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On a Source-Coding Problem with Two Channels and Three Receivers

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This paper treats the problem of communicating a memoryless unit-variance Gaussian source to three receivers. Two channels are available, each with a separate receiver. A third receiver has the outputs of both channels available. We obtain an expression for the simultaneously achievable distortions (mean-squared error). This problem applies to the following situation: Assume that high-quality reproduction of a source is desired at a single receiver which is connected to the source over a pair of links operating in parallel. Further assume that the links are unreliable in that either may fail, and that the source encoder is unaware of the failures. One can then ask how "robust" a system designed for this situation can be. That is, what are the limits on the fidelity achievable when both links are functioning if graceful degradation is required during the failure of either link? An inverse relation between performance in the two modes is obtained in the sense that, as performance in the presence of both links approaches its theoretical optimum, average distortion during failures becomes large. Conversely, if near-ideal performance during link failures is desired, then the distortion achieved when both links operate is far from its optimum value.

I. INTRODUCTION

Consider the following communication problem: An encoder is presented with a sequence of source letters $\{X_k\}$ drawn from alphabet $\mathscr{X}$. We assume the $\{X_k\}$ are independent and identically distributed with probability mass function $p(x)$ (or a probability density function if $\mathscr{X}$ is continuous). For each block of N letters (N arbitrary), two discrete encoder outputs $f_1(\mathbf{X})$ and $f_2(\mathbf{X})$ are produced ($\mathbf{X}$ is a vector of N letters). The cardinalities of f_1 and f_2 are limited by

$$\frac{1}{N}\log \| f_1(\mathbf{X}) \| \le R_i, \qquad i = 1, 2,$$

where the base of the log is arbitrary, but taken to be e in the sequel. Then R_i is the maximum rate at which information can be conveyed over the ith channel, in nats per source letter.

We assume the existence of three receivers which must estimate $\mathbf{X}$ using f_1 alone, f_2 alone, and both f_1 and f_2. The three estimates, denoted by $\hat{\mathbf{X}}_1$, $\hat{\mathbf{X}}_2$, and $\hat{\mathbf{X}}_3$ are N vectors in some reproducing alphabets ($\hat{\mathscr{X}}_1$, $\hat{\mathscr{X}}_2$, $\hat{\mathscr{X}}_3$) which in general may not coincide with each other or with $\mathscr{X}$. Distortions d_1, d_2, and d_3 are incurred at the respective receivers according to

$$d_i = \frac{1}{N} \sum_{k=1}^{N} E[\delta_i(X_k, \hat{X}_{ik})], \qquad i = 1, 2, 3,$$

where $\delta_i(\cdot,\cdot)$ is a nonnegative real-valued function defined on $\mathscr{X}$ and $\hat{\mathscr{X}}_i$. This configuration is summarized in Fig. 1.

The case of only one receiver is the classical rate-distortion problem.[1] Corresponding to the source statistics and the distortion measure, the rate-distortion function is defined by

$$R(d) = \inf_{P_d} I(X; \hat{X}),$$

where $P_d = \{p(\hat{x} \mid x): E[\delta(X, \hat{X})] \leq d\}$ and $I(X; \hat{X})$ is the mutual information between X and $\hat{X}$. A forward coding theorem and its converse exist to the effect that for any d (for which P_d is nonempty) and any $\epsilon > 0$ there is a block length N and a code with at most $e^{R(d)N}$ words such that $\hat{\mathbf{X}}$ (an estimate of $\mathbf{X}$ determined by the encoder output) satisfies

$$\frac{1}{N} \sum_{1}^{N} E[\delta(X_k, \hat{X}_k)] < d + \epsilon.$$

Conversely, for any code with rate less than $R(d)$, the distortion can be no smaller than d. An alternate way of stating this last fact is that if $I(\mathbf{X}; \hat{\mathbf{X}}) \leq NR$, then

$$\frac{1}{N} \sum_{k=1}^{N} E[\delta(X_k, \hat{X}_k)] \geq d^*,$$

where $R(d^*) = R$.

