minimize operating expenses while reserving enough pairs in certain sections to feed future relief jobs. Thus, a heuristic algorithm which uses a linear programming code as a subroutine is designed for solving a multi-time-period nonlinear optimization problem. This promises to be a versatile and efficient algorithm for solving the feeder route allocation problem.

VII. ACKNOWLEDGMENTS

This work has derived much from the concepts developed in earlier work on feeder allocation, particularly by W. N. Bell and B. L. Marsh. I thank W. L. G. Koontz for suggesting the form of the section constraints. Discussions with D. B. Luber were helpful in the formulation of other parts of the model.

APPENDIX A

The feeder route allocation problem is described below with the aid of a simple example. The geographic area which a feeder route serves is divided into allocation areas, or AAS (the number of AAS is NAA), whose number of pairs allocated, z_i, determine the operating cost, $f_i(z_i)$. The symbol f_i is a convex decreasing function of the form $b_i z_i^{-a_i}$. The objective function for the example depicted in Fig. 8 is

$$\text{Minimize} \quad \sum_{i=1}^{4} f_i(z_i). \tag{11}$$

Feeder sections are linear segments of the cable route with uniform numbers of wire pairs along their length. They are the edges between nodes in Fig. 8. The capacity of these sections is the scarce commodity in this problem. From the objective function (11), it is clear that the larger the allocation to each AA, the lower the objective function value will be. However, the allocation of pairs (z_1, z_2, z_3, z_4) must not use more pairs than are available in the feeder sections.

An allocation area is fed by pairs which may not all pass through the same number of sections or follow the same path back to the central office. Thus we subdivide the AA into a smaller unit of demand, the computational unit. A computation unit, CU, is an area served by

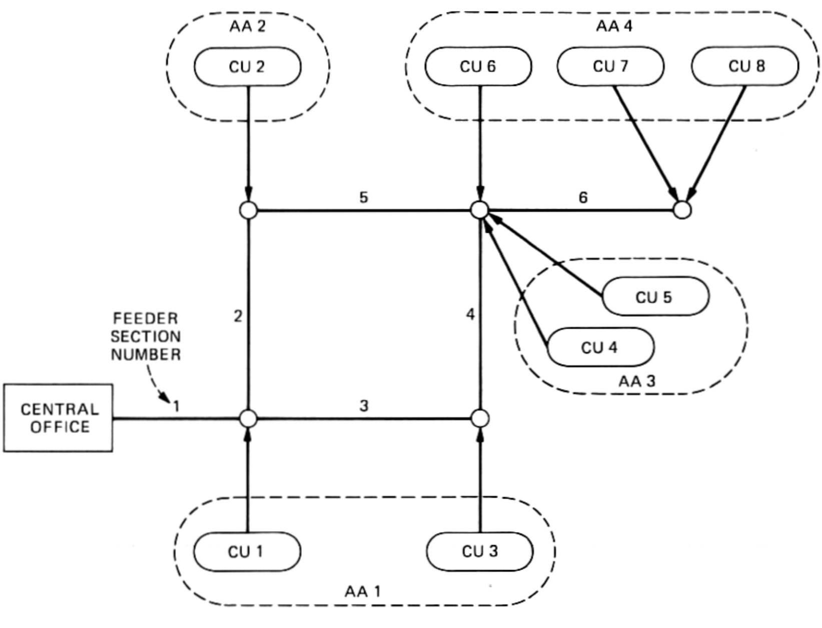

Fig. 8—Example of feeder route allocation problem.

wire pairs which follow a unique path back to the central office. Call the allocation to the NCU computational units $x_1, \cdots, x_8$. From the figure, the following relationships are derived:

$$
\begin{aligned}
z_1 &= x_1 + x_3 \\
z_2 &= x_2 \\
z_3 &= x_4 + x_5 \\
z_4 &= x_6 + x_7 + x_8.
\end{aligned}
\tag{12}
$$

Forecasts of expected growth are subdivided to the CU level. The allocations derived from this model must satisfy the growth forecast over the interval $[0, T]$ where T is the time of the next feeder capacity expansion. Let $\underline{x}_i$ be the maximum expected demand in CU i during $[0, T]$. Then the lower bound constraints

$$\underline{x}_i \leqslant x_i, \qquad i = 1, \ldots, 8 \tag{13}$$

will insure that each CU is allocated enough pairs to meet demand.

The sum of all the CUs whose pairs pass through a particular section j must not exceed the capacity of the section s_j. Thus, the following section constraints are imposed.

