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Integral Equations for Electromagnetic Scattering by Wide Scatterers

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Integral equation formulations of electromagnetic scattering problems have been successful for scatterers which are only a few wavelengths wide. For a large scatterer, the integral equation methods have been uneconomical because too many basis functions are needed to represent the surface current on the scatterer. We use scattering from an infinitely long thin strip to demonstrate that the integral equation methods are economical for wide scatterers if the basis functions are properly chosen. Exact solutions of simple problems, as well as the geometric theory of diffraction, can be used to suggest the appropriate functions. The results demonstrate the feasibility of the techniques used.

I. INTRODUCTION

Integral equation methods for solving electromagnetic scattering problems reduce a two-dimensional problem to a one-dimensional integral equation by formulating the problem in terms of the unknown surface currents on the scattering body or bodies. Similarly, three-dimensional scattering problems are reduced to two-dimensional integral equations.

The unknown surface currents are approximated by a sum of basis functions times unknown coefficients. The basis functions are usually chosen to be simple piecewise constant or linear functions, although higher-order polynomials or piecewise polynomials are sometimes used. The coefficients are chosen to solve (approximately) the integral equation. The resulting electromagnetic field exactly obeys Maxwell's equations and the radiation condition, but obeys the boundary conditions on the scatterer(s) only approximately.

Even if higher-order polynomial basis functions are used, the density of functions on the scatterer must be at least two per wavelength. As a rule of thumb, five to ten functions per wavelength are typical. For scatterers whose dimensions are more than a few wavelengths, the integral equation formulation takes too many of the usual basis functions, and has therefore been thought to be uneconomical.

For scatterers which are many wavelengths wide, the integral equation formulation can be economical if sufficiently good basis functions can be found. This is demonstrated using a two-dimensional problem, consisting of scattering a plane wave from a perfectly conducting, infinitesimally thin strip $x = 0$, $-d/2 \leq y \leq +d/2$, $-\infty < z < \infty$. This geometry has been the subject of considerable analysis, and approximate solutions are available. The exact solution as an infinite series of Mathieu functions is also available, but is difficult to use.

For this model problem, excellent accuracy can be obtained with only 17 basis functions, even for a strip more than 60 wavelengths wide.

If the incident wave depends on z only as $\exp(ik_z z)$, the problem is two dimensional, and the vector electromagnetic scattering problem may be separated into two scalar scattering problems, one for the E-wave, and one for the H-wave. This is done in Section II.

In Section III, we present integral equations to be solved for the currents on the scatterer.

Section IV takes up the difficulties in solving the integral equations. An appropriate set of basis functions may be found by considering the form of the extract solution for scattering from a half-plane. (Some of the integrals are either singular or Cauchy principal-value integrals, and must be done carefully.)

After the approximate currents on the scatterer have been found, the electromagnetic fields far from the scatterer can be calculated. In Section V, the far fields are calculated numerically and compared to those predicted by the geometrical theory of diffraction, which is accurate in the high-frequency (wavelength $\ll$ width of scatterer) limit. Agreement is excellent.

II. FORMULATION OF THE SCATTERING PROBLEM

We assume a time variation of $e^{-i\omega t}$ and suppress the factor throughout. All z dependence of fields is $\exp(ik_z z)$. Then any solution of Maxwell's equation may be written as a superposition of two scalar wave equations (Ref. 1, §11.6). This may be seen as follows. Two of Maxwell's equations are, in rationalized MKS units,

$$\nabla \times \mathbf{E} = i\omega\mu_0 \mathbf{H}, \tag{1}$$

$$\nabla \times \mathbf{H} = -i\omega\epsilon_0 \mathbf{E}. \tag{2}$$

Applying the curl operator to eqs. (1) and (2), and using $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{H} = 0$, we obtain

$$\nabla^2 V + k^2 V = 0, \tag{3}$$

where V is any of E_x, E_y, E_z, H_x, H_y, or H_z, and $k = \omega\sqrt{\mu_0\epsilon_0}$ is the wave number. E_x, E_y, H_x, and H_y may be expressed in terms of E_z and H_z. For example, from (1),

$$i\omega\mu_0 H_y = ik_z E_x - \frac{\partial E_z}{\partial x}, \tag{4}$$

and from (2)

$$-i\omega\epsilon_0 E_x = \frac{\partial H_z}{\partial y} - ik_z H_y. \tag{5}$$

Eliminating H_y and rearranging,

$$E_x = \frac{i}{k^2 - k_z^2}\left(k_z \frac{\partial E_z}{\partial x} + \omega\mu_0 \frac{\partial H_z}{\partial y}\right). \tag{6}$$

We let $k_t^2 = k^2 - k_z^2$. Similarly,

$$E_y = \frac{i}{k_t^2}\left(k_z \frac{\partial E_z}{\partial y} - \omega\mu_0 \frac{\partial H_z}{\partial x}\right), \tag{7}$$

$$H_x = \frac{i}{k_t^2}\left(k_z \frac{\partial H_z}{\partial x} - \omega\epsilon_0 \frac{\partial E_z}{\partial y}\right), \tag{8}$$

$$H_y = \frac{i}{k_t^2}\left(k_z \frac{\partial H_z}{\partial y} + \omega\epsilon_0 \frac{\partial E_z}{\partial x}\right). \tag{9}$$

From (3), E_z obeys the scalar reduced wave equation:

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_t^2 E_z = 0, \tag{10}$$

as does H_z.