A natural problem for the network of Fig. 1 is to characterize the set

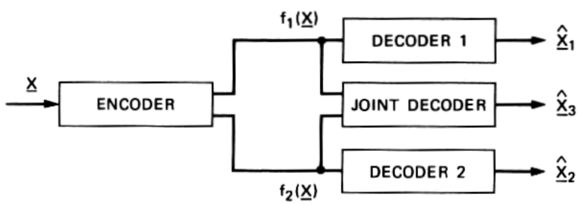

Fig. 1—The channel-splitting problem.

of achievable quintuples $(R_1, R_2, d_1, d_2, d_3)$. Although this problem is as yet unsolved for arbitrary sources and distortion, we have obtained the solution for one important special case—that of a Gaussian source with squared error distortion. In this case, the source and reproduction alphabets are the real line, and

$$\delta_i(x, \hat{x}) = (x - \hat{x})^2, \qquad i = 1, 2, 3.$$

The rate-distortion function for this source and distortion measure is given by Ref. 1, Theorem 4.3.2:

$$R(d) = \frac{1}{2} \log \frac{\sigma_x^2}{d} \quad \text{nats/source letter,} \tag{1}$$

where σ_x^2 is the variance of X, here assumed to be 1.

As noted above, $R(d)$ gives the minimum mutual information per letter required to reproduce souce X with average distortion d. $R(d)$ may be inverted to yield the distortion-rate function [i.e., the solution to $R(d^*) = R$] given by

$$D(R) = e^{-2R}, \tag{1a}$$

which, from the converse to the rate-distortion theorem stated above, is the minimum average distortion achievable in representing N vector $\mathbf{X}$ by $\hat{\mathbf{X}}$, when the average mutual information between vectors $\mathbf{X}$ and $\hat{\mathbf{X}}$ is less than or equal to NR.

To obtain one obvious outer bound to the set of simultaneously achievable $(R_1, R_2, d_1, d_2, d_3)$, observe that estimate $\hat{\mathbf{X}}_i$ is a function of $f_i(\mathbf{X})$ for $i = 1, 2$, and that $\hat{\mathbf{X}}_3$ is a function of $(f_1(\mathbf{X}), f_2(\mathbf{X}))$. Using the data-processing theorem,[1] we get

$$\begin{aligned} I(\mathbf{X}; \hat{\mathbf{X}}_1) &\le I(\mathbf{X}; f_1(\mathbf{X})) \\ &\le H(f_1(\mathbf{X})) \\ &\le NR_1. \end{aligned}$$

Similarly,

$$I(\mathbf{X}; \hat{\mathbf{X}}_2) \le NR_2$$

and

$$I(\mathbf{X}; \hat{\mathbf{X}}_3) \le N(R_1 + R_2).$$

Using (1a) then,

$$\begin{aligned} d_1 &\ge D(R_1) = \exp(-2R_1), \\ d_2 &\ge \exp(-2R_2), \\ d_3 &\ge \exp[-2(R_1 + R_2)]. \end{aligned} \tag{2}$$

In a single-destination problem, the forward part of the rate-distortion theorem implies that the distortion-rate function may be approached arbitrarily closely for sufficiently large block lengths. If the same result applied here, then the inequalities in (2) could be replaced by approximate equalities. In particular, $d_3 = d_1 d_2$ as d_1 and d_2 approach their appropriate lower limits. We will show that this performance is not achievable. The actual set of achievable points is characterized by the following:

Theorem 1: The achievable set of quintuples $(R_1, R_2, d_1, d_2, d_3)$ for Fig. 1 is given by the set of points satisfying

$$\begin{aligned} d_1 &\geq \exp(-2R_1) \\ d_2 &\geq \exp(-2R_2), \\ d_3 &\geq \exp[-2(R_1+R_2)]\,\frac{1}{1-(\sqrt{\Pi}-\sqrt{\Delta})^2}, \end{aligned} \tag{3}$$

where $\Pi = (1-d_1)(1-d_2)$ and $\Delta = d_1 d_2 - \exp[-2(R_1+R_2)]$.

Two simple examples will clarify the behavior of the region specified in the theorem. In the first example, set $R_1 = R_2 = R$ and assume that $d_1 = d_2 = d = e^{-2R}$. That is, the distortion obtained over each side channel is essentially on the appropriate rate-distortion curve. In this case, $\exp[-2(R_1+R_2)] \cong d_1 d_2 = d^2$, $\Delta \cong 0$ and the last inequality in (3) becomes

$$d_3 \geq d^2 \frac{1}{1-(1-d)^2} = \frac{d^2}{2d-d^2} = \frac{d}{2-d},$$

so that the achievable distortion over the joint channel is no better than half the distortion on the side channels. For any interesting (i.e., small) value of d, this is far worse than the value $d_3 \geq d^2$ obtained in (1).

