THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 59, No. 1, January 1980
Printed in U.S.A.

Multiple Commitment of Feeder Capacity in the Loop Plant

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(Manuscript received April 10, 1979)

Under the Serving Area Concept, the area served by the central office is divided into smaller geographical entities known as serving areas. In response to uncertain demand for service, some cable pairs can be multipled, i.e., connected to more than one serving area. In this paper, we develop a stochastic model of the loop plant and demonstrate that the number of multipled pairs can be reasonably limited to about 10 to 20 percent of the total number of pairs available. We also show the robustness of this upper limit.

I. INTRODUCTION

Under the Serving Area Concept (see Bergholm and Koliss[1] and Long[2]), the area served by a central office is divided into smaller geographical entities of 200 to 600 living units known as *serving areas.* Each serving area has an associated terminal box called the *interface.* Feeder pairs, i.e., cable pairs from the central office, are terminated on the *in* side of the interface. Pairs from the living units are terminated on the *out* side of the interface; they are connected to the feeder pairs by means of jumper wires.

Feeder commitment, i.e., the process of physically connecting feeder pairs to the interface, is a complex operation. If too small a number of pairs is committed, frequent rearrangement or addition of cable pairs will be required. If too many pairs are committed, the pairs may lie unused for a long time. Thus, forecasting and optimization techniques are essential in order to economically commit feeder pairs.

However, the uncertainty in forecasting the actual number of pairs that would be required in each serving area is a major problem in feeder commitment. The classical response to uncertain demand is to build flexibility into the feeder plant by means of multipling, i.e., by allowing some feeder pairs to appear in more than one interface.

While multipling builds flexibility into the feeder plant, several problems are associated with multipling, such as:

(*i*) Increased complexity of record keeping.
(*ii*) Need for more terminals on the feeder side of the interface.
(*iii*) Increased craft activity at the interfaces.
(*iv*) Increased difficulty in testing, in fault detection, etc.

The advantages of multipling have therefore to be balanced against the disadvantages to arrive at a suitable level of multipling.

The process of redistributing available feeder pairs is known as *recommitment*, and the addition of new feeder pairs is called *relief*. The main advantage of multipling is that it decreases the probability of having to recommit feeder pairs or advance relief because of variations in customer growth. In this paper, we develop a stochastic model of the loop plant and use it to quantify this benefit of multipling. By studying the impact of multipling on the time of next feeder recommitment or relief, we can note where multipling provides diminishing returns and therefore arrive at reasonable upper limits on the degree of multipling.

II. SATURATING GROWTH MODEL

Consider two serving areas, SA 1 and SA 2. In SA 1, customers requesting telephone service arrive according to a Poisson process with parameter λ_1. Each such customer, on arrival, say, in SA 1, is provided with a feeder pair if one is available, using an intelligent assignment rule, as follows:

(*i*) A spare dedicated pair is assigned if one is available.
(*ii*) If not, a spare multipled pair is assigned.
(*iii*) If both (*i*) and (*ii*) are not possible, we choose a multipled pair that is in use in SA 2, and transfer the service on this pair to a spare dedicated pair in SA 2 if one is available. The multipled pair is thus rendered free and is used in SA 1. If this operation fails, the customer faces a held order.

The customer holds the feeder pair for a random amount of time called the *occupancy time*, which is exponentially distributed with mean Ω_1. After this time, the customer no longer requires service and the feeder pair becomes spare. Let the parameters λ_2 and Ω_2 denote the mean arrival rate and the mean occupancy time in SA 2.

Relief would be required if at any time the total demand in the two serving areas exceeds the total available feeder pairs. Recommitment (or relief) would be required if the demand in either serving area exceeds the sum of the number of pairs dedicated to the area and the number of multipled pairs. We use the expected time to relief or recommitment, denoted by $E(\tau)$, as a criterion for evaluating different alternatives. The random variable τ denotes the actual time of relief or recommitment.

Let

$$X_{1t} = \text{number of working pairs in SA 1 at time } t,$$

$$X_{2t} = \text{number of working pairs in SA 2 at time } t,$$

with the initial conditions $X_{10} = X_{20} = 0$. Both X_{1t} and X_{2t} have Poisson distributions, and the mean value of X_{it} is $\lambda_i\Omega_i(1 - e^{-t/\Omega_i})$ for $i = 1, 2$. The expected number of customers in SA 1 and SA 2 increases in time but soon levels off (saturates) at $\lambda_1\Omega_1$ and $\lambda_2\Omega_2$, respectively. Thus, this model has saturating growth behavior.