The boundary conditions do not couple the E wave ($E_z \neq 0$, $H_z = 0$) and the H wave ($E_z = 0$, $H_z \neq 0$). Let $\mathbf{n}$ be the unit normal vector pointing *into* the scatterer at any point. The scatterer has $n_z = 0$. Since $\mathbf{E}$ and $\mathbf{H}$ are zero inside a perfect conductor, the usual boundary conditions of continuity of tangential $\mathbf{E}$ and normal $\mathbf{H}$ are

$$\mathbf{n} \times E = 0, \tag{11}$$

$$\mathbf{n} \cdot H = 0. \tag{12}$$

The first yields

$$E_z = 0 \tag{13}$$

as the boundary condition for the E wave. Using (8) and (9) we see, since $n_z = 0$, that

$$n_x\left(k_z \frac{\partial H_z}{\partial x} - \omega\epsilon_0 \frac{\partial E_z}{\partial y}\right) + n_y\left(k_z \frac{\partial H_z}{\partial y} + \omega\epsilon_0 \frac{\partial E_z}{\partial x}\right) = 0. \tag{14}$$

Since $E_z = 0$ for the H wave, (14) reduces to

$$\mathbf{n}\cdot\nabla H_z = \frac{\partial H_z}{\partial n} = 0, \tag{15}$$

as the boundary condition for the H wave. If $k_z = 0$, we obtain (15) by combining (6), (7), and (11).

Since the equations and boundary conditions separate, we may solve two scalar scattering problems.

Geometry of the problem

We use standard spherical coordinates, (r, θ, ϕ). Incident quantities are denoted by superscript i and scattered quantities by superscript s. The incident wave vector is

$$\begin{aligned}\mathbf{k}^i &= k(-\cos\phi^i \sin\theta^i, -\sin\phi^i \sin\theta^i, -\cos\theta^i) \\ &= (k_x, k_y, k_z).\end{aligned} \tag{16}$$

Letting $\mathbf{r} = (x, y, z)$, we write an incident E wave as

$$\mathbf{E}_E^i = C_E e^{i\mathbf{k}\cdot\mathbf{r}} \begin{pmatrix} -\cos\phi^i \cos\theta^i \\ \sin\phi^i \cos\theta^i \\ \sin\theta^i \end{pmatrix}, \tag{17}$$

$$\mathbf{H}_E^i = C_E \frac{k}{\omega\mu_0} e^{i\mathbf{k}\cdot\mathbf{r}} \begin{pmatrix} -\sin\phi^i \\ \cos\phi^i \\ 0 \end{pmatrix}, \tag{18}$$

and an incident H wave as

$$\mathbf{E}_H^i = C_H e^{i\mathbf{k}\cdot\mathbf{r}} \begin{pmatrix} \sin\phi^i \\ -\cos\phi^i \\ 0 \end{pmatrix}, \tag{19}$$

$$\mathbf{H}_H^i = C_H \frac{k}{\omega\mu_0} e^{i\mathbf{k}\cdot\mathbf{r}} \begin{pmatrix} -\cos\phi^i \cos\theta^i \\ -\sin\phi^i \cos\theta^i \\ \sin\theta^i \end{pmatrix}, \tag{20}$$

where C_E and C_H are arbitrary complex constants. Any electromagnetic plane wave is the sum of an E wave and an H wave.

III. INTEGRAL EQUATION FORMULATION

To find the z components of the scattered fields, from which the other components may be found, we need to solve two integral equations. The integral equation for E_z^s is fairly standard. First, suppose the scatterer is a perfect conductor with finite thickness; its cross section in the $z = 0$ plane is a thin rectangle, Γ, centered at the origin. Then for $\mathbf{r}$ on Γ, we have [Ref. 2, eq. (4.26)]

$$E_z^i(\mathbf{r}) = -\frac{i}{4}\int_\Gamma H_0^{(1)}(k_t R)\,\frac{\partial E_z(s)}{\partial n}\,ds. \tag{21}$$

Here $E_z = E_z^s + E_z^i$, $H_0^{(1)}$ is a Hankel function[3] representing outgoing waves, R is the distance from $\mathbf{r}$ to any point at arc length s on Γ, and $\mathbf{n}$ is the unit normal vector at s, pointing *into* the scatterer. This formulation ensures that the field obeys Maxwell's equations and that the scattered field obeys the radiation condition.

As the thickness of the strip goes to zero, the contributions of the ends go to zero, and we get

$$E_z^i(0, y, 0) = -i/4 \int_{-d/2}^{d/2} H_0^{(1)}(k_t|y - y'|)\, J_E^*(y')\, dy', \tag{22}$$

where $J_E^*(y')$ is the sum of $\partial E_z/\partial n$ on the $x > 0$ and $x < 0$ sides at point y'.