At the opposite extreme, assume that $d_3 \cong \exp[-2(R_1+R_2)]$. That is, the encoder is designed to provide as good a performance as possible for the joint estimate. From the last inequality in (3), then

$$1-(\sqrt{\Pi}-\sqrt{\Delta})^2 \cong 1,$$

which implies that

$$\begin{aligned} \Pi &\cong \Delta, \\ (1-d_1)(1-d_2) &\cong d_1 d_2 - \exp[-2(R_1+R_2)], \\ 1-d_1-d_2+d_1 d_2 &\cong d_1 d_2 - \exp[-2(R_1+R_2)]. \end{aligned}$$

Therefore,

$$d_1 + d_2 \cong 1 + \exp[-2(R_1+R_2)].$$

Note that the value $d = 1$ can be obtained with no information, merely by always estimating X_k by its mean. In this example, if either d_1 or d_2 is small (not even necessarily near its rate-distortion bound), then the other side-channel distortion must be near 1. In other words, the latter estimate is virtually useless by itself.

To account for these properties intuitively, note that the encodings which lead to $f_1(\mathbf{X})$ and $f_2(\mathbf{X})$ describe partitions of R^N, which we denote by $\{A_m\}_{m=1}^{\exp(NR_1)}$ and $\{B_n\}_{n=1}^{\exp(NR_2)}$, so that knowledge of f_1 or f_2 specifies whether $\mathbf{X}$ falls in A_m or B_n, and knowledge of both f_1 and f_2 specifies where $\mathbf{X}$ falls, in $C_{mn} = A_m \cap B_n$. The distortion achieved is then the moment of inertia of the corresponding set around its centroid.

If d_1 and d_2 are both small, then, on the average, the $\{A_m\}$ and $\{B_n\}$ are highly concentrated around their centroids. Therefore, each A_m can intersect only a few B_n, and the moment of inertia of the average C_{mn} can only be smaller than that of A_m by a moderate fraction. Conversely, if d_3 is close to $\exp[-2(R_1 + R_2)]$, then the joint entropy of f_1 and f_2 must be close to $N(R_1 + R_2)$, which is no smaller than the sum of the individual entropies. Therefore, f_1 and f_2 must be nearly independent. If this is true, then knowledge of A_m must yield very little information about which B_n $\mathbf{X}$ is in. In other words, the average A_m must intersect essentially all the B_n. Therefore, if d_2 is small, implying that the B_n are concentrated, then since they must be distributed to cover R^N, each A_m must have content throughout R^N, and its moment is large.

We now prove the theorem. In the proof that follows, we use the converse to the source-coding theorem cited above, the notation $h(\mathbf{Z})$ to denote the differential entropy of a continuous random vector $\mathbf{Z}$ (Ref. 1, p. 86), and the following two lemmas:

Lemma 1: If a continuous random vector $\mathbf{Z}$ with N components has covariance matrix $\mathbf{\Phi}$, then the differential entropy $h(\mathbf{Z})$ satisfies

$$h(\mathbf{Z}) \le \frac{N}{2} \log 2\pi e |\mathbf{\Phi}|^{1/N} \triangleq Ng(|\mathbf{\Phi}|^{1/N}), \tag{4}$$

where $|\mathbf{\Phi}|$ is the determinant of $\mathbf{\Phi}$. Furthermore, (4) *is satisfied with equality if $\mathbf{Z}$ is Gaussian. In particular, if $N = 1$, then*

$$h(\mathbf{Z}) \le g(\sigma_z^2),$$

where σ_z^2 is the variance of $\mathbf{Z}$. This lemma is proved in Ref. 1 (Theorem 4.5.1).