Each pair committed to SA 1 and SA 2 either can be dedicated to any one serving area or can be multipled between the two areas. Let (see Fig. 1)

$$A_1 = \text{number of pairs dedicated to SA 1}$$

$$A_2 = \text{number of pairs dedicated to SA 2}$$

$$M = \text{number of pairs multipled.}$$

Relief or recommitment will not be required before time t (i.e., $\tau > t$) if all the following conditions are satisfied for all $\psi \in [0, t]$:

(*i*) $X_{1\psi}$ is less than $A_1 + M$, the number of pairs available to SA 1
(*ii*) $X_{2\psi}$ is less than $A_2 + M$, the number of pairs available to SA 2
(*iii*) $X_{1\psi} + X_{2\psi}$ is less than $A_1 + A_2 + M$, the total number of pairs available to both SA 1 and SA 2.

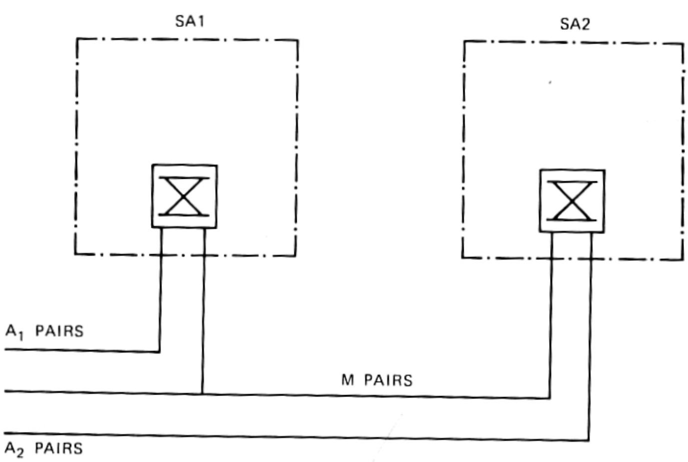

Fig. 1—Multipling under the serving area concept.

Thus the probability, Q_t, that relief or recommitment will not be required before time t can be written as:

$$Q_t = P[\tau > t] = P \begin{cases} X_{1\psi} \leq A_1 + M; \\ X_{2\psi} \leq A_2 + M; \\ X_{1\psi} + X_{2\psi} \leq A_1 + A_2 + M \end{cases} \quad \text{for all } \psi \in [0, t].$$

We approximate this as follows:

$$Q_t \approx P \begin{cases} X_{1t} \leq A_1 + M; \\ X_{2t} \leq A_2 + M; \\ X_{1t} + X_{2t} \leq A_1 + A_2 + M. \end{cases} \tag{1}$$

This relationship is exact if X_{1t} and X_{2t} are nondecreasing in t and is a very good approximation if the downward jumps in X_{1t} and X_{2t} are small. Procedures for computing Q_t and $E(\tau)$ are derived in the appendix.

To study the effect of multipling on $E(\tau)$, we let the total number of feeder pairs allocated to the two serving areas be a fixed number, N. We also let $\lambda_1 = \lambda_2$ and $\Omega_1 = \Omega_2$, so that for any value of M (the number of multipled pairs), A_1 and A_2 (the number of dedicated pairs) can both be set equal to $(N - M)/2$ to maximize $E(\tau)$. We can now vary M from 0 to N and compute $E(\tau)$ for each value of M. The multipling level can be expressed as a percentage of the total number of feeder pairs:

$$\text{multipling level} = \frac{M}{N} \times 100\%.$$

Figure 2 shows a plot of $E(\tau)$ vs multipling level for $N = 150$ pairs, $\lambda_1 = \lambda_2 = 100$ per year, and $\Omega_1 = \Omega_2 = 1$ year. (Unless otherwise specified, these values of the parameters are used in all the following graphs.) The plot shows that $E(\tau)$ increases monotonically with the level of multipling. However, observe that the increase in $E(\tau)$ even at 100 percent multipling is small, i.e., about 10 percent. Also, most of this increase in $E(\tau)$ is obtained in the first 10 to 20 percent of multipling.