We use $E_z^i(\mathbf{r}) = C_E \sin\theta^i e^{i\mathbf{k}\cdot\mathbf{r}}$, and define $u = k_t y$, $v = k_t y'$, $c = k_t d/2$, and $\gamma = k_y/k_t$. Then

$$C_E \sin\theta^i\, e^{i\gamma u} = -\frac{i}{4k_t} \int_{-c}^{c} H_0^{(1)}(|u - v|)\, J_E^*(v/k_t)\, dv. \tag{23}$$

Finally, let $J_E(v) = J_E^*(v/k_t)/4k_t i C_E \sin\theta^i$, and we have the integral equation for $J_E(v)$:

$$e^{i\gamma u} = \int_{-c}^{c} H_0^{(1)}(|u - v|) J_E(v)\, dv.$$

The usual integral equation for $\partial H_z/\partial n$, eq. (4.27) of Ref. 2, is not useful for zero-thickness scatterers. A new integral equation, a generalization of one previously derived for thick scatterers,[4] has been derived by Morrison.[5] His work is directly applicable to thin bodies.

$J_H^*(y)$ is the difference of H_z on the $x > 0$ and $x < 0$ sides. Letting

$$J_H(v) = \frac{J_H^*(v/k_t)}{4(k_x/k_t) C_H (k/\omega\mu_0) \sin\theta^i}, \tag{24}$$

and letting u_1 and u_2 be any values such that

$$-c \leqslant u_1 < u < u_2 \leqslant c, \tag{25}$$

we have the integral equation for $J_H(v)$:

$$\begin{aligned} e^{i\gamma u} = {} & \int_{-c}^{u_1} \frac{H_1^{(1)}(u - v)}{u - v} J_H(v)\, dv + \int_{u_2}^{c} \frac{H_1^{(1)}(v - u)}{v - u} J_H(v)\, dv \\ & + \int_{u_1}^{u_2} H_0^{(1)}(|u - v|) J_H(v)\, dv \\ & + \int_{u_1}^{u_2} H_1^{(1)}(|u - v|)\, \mathrm{sgn}(v - u)\, \frac{dJ_H(v)}{dv}\, dv \\ & - J_H(u_2) H_1^{(1)}(u_2 - u) - J_H(u_1) H_1^{(1)}(u - u_1). \end{aligned} \tag{26}$$

Here $H_1^{(1)}$ is a Hankel function,[3] and sgn is the sign function: $\mathrm{sgn}(u) = +1$ if $u > 0$, and -1 if $u < 0$. The final integral in the integral equation is a Cauchy principal-value integral. The equation simplifies considerably in form, if $u_1 = -c$ and $u_2 = c$, since $J_H(-c) = J_H(c) = 0$. For the numerical examples in Section VI, u_1 and u_2 are taken to be close to u. The equation looks more complicated than if $u_1 = -c$ and $u_2 = c$, but the integrals are no harder.

Far fields

Once the currents on the strip have been found, the field can be calculated at points off the strip by the usual formulas:[2]

$$E_z^s(\mathbf{r}) = \exp(ik_z z)\,\frac{i}{4}\int_{-d/2}^{d/2} H_0^{(1)}(k_t R) J_E^*(y')\,dy', \tag{27}$$

$$H_z^s(\mathbf{r}) = -\exp(ik_z z)\,\frac{ik_t}{4}\int_{-d/2}^{d/2} H_1^{(1)}(k_t R)\,\frac{\mathbf{n}\bullet\mathbf{R}}{R}\,J_H^*(y')\,dy', \tag{28}$$

where $\mathbf{R}$ is the vector from the point $(x, y, 0)$ to the point on the strip at $(0, y', 0)$. The other field components can be found from eqs. (6)–(9).

For the far fields, $R \gg d$, the equations simplify greatly, since the asymptotic form of the Hankel functions can be used,[3]

$$H_n^{(1)}(\xi) = (2/\pi\xi)^{1/2} e^{i(\xi - n\pi/2 - \pi/4)} + O(\xi^{-1}). \tag{29}$$

We let $\rho = (x^2 + y^2)^{1/2}$. For $R \gg d$,

$$R = [x^2 + (y - y')^2]^{1/2} = \rho - \frac{yy'}{\rho} + O\left(\frac{y'}{\rho}\right).$$

Only the first two terms need be retained; the second term need be retained only in the exponential. Thus

$$H_n^{(1)}(k_t R) = \left(\frac{2}{\pi k_t \rho}\right)^{1/2} \cdot\exp[i(k_t\rho - k_t y y'/\rho - n\pi/2 - \pi/4)] + O(d/\rho). \tag{30}$$

We use the auxiliary function F,

$$F(\xi) = \left(\frac{2}{\pi\xi}\right)^{1/2} e^{i(\xi - \pi/4)}. \tag{31}$$

Finally, using the definitions of J_E and J_H, changing variables to $v = k_t y'$, and using a superscript f to denote far fields, we get

$$E_z^f(\mathbf{r}) = -\exp(ik_z z)\, C_E \sin\theta^i F(k_t\rho) \int_{-c}^{c} e^{-ivy/\rho} J_E(v)\,dv, \tag{32}$$

$$H_z^f(\mathbf{r}) = -\exp(ik_z z)\, C_H \sin\theta^i \frac{k_x}{k_t}\frac{x}{\rho} F(k_t\rho) \int_{-c}^{c} e^{-ivy/\rho} J_H(v)\,dv. \tag{33}$$

The integrals may be evaluated by the methods described in Section IV.