Lemma 2: Let $W \to \mathbf{X} \to \mathbf{Y}$ be a Markov chain, where

$$\mathbf{Y}_k = \mathbf{X}_k + \mathbf{Z}_k, \qquad k = 1, N,$$

and $\{Z_k\}$ *are independent of* W. *Then*

$$\exp\left(\frac{2}{N}h(\mathbf{Y}\mid W)\right) \geq \exp\left(\frac{2}{N}h(\mathbf{X}\mid W)\right) + \exp\left(\frac{2}{N}h(\mathbf{Z})\right). \tag{5}$$

In particular, if the $\{Z_k\}$ are independent and identically distributed (IID) Gaussian with variance σ^2, then

$$\exp\left(\frac{2}{N}h(\mathbf{Y}\mid W)\right) \geq \exp(\left(\frac{2}{N}h(\mathbf{X}\mid W)\right) + 2\pi e\sigma^2.$$

To prove Lemma 2, we note that the unconditional form is due to Blachman.[2] Inequality (5) then holds pointwise on W. Taking logs,

$$\frac{2}{N}h(\mathbf{Y}\mid W=w) \geq \log\left[\exp\left(\frac{2}{N}h(\mathbf{X}\mid W=w)\right) + \exp\left(\frac{2}{N}h(\mathbf{Z})\right)\right].$$

The function $\log(e^x + k)$ is convex in x, so we can average both sides over W and preserve the direction of the inequality using Jensen's inequality. Exponentiating yields Lemma 2.

II. CONVERSE PART OF THEOREM 1

The mutual information between source block $\mathbf{X}$ and the joint-channel estimate of $\mathbf{X}$, denoted by $\hat{\mathbf{X}}_3$, satisfies the following inequality:

$$\begin{aligned}
I(\mathbf{X}; \hat{\mathbf{X}}_3) &\overset{(a)}{\leq} I(\mathbf{X}; f_1(\mathbf{X}) f_2(\mathbf{X})) \\
&\leq H(f_1(\mathbf{X}), f_2(\mathbf{X})) \\
&= H(f_1(\mathbf{X})) + H(f_2(\mathbf{X})) - I(f_1(\mathbf{X}); f_2(\mathbf{X})) \\
&\overset{(b)}{=} I(\mathbf{X}; f_1(\mathbf{X})) + I(\mathbf{X}; f_2(\mathbf{X})) - I(f_1(\mathbf{X}); f_2(\mathbf{X})) \\
&\overset{(c)}{\leq} N(R_1 + R_2) - I(f_1(\mathbf{X}); f_2(\mathbf{X})) \\
&\overset{(a)}{\leq} N(R_1 + R_2) - I(\hat{\mathbf{X}}_1; \hat{\mathbf{X}}_2),
\end{aligned} \tag{6}$$

where the steps labeled (a) follow from the data-processing theorem, (b) from the fact that f_1 and f_2 are determined by $\mathbf{X}$, and (c) from the channel constraints.

By the converse to the source-coding theorem,

$$d_3 \geq D\left(\frac{1}{N}I(\mathbf{X}; \hat{\mathbf{X}}_3)\right).$$

Using eq. (1a) for $D(R)$, we have

$$\begin{aligned}
d_3 &\geq \exp\left(-\frac{2}{N}I(\mathbf{X}; \hat{\mathbf{X}}_3)\right) \\
&\geq \exp[-2(R_1 + R_2)]\exp\left(\frac{2}{N}I(\hat{\mathbf{X}}_1; \hat{\mathbf{X}}_2)\right),
\end{aligned} \tag{7}$$

where the second inequality follows from (6).

We need now to lower-bound the second exponent in (7). To do this, we define an artificial random vector $\mathbf{Y}$, formed by adding to $\mathbf{X}$ a zero-mean Gaussian vector $\mathbf{Z}$, whose components are independent and have common variance ϵ. Although $\mathbf{Y}$ is independent of $\hat{\mathbf{X}}_1$ and $\hat{\mathbf{X}}_2$, given $\mathbf{X}$, and plays no apparent intuitive role in the encoding/decoding process, $\mathbf{Y}$ provides the crucial lower bound in the proof.