In addition to $E(\tau)$, Q_t is also an important measure of the effectiveness of multipling. For example, if t is the planned future relief or recommitment date, then Q_t is the probability that premature (unplanned) recommitment or relief work will not be required before that date.

Figure 3 shows a plot of Q_t vs multipling level for various values of t. For all values of t, Q_t increases monotonically with the level of multipling. Again, most of the increase in the value of Q_t is obtained in the first 10 to 20 percent of multipling.

The observations made above indicate that the incremental usefulness of multipling, as measured by $E(\tau)$ and Q_t, decreases rapidly after the first 10 to 20 percent. Since the problems associated with multipling

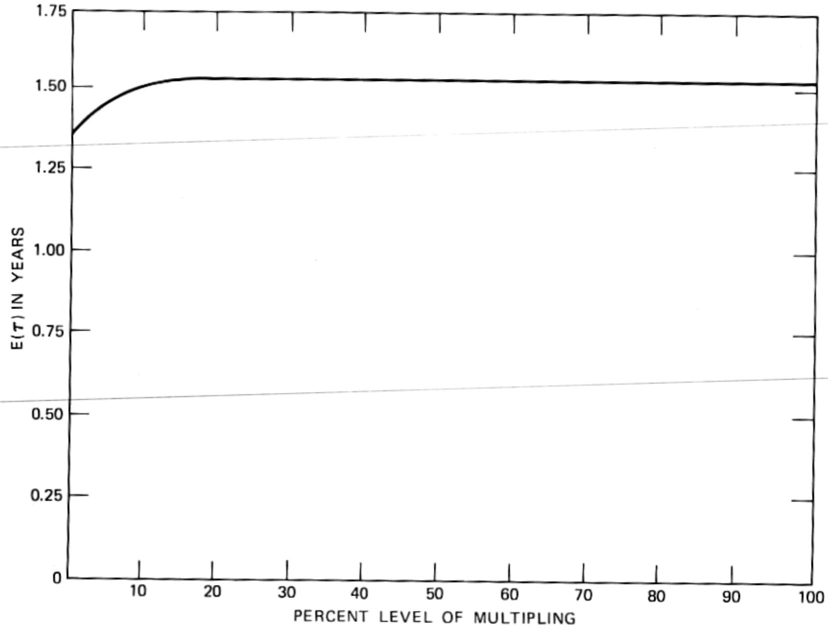

Fig. 2—Mean plant life vs level of multipling.

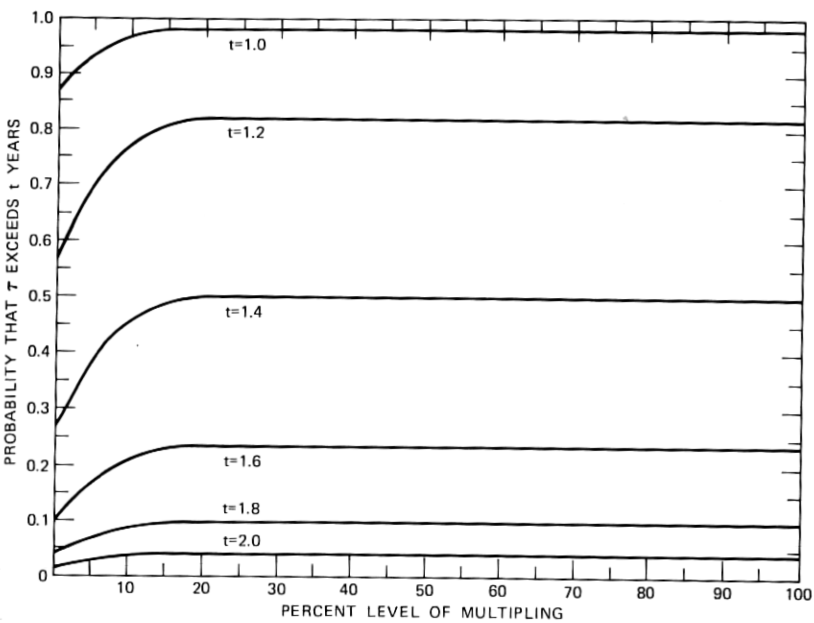

Fig. 3—Probability that plant life exceeds t years vs level of multipling.

can be expected to increase with the level of multipling, it appears reasonable to limit multipling to 10 to 20 percent.