For incident waves with $\theta^i = \pi/2$ ($k_z = 0$), far-field coefficients may be defined (Ref. 6, p. 6):

$$\begin{aligned} &E \text{ wave:} \quad P_E = E_z^f(\mathbf{r})/C_e F(k_t\rho), \\ &H \text{ wave:} \quad P_H = H_z^f(\mathbf{r})/C_H F(k_t\rho). \end{aligned}$$

IV. NUMERICAL SOLUTION OF THE INTEGRAL EQUATIONS

We first give an overview of the problem, and then discuss various aspects of the solution method.

4.1 Overview

The problem has now been reduced to solving two uncoupled one-dimensional linear integral equations on the boundary of the scatterer. The usual method to solve such integral equations approximately is first to approximate the unknown J by a sum of basis functions, $\{f_n\}$:

$$J(v) = \sum_{n=1}^{N} a_n f_n(v). \tag{34}$$

The choice of the $\{f_n\}$ will be discussed later. Then the integrals are done, reducing the integral equation to

$$e^{i\gamma u} = \sum_{n=1}^{N} a_n I_n(u), \tag{35}$$

where $I_n(u)$ is defined by replacing $J(v)$ by $f_n(v)$ in the integral equation, and performing the integrals. For the E equation, for example,

$$I_n(u) = \int_{-c}^{c} H_0^{(1)}(|u - v|) f_n(v)\, dv. \tag{36}$$

If the basis functions are simple piecewise polynomials, the integrals can be done simply by (say) the trapezoidal rule, or by Gaussian quadrature. For the basis functions which we will use, the integrals must be done much more carefully.

For finite N, eq. (35) cannot hold for all u in $(-c, c)$. We choose M points $\{u_j\}$, $M > N$, at which to make (35) hold approximately. For ease of computation, the M equations in N unknowns are usually solved in a least-squares sense. Other approximate solutions are also reasonable, though more difficult to calculate. $M \approx 2N$ is a common choice, and it is used for the calculations in Section V.

4.2 Basis functions

In scattering problems where the scatterer is not many wavelengths wide, a common practice is to let the basis functions $\{f_n\}$ be piecewise constant or piecewise linear functions, zero over all but a small part of the scatterer. Then the integrands of the integrals defining the $\{I_n\}$ are nonzero over only a small part of the scatterer, and evaluating the integral is relatively easy. Frequently the integrals are evaluated by a simple midpoint rule, except for the part of the integral with v near u, where the integrand is singular. When higher-order piecewise polynomials (splines) are used, the integrals are more difficult, but fewer basis functions are needed.

We are interested in scatterers whose widths are many wavelengths, perhaps hundreds. We see shortly that the correct currents J have some components that vary as $e^{i\gamma u}$ and others as $e^{\pm iu}$. If splines are used as basis functions, at least two splines per cycle of a sinusoid are needed, and more if reasonable accuracy is to be obtained.

In addition, the currents near the edges of the scatterer are not well approximated by polynomials, and many additional basis functions would be needed there.

Basis functions which closely mimic the true currents are needed, so that only a few functions will suffice. Of course, for most problems the true currents are not known. (For the thin strip, the true currents *are* known analytically,[7] but only as an infinite sum of Mathieu functions. The analytic solution is of less use than a purely numerical solution, since the individual terms in the sum are hard to evaluate, and since the sum converges slowly.)

One approximate component of the true current is known, the physical-optics approximation to the current, or the current that would flow in an infinitely-wide scatterer. Far from the edges of the scatterer, the current should be approximately equal to the physical-optics current, which is proportional to $e^{i\gamma u}$. Therefore, a reasonable basis function is $e^{i\gamma u}$. To better approximate the physical-optics-like part of the current, we may also include a few $(u/c)e^{i\gamma u}$, $(u/c)^2 e^{i\gamma u}$, $\cdots$. These terms are equivalent to approximating part of the current by $J_{po}(u)e^{i\gamma u}$, and expanding $J_{po}(u)$ in simple smooth basis functions, since $J_{po}(u)$ is not expected to vary rapidly.

Other approximate components of the current can be obtained from the exact solution to a simpler problem, reflection of a plane wave from a thin half-plane. Scattering from a half-plane was solved by Sommerfeld, for a plane wave incident at $\theta^i = \pi/2$. The solutions are readily accessible in Ref. 7 and in Chapter 11.5 of Ref. 1. The currents are conveniently written in terms of the angle $\alpha = \pi/2 - \phi^i$, the angle between the incident wave and the positive y axis:

$$J_E^{\ddagger}(y) = \sin\alpha\, e^{-iky\cos\alpha} - \frac{e^{i(ky-\pi/4)}}{\sqrt{\pi}} \{2 \sin\alpha G[\sqrt{2ky}\cos(\alpha/2)] - i\sqrt{2/k_y}\sin(\alpha/2)\}. \quad (37)$$