It is true that

$$\begin{aligned} I(\hat{\mathbf{X}}_1; \hat{\mathbf{X}}_2\mathbf{Y}) &= I(\hat{\mathbf{X}}_1; \hat{\mathbf{X}}_2 \,|\, \mathbf{Y}) + I(\hat{\mathbf{X}}_1; \mathbf{Y}) \\ &= I(\hat{\mathbf{X}}_1; \hat{\mathbf{X}}_2) + I(\hat{\mathbf{X}}_1; \mathbf{Y} \,|\, \hat{\mathbf{X}}_2). \end{aligned}$$

Therefore,

$$\begin{aligned} I(\hat{\mathbf{X}}_1; \hat{\mathbf{X}}_2) &= I(\hat{\mathbf{X}}_1; \hat{\mathbf{X}}_2 \,|\, \mathbf{Y}) + I(\hat{\mathbf{X}}_1; \mathbf{Y}) - I(\hat{\mathbf{X}}_1; \mathbf{Y} \,|\, \hat{\mathbf{X}}_2) \\ &\geq I(\hat{\mathbf{X}}_1; \mathbf{Y}) - I(\hat{\mathbf{X}}_1; \mathbf{Y} \,|\, \hat{\mathbf{X}}_2) \\ &= I(\hat{\mathbf{X}}_1; \mathbf{Y}) + I(\hat{\mathbf{X}}_2; \mathbf{Y}) - I(\hat{\mathbf{X}}_1\hat{\mathbf{X}}_2; \mathbf{Y}), \end{aligned} \tag{8}$$

where the inequality follows from the nonnegativity of mutual information, and all other steps follow from the identity $I(A; BC) = I(A; B) + I(A; C \,|\, B)$. Now for $i = 1, 2$,

$$\begin{aligned} \frac{1}{N}\sum_{k=1}^{N} E[(\hat{X}_{ik} - Y_k)^2] &= \frac{1}{N}\sum_{1}^{N} E[(\hat{X}_{ik} - X_k + X_k - Y_k)^2] \\ &= \frac{1}{N}\sum_{1}^{N} [E(\hat{X}_{ik} - X_k)^2 + E(X_k - Y_k)^2] \\ &= d_i + \epsilon, \end{aligned}$$

where the cross term vanishes since $\{Z_k\}$ are independent of all else. Also $\mathbf{Y}$ is a Gaussian vector with independent components, each of variance $1 + \epsilon$. The rate-distortion function for Y is then given by (1):

$$R_y(d) = \frac{1}{2}\log\frac{1+\epsilon}{d}.$$

So by the converse to the source-coding theorem,

$$\frac{1}{N} I(\hat{\mathbf{X}}_i; \mathbf{Y}) \geq \frac{1}{2}\log\frac{1+\epsilon}{d_i + \epsilon}, \qquad i = 1, 2. \tag{9}$$

As for the last term in (8),

$$\begin{aligned} I(\hat{\mathbf{X}}_1\hat{\mathbf{X}}_2; \mathbf{Y}) &= h(\mathbf{Y}) - h(\mathbf{Y} \,|\, \hat{\mathbf{X}}_1\hat{\mathbf{X}}_2) \\ &= Ng(1+\epsilon) - h(\mathbf{Y} \,|\, \hat{\mathbf{X}}_1\hat{\mathbf{X}}_2) \qquad \text{(Lemma 1)} \\ &\leq Ng(1+\epsilon) - \frac{N}{2}\log\left[\exp\left(\frac{2}{N} h(\mathbf{X} \,|\, \hat{\mathbf{X}}_1\hat{\mathbf{X}}_2)\right) + 2\pi e\epsilon\right]. \\ &\qquad \text{(Lemma 2)} \end{aligned}$$

But

$$\begin{aligned} h(\mathbf{X} \mid \hat{\mathbf{X}}_1\hat{\mathbf{X}}_2) &= h(\mathbf{X} \mid \hat{\mathbf{X}}_1\hat{\mathbf{X}}_2) - h(\mathbf{X}) + h(\mathbf{X}) \\ &= -\mathrm{I}(\mathbf{X}; \hat{\mathbf{X}}_1\hat{\mathbf{X}}_2) + h(\mathbf{X}) \\ &= h(\mathbf{X}) - H(\hat{\mathbf{X}}_1\hat{\mathbf{X}}_2) + H(\hat{\mathbf{X}}_1\hat{\mathbf{X}}_2 \mid \mathbf{X}) \\ &\overset{(a)}{=} h(\mathbf{X}) - H(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2) \\ &= h(\mathbf{X}) - H(\hat{\mathbf{X}}_1) - H(\hat{\mathbf{X}}_2) + I(\hat{\mathbf{X}}_1; \hat{\mathbf{X}}_2) \\ &= Ng(1) - H(\hat{\mathbf{X}}_1) - H(\hat{\mathbf{X}}_2) + I(\hat{\mathbf{X}}_1; \hat{\mathbf{X}}_2) \qquad \text{(Lemma 1)} \\ &\geq N(g(1) - R_1 - R_2) + I(\hat{\mathbf{X}}_1; \mathbf{X}_2), \qquad \text{(channel constraint)} \end{aligned}$$