III. COMMITMENT ERRORS AND MULTIPLING

In the foregoing discussion, we assumed that the growth parameters λ_1, λ_2, Ω_1, and Ω_2 are known sufficiently accurately. We also assumed that $\lambda_1 = \lambda_2$ and $\Omega_1 = \Omega_2$ so that, for any multipling level, optimal values of A_1 and A_2 can be determined easily. Often these two assumptions do not hold, and two types of errors can occur:

(*i*) Errors in Forecasting: The parameters λ_1, λ_2, Ω_1, and Ω_2 are not known accurately enough to determine optimal values of A_1 and A_2.

(*ii*) Errors in Commitment Strategy: Even if the parameters are known accurately, if $\lambda_1 \neq \lambda_2$ or $\Omega_1 \neq \Omega_2$, it is not clear how the optimal values of A_1 and A_2 can be determined, and thus there can be error in the commitment strategy.

The effect of these two errors is that the values of A_1 and A_2 chosen may not be optimal, resulting in a smaller $E(\tau)$. The effect of non-optimal commitment can be studied by varying A_1 and A_2 at different levels of multipling and computing $E(\tau)$ in each case.

Figure 4 shows $E(\tau)$ plotted vs $A_1/(A_1 + A_2)$, the fraction of non-multipled pairs committed to SA 1, at different levels of multipling. The plot shows that $E(\tau)$ is quite sensitive to variations in pair

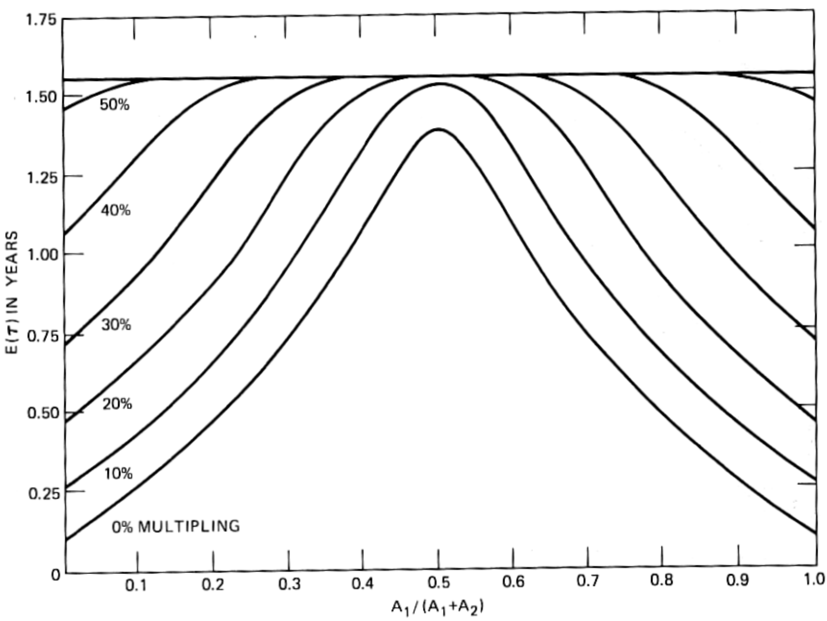

Fig. 4—Mean plant life vs fraction of pairs committed to SA 1.

commitments when there is no multipling, but this sensitivity decreases remarkably as the level of multipling increases. Thus, multipling provides some "buffering" against commitment errors.

Let A_1^* and A_2^* be the optimal commitments at a multipling level M. Also let $\hat{A}_1$ and $\hat{A}_2$ be the actual commitments, which are possibly different from A_1^* and A_2^* because of the two types of error mentioned above. We can express the error in pair commitment, ϵ, as

$$\epsilon = \left(\frac{|A_1^* - \hat{A}_1|}{A_1^*} + \frac{|A_2^* - \hat{A}_2|}{A_2^*} \right) \times 100\%.$$

Figure 5 shows a plot of $E(\tau)$ vs multipling level at various values of ϵ. Here it is assumed that the optimal values A_1^* and A_2^* follow from the condition $A_1^* = A_2^*$. By varying the values of $\hat{A}_1$ and $\hat{A}_2$, we get different values of ϵ. From the plot we see that, at high levels of commitment error, multipling does provide substantial increase in the value of $E(\tau)$.