Here $G(\xi)$ is the Fresnel function,[1]

$$G(\xi) = e^{-i\xi^2} \int_{\xi}^{\infty} e^{i\mu^2}\, d\mu. \tag{38}$$

For large positive ξ,

$$G(\xi) = \frac{i}{2\xi} + O(\xi^{-3}). \tag{39}$$

For small positive ξ,

$$G(\xi) = \frac{1}{2}\sqrt{\pi} e^{i\pi/4} - \xi + O(\xi^2). \tag{40}$$

The H current is

$$J_H^*(y) = e^{-iky\cos\alpha} - \frac{2}{\sqrt{\pi}} e^{i(ky-\pi/4)} G[\sqrt{2ky}\cos(\alpha/2)]. \tag{41}$$

For large y, away from the edge of the reflecting half-plane[1]

$$J_E^*(y) = \sin\alpha\, e^{-iky\cos\alpha} + e^{iky} \cdot O[(ky)^{-3/2}]. \tag{42}$$

The current is equal to the physical-optics current plus correction terms which oscillate and decay away from the edge. Note that the oscillation of the correction term is e^{iky} rather than $e^{iky\cos\alpha}$; the correction terms correspond to waves propagating along the surface.

Far away from the edge, the H current is

$$J_H^*(y) = e^{-iky\cos\alpha} - \frac{\sec(\alpha/2)}{\sqrt{2\pi}} e^{i(ky+\pi/4)} \{(ky)^{-1/2} + O[(ky)^{-3/2}]\}. \tag{43}$$

The correction to the physical-optics current decays more slowly with distance from the edge than does the corresponding correction to the E current.

Near the edge,

$$J_E^*(y) = (2/\pi)^{1/2} e^{i(ky-\pi/4)} \sin(\alpha/2)\{(ky)^{-1/2} + 4\cos^2(\alpha/2)(ky)^{1/2} + O[(ky)^{3/2}]\}, \tag{44}$$

which is infinite at the edge itself. The H current is similar, but it is not infinite at the edge:

$$J_H^*(y) = 2\left(\frac{2}{\pi}\right)^{1/2} \cos(\alpha/2) e^{i(ky-\pi/4)} \{(ky)^{1/2} + O[(ky)^{3/2}]\}. \tag{45}$$

Suitable basis functions for the strip can be obtained by using similar functions at each edge of the strip. We use different basis functions near the edges and away from the edges. Near the left edge, in $(-c, -c + \delta)$, we use basis functions

$$e^{i(c+v)} \left(\frac{c+v}{\delta} \right)^{m-1/2}, \tag{46}$$

where $m = 0, 1, 2, \cdots$ for J_E and $m = 1, 2, \cdots$ for J_H. The physical optics current is not used near the edges. Near the right edge, in $(c - \delta, c)$, we use basis functions

$$e^{i(c-v)} \left(\frac{c-v}{\delta} \right)^{m-1/2}. \tag{47}$$

Away from the edges, in $(\delta - c, c - \delta)$, we use three families of basis functions: physical optics, decaying from the left edge, and decaying from the right edge. With $m = 0, 1, 2, \cdots$, these are

$$\text{physical optics:} \quad \left(\frac{v}{c} \right)^{m} e^{i\gamma v},$$

$$\text{decaying from left edge:} \quad \left(\frac{\delta}{c+v} \right)^{m+1/2} e^{i(c+v)},$$

$$\text{decaying from right edge:} \quad \left(\frac{\delta}{c-v} \right)^{m+1/2} e^{i(c-v)}.$$

The interference between waves scattered at the two edges causes the coefficients to be different from the corresponding half-plane coefficients, but the same forms suffice. The solution is made more complicated by having several different kinds of basis functions, but only a few of each will be needed.

The same basis functions would be obtained by taking the surface current calculated by geometrical diffraction theory,[7] and expanding the current both near and far from the edges of the strip.

4.3 *Integrals*

There are two sources of difficulty in performing any of the integrals to obtain any $I_n(u)$ for a given point u. [We assume none of the points $\pm c$, $\pm(c - \delta)$ will be chosen for a u point.] First, there are singularities, both in the Hankel functions at $u = v$ and in the basis functions at $v = \pm c$. Second, both the Hankel functions and the basis functions are oscillatory, with many cycles present in some of the integrals. Many kinds of integrals must be done; we discuss only a few of them.

We first consider integrals over the basis functions near the edges. As an example, consider

$$\int_{-c}^{\delta-c} H_0^{(1)}(|u - v|) e^{i(c+v)} \left(\frac{c+v}{\delta} \right)^{m-1/2} dv, \tag{48}$$

for $u > \delta - c$. The Hankel function is not singular in the interval. If $w = (c + v)/\delta$, the integral is

$$\delta \int_0^1 H_0^{(1)}(u + c - w\delta) e^{iw\delta} w^{m-1/2}\, dw. \tag{49}$$

Since $m \geqslant 0$, this integral is done by Gauss quadrature formulas with weight function $w^{-1/2}$. A series of Gauss quadrature formulas with increasing numbers of points is used, until two successive values agree sufficiently closely. If u is within the interval, the singular part centered around $u = v$ is handled separately, and the part nearer the edge is handled similarly to the above method.