where step (a) follows from the fact that $\hat{\mathbf{X}}_1$ and $\hat{\mathbf{X}}_2$ are determined by $\mathbf{X}$. Therefore,

$$I(\hat{\mathbf{X}}_1\hat{\mathbf{X}}_2; \mathbf{Y}) \leq Ng(1+\epsilon) - \frac{N}{2}\log\left[\exp[2(g(1) - R_1 - R_2)]\exp\left(\frac{2}{N} I(\hat{\mathbf{X}}_1; \hat{\mathbf{X}}_2)\right) + 2\pi e\epsilon\right].$$

Since $e^{2g(1)} = 2\pi e$,

$$I(\hat{\mathbf{X}}_1\hat{\mathbf{X}}_2; \mathbf{Y}) \leq Ng(1+\epsilon) - Ng\left[\exp[-2(R_1 + R_2)]\exp\left(\frac{2}{N} I(\hat{\mathbf{X}}_1; \hat{\mathbf{X}}_2)\right) + \epsilon\right]. \qquad (10)$$

Combining (8), (9), and (10), and defining $t \triangleq \exp[2I(\hat{\mathbf{X}}_1; \hat{\mathbf{X}}_2)/N]$, we have

$$t \geq \frac{(1+\epsilon)\{\exp[-2(R_1 + R_2)]t + \epsilon\}}{(\epsilon + d_1)(\epsilon + d_2)}.$$

Isolating t,

$$t\{(\epsilon + d_1)(\epsilon + d_2) - (1+\epsilon)\exp[-2(R_1 + R_2)]\} \geq \epsilon(1+\epsilon)$$

$$t(\epsilon^2 + \epsilon\{d_1 + d_2 - \exp[-2(R_1 + R_2)]\} + d_1 d_2 - \exp[-2(R_1 + R_2)]) \geq \epsilon(1+\epsilon).$$

Since $d_i \geq \exp(-2R_i)$, the quadratic on the left-hand side is always nonnegative (as long as ϵ is). Define Δ and Π as in the statement of Theorem 1, so that

$$t \geq \frac{\epsilon(1+\epsilon)}{\epsilon^2 + \epsilon(1 + \Delta - \Pi) + \Delta}. \qquad (11)$$

This inequality holds for any $\epsilon \geq 0$. In particular, choose that ϵ

which maximizes the right-hand side. Taking derivatives and setting to zero, it is readily shown that the maximizing ϵ is given by

$$\epsilon = \frac{\sqrt{\Delta}}{\sqrt{\Pi} - \sqrt{\Delta}}.$$

The numerator of (11) is

$$\epsilon(1+\epsilon) = \frac{\sqrt{\Delta}}{\sqrt{\Pi} - \sqrt{\Delta}}\left(1 + \frac{\sqrt{\Delta}}{\sqrt{\Pi} - \sqrt{\Delta}}\right) = \frac{\sqrt{\Pi\Delta}}{(\sqrt{\Pi} - \sqrt{\Delta})^2}.$$

The denominator is

$$\begin{aligned}
&\epsilon^2 + \epsilon(1 + \Delta - \Pi) + \Delta \\
&\quad = \frac{\Delta}{(\sqrt{\Pi} - \sqrt{\Delta})^2} + \frac{\sqrt{\Delta}}{(\sqrt{\Pi} - \sqrt{\Delta})}(1 + \Delta - \Pi) + \Delta \\
&\quad = \frac{1}{(\sqrt{\Pi} - \sqrt{\Delta})^2} \\
&\qquad \cdot [\Delta + \sqrt{\Delta}(\sqrt{\Pi} - \sqrt{\Delta})(1 + \Delta - \Pi) + \Delta\,(\sqrt{\Pi} - \sqrt{\Delta})^2] \\
&\quad = \frac{1}{(\sqrt{\Pi} - \sqrt{\Delta})^2} \\
&\qquad \cdot [\Delta + \sqrt{\Pi\Delta} - \Delta - \sqrt{\Delta}(\sqrt{\Pi} - \sqrt{\Delta})(\Pi - \Delta) + \Delta\,(\sqrt{\Pi} - \sqrt{\Delta})^2] \\
&\quad = \frac{1}{(\sqrt{\Pi} - \sqrt{\Delta})^2} \\
&\qquad \cdot [\sqrt{\Pi\Delta} - \sqrt{\Delta}(\sqrt{\Pi} - \sqrt{\Delta})^2(\sqrt{\Pi} + \sqrt{\Delta}) + \Delta(\sqrt{\Pi} - \sqrt{\Delta})^2] \\
&\quad = \frac{1}{(\sqrt{\Pi} - \sqrt{\Delta})^2} \cdot [\sqrt{\Pi\Delta} - \sqrt{\Pi\Delta}(\sqrt{\Pi} - \sqrt{\Delta})^2] \\
&\quad = \frac{\sqrt{\Pi\Delta}}{(\sqrt{\Pi} - \sqrt{\Delta})^2}[1 - (\sqrt{\Pi} - \sqrt{\Delta})^2].
\end{aligned}$$