$E(\tau)$ attains its maximum value, denoted by $E_{100}(\tau)$, at 100-percent multipling. But this value can also be achieved at levels less than 100 percent. Define μ_{100} to be the smallest multipling level at which $E(\tau) = E_{100}(\tau)$, and μ_{90} to be the smallest multipling level at which $E(\tau) =$ 90% of $E_{100}(\tau)$. The values μ_{100} or μ_{90} can be thought of as the maximum useful multipling level.

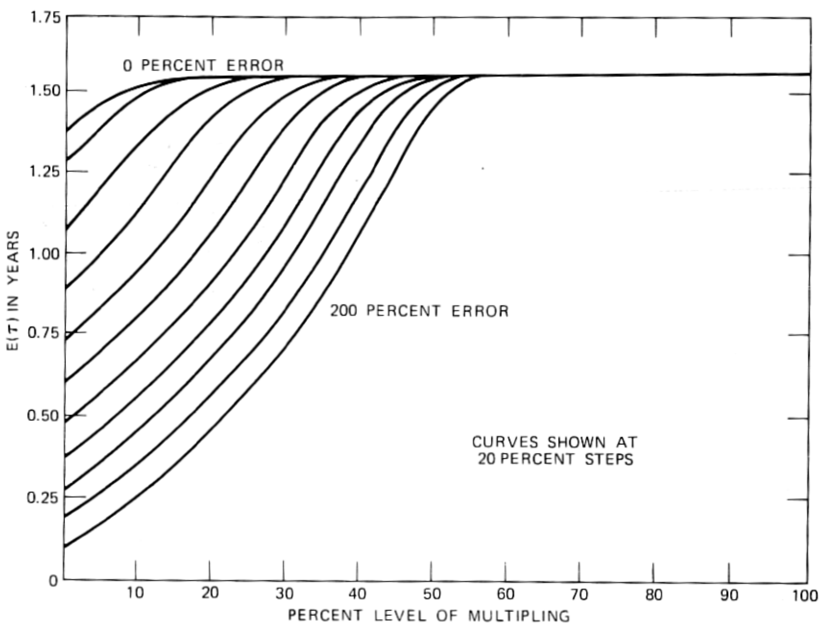

Fig. 5—Mean plant life vs level of multipling at different error levels.

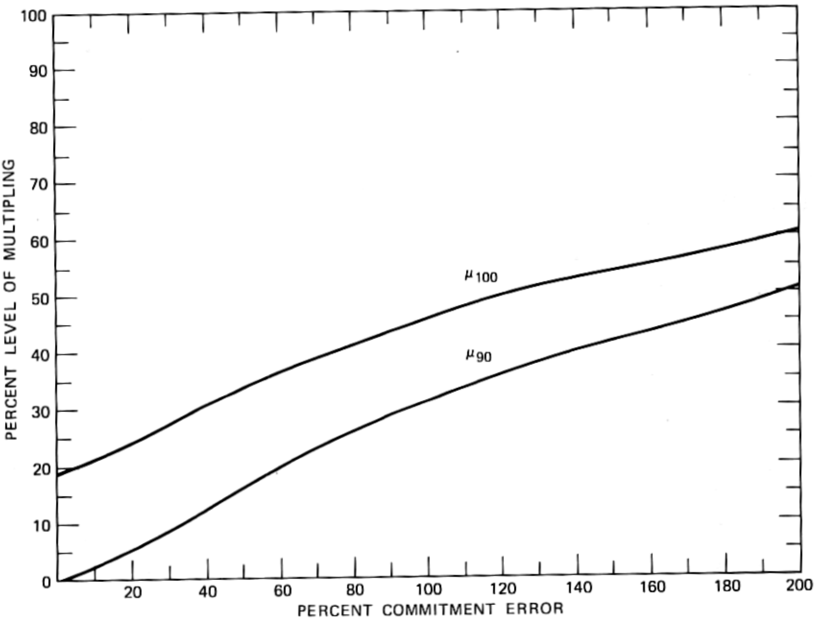

Fig. 6—Level of multipling vs commitment error.