As another example, consider

$$\int_u^{u_2} H_0^{(1)}(|u - v|) f(v)\, dv, \tag{50}$$

for some $u_2 > u$. (In practice we choose $u_2 \leqslant u + 1$.) We let $w^2 = v - u$, obtaining

$$2 \int_0^{(u_2-u)^{1/2}} w H_0^{(1)}(w^2) f(u + w^2)\, dw. \tag{51}$$

The integrand is no more singular than $w \ln(w)$. We use a complex version of Patterson's automatic quadrature program.[8]

The small argument behaviors of the Hankel functions are[3]

$$H_0^{(1)}(\xi) = \frac{2i}{\pi} \ln(\xi) + O(1) + O[\xi^2 \ln(\xi)], \tag{52}$$

$$H_1^{(1)}(\xi) = -\frac{2i}{\pi\xi} + \bar{H}_1^{(1)}(\xi), \tag{53}$$

$$\bar{H}_1^{(1)}(\xi) = \frac{i\xi}{\pi} \ln(\xi) + O(\xi) + O[\xi^3 \ln(\xi)]. \tag{54}$$

For $0 < \xi < 8$, $H_0^{(1)}(\xi)$ and $\bar{H}_1^{(1)}(\xi)$ are evaluated using the approximations in Ref. 9. For example,

$$H_0^{(1)}(\xi) = J_0(\xi) + iY_0(\xi) = J_0(\xi) + i\left(\frac{2}{\pi} J_0(\xi) \ln(x) + \bar{Y}_0(\xi)\right). \tag{55}$$

Both $J_0(\xi)$ and $\bar{Y}_0(\xi)$ are approximated by rational functions of $(\xi/8)^2$. In the example of Section V, numerator and denominator are taken to be quintic polynomials, giving an absolute error less than 10^{-8} in $J_0(\xi)$ and $\bar{Y}_0(\xi)$.

For large values of ξ, the Hankel functions decay slowly and oscillate.[3] For $\xi > 8$, other approximations from Ref. 9 are used. For example,

$$H_0^{(1)}(\xi) = (2/\pi\xi)^{1/2}[P_0(\xi) + iQ_0(\xi)]e^{i(\xi-\pi/4)}. \tag{56}$$

Both P_0 and Q_0 are approximated by polynomial functions of $(8/\xi)^2$. In the examples of Section V, cubic polynomials are used, giving an absolute error less than 10^{-8}.

Integrals such as

$$\int_{u+1}^{u+8} H_0^{(1)}(v-u)f(v)\,dv, \tag{57}$$

are computed by the Patterson method.

For scatterers which are many wavelengths wide, $c \gg 1$, in integrals such as

$$\int_{u+8}^{c-8} H_0^{(1)}(v-u)f(v)\,dv, \tag{58}$$

the Hankel function may have many oscillations, as may $f(v)$. If $f(v)$ is a basis function decaying from the right, the $e^{i(v-u)}$ factor of the Hankel function just cancels the oscillations of the $e^{i(c-v)}$ factor of the basis function. If $f(v)$ is a basis function decaying from the left, the oscillations reinforce each other. If $f(v)$ is a physical-optics basis function, partial cancellation or reinforcement occurs.

If the total oscillation in the exponential part of the integral is small, the integral is computed by the Patterson method. Any cancellation in the exponential is done explicitly.

If the total oscillation in the exponential part of the integral is large, a different method must be used to be economical and to avoid too much error from accumulation of round-off errors. Many methods have been suggested for doing integrals of the type

$$\int_a^b g(v)e^{i\omega v}\,dv, \tag{59}$$

where $\omega|b-a| \gg 1$. The most widely known is Filon's method[10]; (a, b) is divided into an even number of intervals, each of length h, and g is evaluated at the endpoints of the intervals. The intervals are paired, and on each pair the three values of g are interpolated by a quadratic polynomial. No continuity is imposed on adjacent polynomials. The polynomials times $e^{i\omega v}$ are integrated analytically. Because no continuity is imposed, the error is proportional to ωh; for good accuracy $\omega h \leqslant 1$ is needed. Thus the number of sampling points must increase as $\omega|b-a|$ increases.

Methods which do not require the number of sampling points to increase, involve approximating $g(v)$ on the whole interval. A series of papers[11–16] deal with approximating g by a sum of either Legendre or Chebyshev polynomials orthogonal on (a, b). The polynomials times $e^{i\omega v}$ are then integrated. For Legendre polynomials, the integration can be done analytically, but care must be taken in doing the summation to produce a stable method. For Chebyshev polynomials, the integration cannot be done analytically, except in terms of infinite series, but no stability difficulty arises.

Many of the particular types of $g(v)$ that we need, decay away from one or both ends of the interval towards the middle, and have a singularity just outside the range of integration. For this type of function, splines are better suited than polynomials; fewer terms are needed for a prescribed accuracy of approximation.

Splines have been used before for integrals like (59)[17]; however, a uniform spacing of sampling points was used, which is uneconomical for our usual integrands. Instead of a uniform mesh, we use an adaptively generated nonuniform mesh.[18] The splines times $e^{i\omega v}$ are integrated analytically. Typical values of ωh, where h is a mesh spacing, range from 1 to 50. For $c = 200$ and $\delta = 1$, typical meshes have 20 to 40 intervals.