Therefore, (11) becomes

$$t \geq \frac{1}{1 - (\sqrt{\Pi} - \sqrt{\Delta})^2}.$$

Substituting into (7) yields the third inequality of Theorem 1. The first two inequalities in Theorem 1 are, of course, trivial. Theorem 1 (converse) is proved.

III. FORWARD PART OF THEOREM 1

To prove the forward theorem we evaluate the following achievable region for the general case of Fig. 1, found by El Gamal and Cover.[3] Consider $\{X_k\}$ drawn IID from alphabet $\mathscr{X}$ according to probability assignment $p(x)$. Let $\hat{\mathscr{X}}_1$, $\hat{\mathscr{X}}_2$, $\hat{\mathscr{X}}_3$ be the appropriate reproducing alphabets at the three receivers in Fig. 1, and let $d_1(\cdot,\cdot)$, $d_2(\cdot,\cdot)$, $d_3(\cdot,\cdot)$, be the respective (single-letter) distortion measures. Consider a test encoder of the form shown in Fig. 2. That is, let auxiliary random variables $U \in \mathscr{U}$ and $V \in \mathscr{V}$ be arbitrarily jointly distributed, given X. For any three decoding functions,

$$g_1: \mathscr{U} \rightarrow \hat{\mathscr{X}}_1,$$
$$g_2: \mathscr{V} \rightarrow \hat{\mathscr{X}}_2,$$
$$g_3: \mathscr{U} \times \mathscr{V} \rightarrow \hat{\mathscr{X}}_3,$$

average distortions

$$\begin{aligned} d_1 &= E[d_1(X, g_1(U))], \\ d_2 &= E[d_1(X, g_2(V))], \\ d_3 &= E[d_3(X, g_3(U, V))], \end{aligned} \tag{12}$$

are achievable if

$$\begin{aligned} R_1 &\geq I(U; X), \\ R_2 &\geq I(V; X), \\ R_1 + R_2 &\geq I(UV; X) + I(U; V). \end{aligned} \tag{13}$$

Applying this result to our problem, let

$$U = X + N_1,$$
$$V = X + N_2,$$

where N_1 and N_2 are jointly zero-mean Gaussian with covariance matrix

$$\mathbf{\Phi}_1 = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{bmatrix}.$$

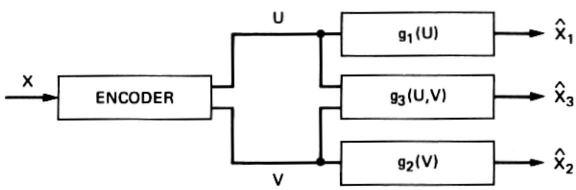

Fig. 2—Test encoder for achievable regions.