Figure 6 shows μ_{100} and μ_{90} plotted against ϵ. Observe that both μ_{100} and μ_{90} increase (approximately) linearly with ϵ. Also note that, to obtain a 10-percent increase in $E(\tau)$, the multipling level has to be increased from μ_{90} to μ_{100}, which is a substantial increase. For error levels of up to about 40 percent, μ_{90} is less than 15 percent. Thus, even in the presence of moderate commitment errors, 10 to 20 percent is a reasonable upper limit in the level of useful multipling.

IV. SENSITIVITY ANALYSIS

To test the robustness of the results presented above, several perturbations were made in the model. The results of the sensitivity analysis are described below.

V. OCCUPANCY TIME DISTRIBUTION

In the original model, we assumed that the occupancy times were exponentially distributed. One possible perturbation would be to assume different distributions for occupancy times. We consider three distributions.

The gamma distribution is an excellent choice, and is preferred to the exponential distribution whenever analytically tractable. The gamma density has a peak at a positive value and a thicker tail.

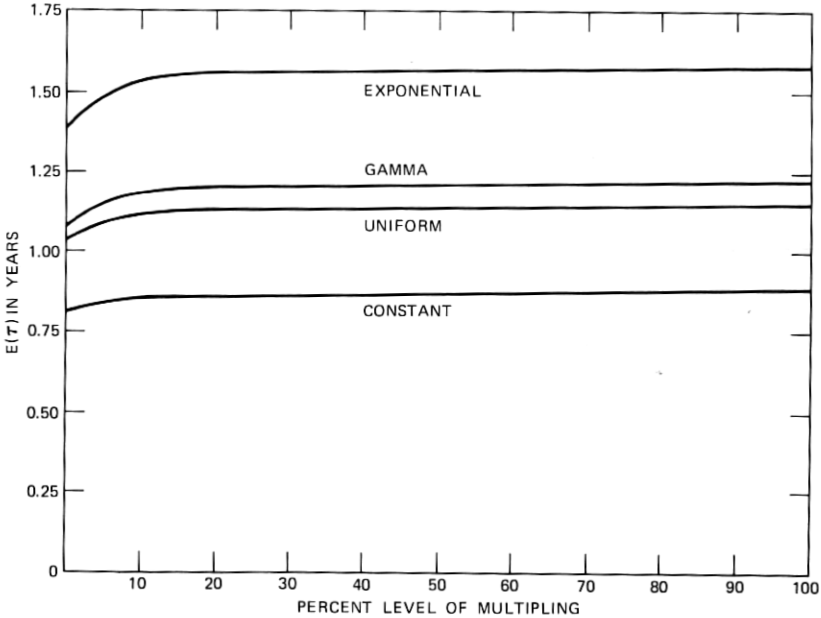

Fig. 7—Mean plant life vs level of multipling for various occupancy time distributions.

A constant occupancy time of one year is also a good choice in certain areas, such as those which contain a large number of apartments or areas near colleges and universities.

We also consider the uniform distribution (from 0 to 2 years) for the sake of completeness. Changes in the expressions for Q_t for these distributions are discussed in the appendix.

Figure 7 shows plots of $E(\tau)$ vs multipling level for the four distributions—exponential, gamma, constant, and uniform. Here again we see the "diminishing returns" nature of multipling independent of the occupancy time distribution. Note, however, that at any given multipling level $E(\tau)$ does depend on the distribution chosen.

VI. LINEAR GROWTH

If we let the mean occupancy time become very large, then the demand for service in the serving area grows linearly, independent of the occupancy time distribution. Most models of the loop plant assume linear growth characteristics.

Under the linear growth assumption, both X_{1t} and X_{2t} (the number of working pairs in SA 1 and SA 2 at time t) increase monotonically in t, and therefore approximation (1) for Q_t is exact.

$E(\tau)$ is plotted against multipling level in Fig. 8. We see that the linear growth model further confirms the upper limit of 10 to 20 percent.