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 &\leqslant s_1 \\
x_2 \qquad\qquad + x_5 + x_6 + x_7 \qquad &\leqslant s_2 \\
x_3 + x_4 \qquad\qquad\qquad + x_8 &\leqslant s_3 \\
x_4 \qquad\qquad\qquad + x_8 &\leqslant s_4 \\
x_5 + x_6 + x_7 \qquad &\leqslant s_5 \\
x_7 + x_8 &\leqslant s_6.
\end{aligned}
\tag{14}
$$

The routing of the pairs serving a particular CU can be derived by examining the constraints (14). The pairs serving CU 5, for example, pass through sections 5, 2, and 1 because x_5 appears in the constraints for sections 5, 2, and 1.

The objective function (11) and the constraints (12), (13), and (14) define the mathematical program whose solution is discussed in Section IV.

APPENDIX B

An example is presented to show that the constraint matrix for the feeder route model presented in this paper need not be totally unimodular. This example route has only five sections and is concerned with three demand points or computational units (see Fig. 9).

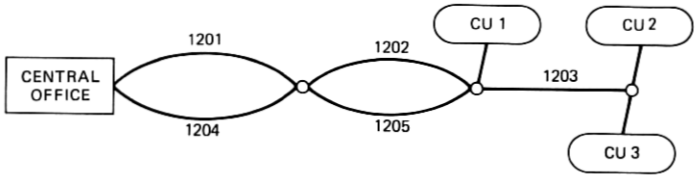

Fig. 9—Feeder route model.

The routing of demand for the computational units is

CU 1: Sections 1202, 1201
CU 2: Sections 1203, 1202, 1204
CU 3: Sections 1203, 1205, 1201.

The gauge requirement for each CU is 26 gauge, so no multiple-gauge considerations are relevant.

The constraints for sections 1201, 1203, and 1205 are

Section	x_1	x_2	x_3	
1201	1	0	1	$\leqslant t_1$
1202	1	1	0	$\leqslant t_2$
1203	0	1	1	$\leqslant t_3$
1204	0	1	0	$\leqslant t_4$
1205	0	0	1	$\leqslant t_5$.

The square submatrix created by ignoring the last two rows has a determinant of 2. Thus, the constraint matrix is not totally unimodular. It is clear from this example that the fixed routing of demand prevents the use of network flow algorithms, with their efficiencies of speed and storage, for the solution of the allocation subproblems.

REFERENCES

1. B. L. Marsh, "The Feeder Allocation Process," B.S.T.J., *57*, No. 4 (April 1978), pp. 869–890.
2. W. L. G. Koontz, "An Approach to Modeling Operating Costs in the Loop Network," B.S.T.J., *57*, No. 4 (April 1978), pp. 891–910.
3. W. L. G. Koontz, "Economic Evaluation of Loop Feeder Relief Alternatives," B.S.T.J., *59*, No. 3 (March 1980), pp. 277–293.
4. J. Friedenfelds and C. D. McLaughlin, "A Heuristic Branch-and-Bound Algorithm for Telephone Feeder Capacity Expansion," Operations Research, *27* (1979), pp. 567–582.
5. N. G. Long, "Loop Plant Modeling: Overview," B.S.T.J., *57*, No. 4 (April 1978), pp. 797–806.
6. K. Truemper, "Algebraic Characterizations of Totally Unimodular Matrices," Working Paper, University of Texas of Dallas, 1975.
7. G. W. Aughenbaugh and N. H. Noe, "Mechanized Activity Tracking and Fill Monitoring in the Loop Plant," International Symposium on Subscriber Loops and Services, Atlanta, Georgia, 1978.
8. L. S. Lasdon, R. L. Fox and M. W. Ratner, "Nonlinear Optimization Using the Generalized Reduced Gradient Method," R.A.I.R.O., *3* (1974), pp. 73–104.

9. J. B. Rosen, "The Gradient Projection Method for Nonlinear Programming: Part I—Linear Constraints," SIAM J. Appl. Math., *8* (1960), pp. 181–217.
10. M. Collins, L. Cooper, R. Helgason, J. Kennington, and L. LeBlanc, "Solving the Pipe Network Analysis Problem Using Optimization Techniques," Management Science, *24* (March 1978), pp. 747–760.
11. G. B. Dantzig, *Linear Programming and Extensions*, Princeton: Princeton University Press, 1963.
12. R. R. Meyer, "A Class of Nonlinear Integer Programs Solvable by a Single Linear Program," SIAM J. Control and Optimization, *15*, pp. 935–946.
13. J. K. Ho and A. S. Manne, "Nested Decomposition for Dynamic Models," Mathematical Programming, *6* (1974), pp. 121–140.
14. A. F. Perold and G. B. Dantzig, "A Basis Factorization Method for Block Triangular Linear Programs," Technical Report SOL 78-7, Systems Optimization Laboratory, Stanford University, April 1978.