4.4 Solving for the coefficients $\{a_n\}$

By methods like those just described, each $I_n(u)$ may be calculated for any prescribed u. We choose M points $\{u_j\}$, $M > N$, and require (35) to hold in a least-squares sense at the M points. A standard subroutine[19] is used to solve the equations for the $\{a_n\}$. Continuity of the $J(v)$ at $\pm(c - \delta)$ is enforced by including extra equations and weighting them heavily.[20]

Apparently, no useful theory exists for choosing the $\{u_j\}$. Each u_j effectively "samples" strongly near u_j, and less strongly far from u_j. It is clear that sampling must be done in the region where each of the basis functions has appreciable magnitude. Some basis functions are nonzero only within distance δ from an edge, and sampling points must be clustered there. Other basis functions decay away from the edges, starting at distances larger than δ, and need sampling points not too far from the edges. Finally, the physical-optics basis functions are less localized; a good distribution of sampling points for the decaying basis functions should also suffice for the physical-optics basis functions.

For the numerical examples of Section V, the sampling points in $(-c, \delta - c)$ and $(c - \delta, c)$ are evenly spaced. In the left half of the central region, the distances of the sampling points from $u = \delta - c$ increase geometrically, starting with $u = 3\delta/2 - c$, and ending with $u = 0$. The points in the right half are symmetrically placed. If c is small enough, uniform spacing is used.

Table I—RMS and ERR for E and H waves, M varied

m_e	m_c	M	E wave RMS	E wave ERR	H wave RMS	H wave ERR
4	9	17	1.4 (−3)	5.1 (−4)	3.7 (−4)	3.2 (−4)
5	11	21	1.7 (−3)	1.4 (−3)	2.1 (−3)	7.7 (−4)
7	17	31	2.0 (−3)	9.7 (−4)	3.6 (−3)	5.9 (−4)
9	21	39	2.1 (−3)	5.3 (−4)	4.4 (−3)	2.4 (−4)
11	25	47	2.3 (−3)	1.9 (−4)	4.9 (−3)	2.8 (−4)

V. NUMERICAL RESULTS

In this section we present selected numerical results for scattering from a thin, infinitely long strip. All programs were run in single precision on a Honeywell 6000 series computer, whose relative roundoff error is 1.5×10^{-8}.

In order to compare our results with previous work, all results are presented for a plane wave incident at $\theta^i = \pi/2$ and $\phi^i = \pi/4$.

We let n_e by the number of edge basis functions in each edge, n_d the number of decaying basis functions from each edge, and n_p the number of physical-optics basis functions. The total number of unknown coefficients is $N = 2n_e + 2n_d + n_p$.

For $c \gg 1$, the geometric theory of diffraction can be used to compare far fields. Expansions for the E wave far field are given in Ref. 7, p. 200, and those for the H wave in Ref. 7, p. 218:

$$P_E/c = p_{0E} + p_{1E}c^{-5/2} + O(c^{-7/2}), \tag{60}$$

$$P_H/c = p_{0H} + p_{1H}c^{-3/2} + O(c^{-2}). \tag{61}$$

We used two terms of the expansion of P_H and one term for P_E, for $c \geqslant 50$. The maximum value of P/c is approximately $\sqrt{2}/2$, at specular reflection, $\phi^s = -\pi/4$. We checked values of P at 33 evenly-spaced angles in the range $-\pi/4 - 4\pi/c \leqslant \phi^s \leqslant -\pi/4 + 4\pi/c$.

With $c = 100$, $\delta = 1$, $n_e = n_d = 4$, $n_p = 1$, so that $N = 17$, we varied M, the number of sampling points (Table I). Letting m_e be the number of points in each edge, and m_c the number in the central region, $M = 2m_e + m_c$.

In Tables I to V, RMS indicates the root-mean-square error in solving

Table II—RMS and ERR for E and H waves, n_p varied

n_p	E wave RMS	E wave ERR	H wave RMS	H wave ERR
1	2.1 (−3)	7.1 (−4)	4.1 (−3)	3.5 (−4)
2	2.1 (−3)	6.6 (−4)	4.0 (−3)	2.6 (−4)
3	1.8 (−3)	7.8 (−4)	3.9 (−3)	3.4 (−4)
4	2.2 (−3)	8.4 (−4)	3.8 (−3)	2.4 (−3)

Table III—RMS and ERR for E and H waves, δ varied

δ	E wave RMS	E wave ERR	H wave RMS	H wave ERR
0.5	3.6 (−3)	3.5 (−3)	5.3 (−3)	1.5 (−3)
1.0	2.4 (−3)	2.0 (−4)	3.9 (−3)	2.9 (−4)
1.5	1.3 (−3)	2.7 (−3)	1.4 (−2)	2.1 (−3)
2.0	2.4 (−3)	5.2 (−3)	3.3 (−2)	1.3 (−2)

the over-determined linear system of equations. [Because of the two constraint equations which impose continuity of J at $\pm(c - \delta)$, the equations are over-determined even for $M = N$.) ERR indicates the maximum absolute difference between the calculated far field and the geometrical theory of diffraction far field, at the 33 angles checked. We write 1.4 (−3) for 1.4×10^{-3}. The errors are not strongly dependent on M/N.

With $c = 100$, $\delta = 1$, $n_e = n_d = 4$, $m_e = 8$, and $m_c = 17 + 2_{np}$ (so that $M \approx 2N$), we varied n_p (Table II). Adding additional physical-optics basis functions beyond the first is essentially useless.