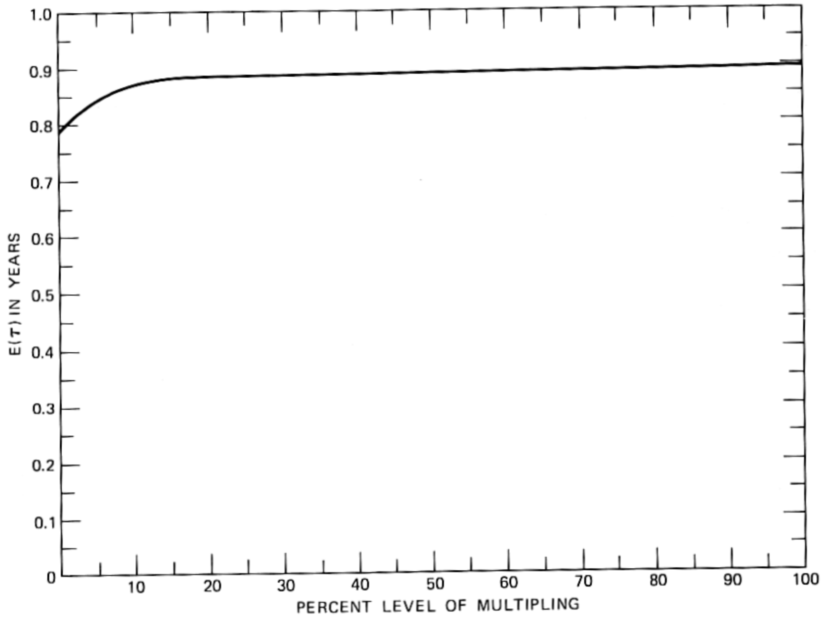

Fig. 8—Mean plant life vs level of multipling—linear growth case.

Since approximation (1) is exact in this case, and the results are identical to the previous cases considered, we conclude that (1) gives a good approximation for Q_t.

VII. ALLOCATION

The last perturbation in the model we study is in N, the total number of feeder pairs allocated to the two serving areas.

Figure 9 shows $E(\tau)$ plotted against the level of multipling for various values of N for the saturating growth case with exponential occupancy time distribution. Observe that $E(\tau)$ increases rapidly as N increases. Also, multipling is less effective for small values of N, i.e., when the feeder pairs are underallocated. The 10- to 20-percent upper limit holds, again, at all allocation levels.

VIII. CONCLUSIONS

A probabilistic model of the interfaced loop plant is developed here. Using this model, the distribution of the time until relief or recommitment can be determined at any arbitrary level of multipling.

Two objective functions for maximization can be defined:

(*i*) Expected time until relief or recommitment.
(*ii*) Probability of no premature relief or recommitment.

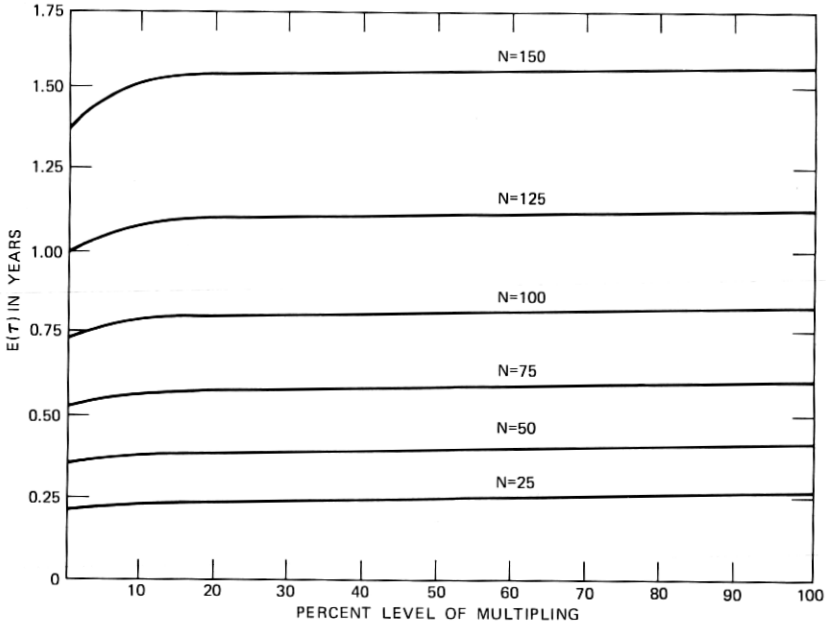

Fig. 9—Mean plant life vs level of multipling at various allocation levels.

For both objectives, we demonstrate the "diminishing returns" nature of multipling. From this, we conclude that a reasonable upper limit for multipling between serving areas is in the range 10 to 20 percent.

The results presented here are robust in the sense that the conclusions hold under very general conditions—under different occupancy time distributions, growth rates, and allocation levels.