Φ_1 is also the covariance matrix of U and V given X. Without the conditioning on X, U and V are jointly Gaussian with covariance matrix

$$\Phi_2 = \begin{bmatrix} 1+\sigma_1^2 & 1+\sigma_1\sigma_2\rho \\ 1+\sigma_1\sigma_2\rho & 1+\sigma_2^2 \end{bmatrix}.$$

Lemma 1 allows us to evaluate the right-hand side of (13) as

$$\begin{aligned} I(U;X) &= h(U) - h(U|X) \\ &= \frac{1}{2}\log\frac{1+\sigma_1^2}{\sigma_1^2}, \\ I(V;X) &= h(V) - h(V|X) \\ &= \frac{1}{2}\log\frac{1+\sigma_2^2}{\sigma_2^2}, \\ I(UV;X) + I(U;V) &= h(UV) - h(UV|X) + h(U) + h(V) - h(UV) \\ &= \frac{1}{2}\log\frac{(1+\sigma_1^2)(1+\sigma_2^2)}{\sigma_1^2\sigma_2^2(1-\rho^2)}. \end{aligned}$$

Clearly, the best $g_1(u)$, $g_2(v)$, and $g_3(u, v)$ are the minimum mean-squared-error estimates of x, given the respective arguments. These are given by

$$g_1(u) = \frac{1}{\overline{U^2}}u,$$

$$g_2(v) = \frac{1}{\overline{V^2}}v,$$

and

$$g_3(u, v) = \frac{\overline{V^2} - \overline{UV}}{\overline{U^2}\,\overline{V^2} - (\overline{UV})^2}u + \frac{\overline{U^2} - \overline{UV}}{\overline{U^2}\,\overline{V^2} - (\overline{UV})^2}v.$$

Evaluating the various expressions and substituting into (12) yields

$$\begin{aligned} d_1 &= \frac{\sigma_1^2}{1+\sigma_1^2}, \\ d_2 &= \frac{\sigma_2^2}{1+\sigma_2^2}, \\ d_3 &= \frac{\sigma_1^2\sigma_1^2(1-\rho^2)}{\sigma_1^2\sigma_2^2(1-\rho^2) + \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}. \end{aligned} \tag{14}$$

The constraints (13) become

$$R_1 \geq \frac{1}{2} \log \frac{1}{d_1},$$

$$R_2 \geq \frac{1}{2} \log \frac{1}{d_2},$$

$$R_1 + R_2 \geq \frac{1}{2} \log \frac{1}{d_1 d_2 (1 - \rho^2)}.$$

We can therefore choose ρ arbitrarily, so long as

$$d_1 d_2 (1 - \rho^2) \geq \exp[-2(R_1 + R_2)],$$

$$\rho^2 \leq \frac{d_1 d_2 - \exp[-2(R_1 + R_2)]}{d_1 d_2}.$$

Choose

$$\rho = -\frac{\sqrt{d_1 d_2 - \exp[-2(R_1 + R_2)]}}{\sqrt{d_1 d_2}}.$$

Substituting this value for ρ, and using the fact that $\sigma_i^2 = d_i/(1 - d_i)$ for $i = 1, 2$ [obtained from the first two parts of (14)], the last equation in (14) can be written as

$$d_3 = \frac{\exp[-2(R_1 + R_2)]}{D}, \tag{15}$$

where

$$\begin{aligned} D &= \exp[-2(R_1 + R_2)] + d_1(1 - d_2) + d_2(1 - d_1) \\ &\quad + 2\sqrt{1 - d_1}\,\sqrt{1 - d_2}\,\sqrt{d_1 d_2 - \exp[-2(R_1 + R_2)]} \\ &= \exp[-2(R_1 + R_2)] - d_1 d_2 + d_1 + d_2 - d_1 d_2 \\ &\quad + 2\sqrt{1 - d_1}\,\sqrt{1 - d_2}\,\sqrt{d_1 d_2 - 1 \exp[-2(R_1 + R_2)]} \\ &= -\Delta - \Pi + 1 + 2\sqrt{\Pi\Delta} \\ &= 1 - (\sqrt{\Pi} - \sqrt{\Delta})^2, \end{aligned}$$

where Π and Δ are as before. Equation (15) thus reduces to the last part of eq. (3), and Theorem 1 is proved.

IV. CONCLUSIONS

We have obtained the solution to the channel-splitting problem described in the introduction and depicted in Fig. 1, for the case where the input letters are IID Gaussian and the distortion measure of interest is the mean-squared error. So far, no complete solution is known for any other source or distortion measure. Wolf et al.[4] have obtained an outer bound for the case of a binary symmetric source with Hamming (i.e., probability of error) distortion and have compared it in one case to the achievable region of Cover and El Gamal, but the bound exceeds the achievable point. Also, Witsenhausen[5] has considered a version of the binary problem and, in particular, has obtained, under slightly different assumptions, a stronger outer bound at one extreme point.

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