Table IV—RMS and ERR for E and H waves, n_e varied

$n_e = n_d$	E wave RMS	E wave ERR	H wave RMS	H wave ERR
2	2.9 (−2)	5.3 (−3)	1.4 (−1)	2.1 (−2)
3	5.6 (−3)	1.0 (−3)	2.9 (−2)	1.3 (−3)
4	2.4 (−3)	2.0 (−4)	3.9 (−3)	2.9 (−4)
5	1.7 (−3)	1.8 (−4)	7.6 (−4)	2.2 (−4)

With $c = 100$, $n_e = n_d = 4$, $n_p = 1$, $m_e = 8$, and $m_c = 25$, we varied δ (Table III). The best value is approximately 1. Similar results are obtained for $c = 50$ and $c = 200$.

With $c = 100$, $\delta = 1$, and $n_p = 1$, we varied n_e and n_d, keeping $m_e = 2n_e$ and $m_c = 4m_d + 3$, so that $M \approx 2N$ (Table IV). This illustrates convergence with increasing N.

Finally, with $\delta = 1$, $n_e = n_d = 4$, $n_p = 1$, $m_e = 8$, and $m_c = 19$, we varied c (Table V). Convergence is similar for $c = 50$, 100, and 200.

Table V—RMS and ERR for E and H waves, c varied

c	E wave RMS	E wave ERR	H wave RMS	H wave ERR
50	1.7 (−3)	3.0 (−4)	3.6 (−3)	7.0 (−4)
100	2.4 (−3)	2.0 (−4)	3.9 (−3)	2.9 (−4)
200	1.8 (−3)	0.3 (−4)	3.5 (−3)	2.2 (−4)

REFERENCES

1. M. Born and E. Wolf, *Principles of Optics*, London: Pergamon, 1959.
2. A. J. Poggio and E. K. Miller, "Integral Equation Solutions of Three-Dimensional Scattering Problems," in *Computer Techniques for Electromagnetics*, R. Mittra, ed., Oxford: Pergamon, 1973.
3. F. W. J. Olver, "Bessel Functions of Integral Order," in *Handbook of Mathematical Functions*, M. Abramowitz and I. A. Stegun, eds., New York: Dover, 1965.
4. K. M. Mitzner, "Acoustic Scattering from an Interface between Media of Greatly Different Density," J. Math. Phys., *7* (1966), pp. 2053–2060.
5. J. A. Morrison, "Integral Equations for Electromagnetic Scattering by Perfect Conductors with Two-Dimensional Geometry," B.S.T.J., *58,* No. 2 (February 1979), pp. 409–425.
6. J. J. Bowman, T. B. A. Senior, and P. L. E. Uslenghi, eds., *Electromagnetic and Acoustic Scattering by Simple Shapes*, Amsterdam: North-Holland, 1969.
7. J. S. Asvetas and R. C. Kleinman, "The Strip," Chapter 8 of Ref. 6.
8. T. N. L. Patterson, "Algorithm 468, Algorithm for Automatic Numerical Integration over a Finite Interval," Comm. ACM, *16* (1973), pp. 694–699.
9. J. F. Hart et al., *Computer Approximations*, New York: Wiley, 1968.
10. L. N. G. Filon, "On a Quadrature Formula for Trigonometric Integrals," Proc. Roy. Soc. Edinburgh, *49* (1928), pp. 38–47.
11. N. S. Bakhalov and L. G. Vasil'eva, "Evaluation of the Integrals of Oscillating Functions by Interpolation at Nodes of Gaussian Quadratures," U.S.S.R. Comp. Math. Phys., *8* (1968), pp. 241–249.
12. R. Piessens and F. Poleunis, "A Numerical Method for the Integration of Oscillatory Functions," BIT, *11* (1971), pp. 317–327.
13. T. Havie, "Remarks on an Expansion for Integrals of Rapidly Oscillating Functions," BIT, *13* (1973), pp. 16–29.
14. A. Alaylioglu, G. A. Evans, and J. Hyslop, "The Use of Chebyshev Series for the Evaluation of Oscillatory Integrals," Comput. J., *19* (1976), pp. 258–267.
15. R. K. Littlewood and V. Zakian, "Numerical Evaluation of Fourier Integrals," J. Inst. Math. Appl., *18* (1973), pp. 331–339.
16. T. N. L. Patterson, "On High Precision Methods for the Evaluation of Fourier Integrals with Finite and Infinite Limits," Numer. Math., *27* (1976), pp. 41–52.
17. B. Einarsson, "Numerical Calculation of Fourier Integrals with Cubic Splines," BIT, *8* (1968), pp. 279–286.
18. A. Curtis and M. J. D. Powell, "Using Cubic Splines to Approximate Functions of One Variable to Prescribed Accuracy," Atomic Energy Research Establishment, Harwell, Report AERE-R5602, 1967.
19. P. Businger and G. H. Golub, "Linear Least Squares Solutions by Householder Transformations," in *Handbook for Automatic Computation, Volume II—Linear Algebra*, J. H. Wilkinson and C. Reinsch, eds., New York: Springer-Verlag, 1971.
20. C. L. Lawson and R. J. Hanson, *Solving Least Squares Problems*, Englewood Cliffs: Prentice-Hall, 1974.