Commitment error, which is a result of errors in forecasting or in commitment strategy, largely determines the level of multipling that is to be used. This level exceeds 15 percent only under severe error conditions.

IX. ACKNOWLEDGMENTS

Many people have contributed to the results presented in this paper, and their contributions are gratefully acknowledged. The author wishes especially to thank Warren Koontz, who suggested this area of work, Bill Bell, and Davinder Sethi.

APPENDIX

A.1 Distributions of X_{1t} and X_{2t}

Let $G_1(\cdot)$ denote the distribution of the occupancy time and λ_1 be the Poisson arrival rate in serving area SA 1. Then X_{1t}, the number of working pairs in SA 1, can be shown[3] to have the Poisson distribution with mean $\alpha_1 t$, where

$$\alpha_1 = \lambda_1 \int_0^t \frac{1 - G(x)}{t}\, dx.$$

Case 1: G_1 is exponential with mean Ω_1, i.e.,

$$G_1(t) = 1 - e^{-t/\Omega_1}.$$

α_1 is given by

$$\alpha_1 = \frac{\lambda_1 \Omega_1}{t}\,(1 - e^{-t/\Omega_1}).$$

Case 2: G_1 is gamma with mean Ω_1 and shape parameter 2, i.e.,

$$G_1(t) = 1 - \frac{1}{\Omega_1}\, e^{-2t/\Omega_1}(2t + \Omega_1).$$

This yields

$$\alpha_1 = \frac{\lambda_1 \Omega_1}{t}\,\{1 - e^{-2t/\Omega_1}(1 + t/\Omega_1)\}.$$

Case 3: G_1 is constant at Ω_1, i.e.,

$$G_1(t) = \begin{cases} 0 & \text{for } t < \Omega_1 \\ 1 & \text{for } t \geq \Omega_1. \end{cases}$$

From this, we obtain

$$\alpha_1 = \begin{cases} \lambda_1 & \text{for } t < \Omega_1 \\ \dfrac{\lambda_1 \Omega_1}{t} & \text{for } t \geq \Omega_1. \end{cases}$$

Case 4: G_1 is uniform $[0, 2\Omega_1]$, i.e.,

$$G_1(t) = \begin{cases} \dfrac{t}{2\Omega_1} & \text{for } 0 \leq t < 2\Omega_1 \\ 1 & \text{for } t \geq 2\Omega_1. \end{cases}$$

This yields

$$\alpha_1 = \begin{cases} \lambda_1 \left(1 - \dfrac{t}{4\Omega_1}\right) & \text{for } 0 \leq t < 2\Omega_1 \\ \dfrac{\lambda_1 \Omega_1}{t} & \text{for } t \geq 2\Omega_1 \end{cases}$$

The distributions of X_{2t} for these cases can be similarly derived.

A.2 Expression for Q_t

We have

$$Q_t = P\{\tau > t\}$$

$$= P\begin{cases} X_{1t} \leqslant A_1 + M; \\ X_{2t} \leqslant A_2 + M; \\ X_{1t} + X_{2t} \leqslant A_1 + A_2 + M, \end{cases}$$

where X_{1t} is Poisson-distributed with mean $\alpha_1 t$ and X_{2t} is Poisson-distributed with mean $\alpha_2 t$. Conditioning on the event

$$X_{1t} = k,$$

we have

$$Q_t = \sum_{k=0}^{A_1+M} P\{X_{1t} = k\}.P\begin{cases} X_{2t} \leqslant A_2 + M; \\ \\ X_{2t} + k \leqslant A_1 + A_2 + M. \end{cases}$$

$$= \sum_{k=0}^{A_1+M} \left\{ \frac{e^{-\alpha_1 t}(\alpha_1 t)^k}{k!} \cdot \left(\sum_{i=0}^{\theta_k} \frac{e^{-\alpha_2 t}(\alpha_2 t)^i}{i!} \right) \right\}$$

where

$$\theta_k = \min(A_2 + M, \quad A_1 + A_2 + M - k).$$

Computation of Q_t is now straightforward.

A.3 Mean time till blockage, $E(\tau)$

The time till blockage τ is a positive-valued random variable. Therefore, we can write

$$E(\tau) = \int_0^\infty P\{\tau > t\}\, dt$$

$$= \int_0^\infty Q_t\, dt.$$

Thus, $E(\tau)$ can be evaluated using numerical integration methods.

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