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A Note Concerning Optical-Waveguide Modulation Transfer Functions

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A necessary and sufficient condition is given for the modulation-transfer-function of certain multimode optical fiber guides to be zero free in the closed right-half of the complex plane, and to be structurally stable with respect to that property. The condition is of interest, for example, in connection with the possibility of determining the phase of a modulation-transfer-function from its amplitude.

I. INTRODUCTION AND PRELIMINARIES

Reference 1 considers the range of validity of a Hilbert-transform approach in which the measured magnitude of the modulation-transfer-function of an optical fiber guide is used to compute the guide's impulse response.* It is argued there that a key "minimum-phase assumption" can fail to be satisfied in important cases, and a few closely related experimental and analytical results are presented.

The purpose of this note is to report on a result along the same lines as a proposition given in Ref. 1 to the effect that, for a fiber guide that can propagate a finite number of discrete modes without mode mixing, the modulation-transfer-function (more precisely, the Laplace transform version of the modulation-transfer-function) is zero-free in the closed right half of the complex plane, and that property is structurally stable in a certain sense, if and only if a certain condition is met. The theorem described in Section II is concerned with a more realistic and far more interesting case in which mode mixing is not ruled out. In particular, the result provides further detailed support for the conclusion reached in

* By "the guide's impulse response" is meant the output power of the guide excited by a unit impulse of optical power. The modulation-transfer-function $G(\omega)$ is the envelope response of the fiber guide to an incoherent optical signal sinusoidally modulated at angular frequency ω. To the extent that certain approximations hold, (Ref. 2), the impulse response is the Fourier transform of the modulation-transfer-function. The reason for considering the Hilbert transform approach is that it is often desirable to determine the impulse response of fiber guides by methods other than direct time-domain measurement, and, while $|G(\omega)|$ can be measured easily, it is at the present time difficult to accurately measure the phase of $G(\omega)$. (See Refs. 3 and 4.)

Ref. 1. (i.e., that "nonminimum-phase behavior" is likely to arise, and can arise, in important actual cases). As in Ref. 1, it is the introduction and use of the concept of structural stability of a mathematical property which, for the case considered, enables a result to be obtained that is complete and easy to interpret.*

We consider, as in Ref. 1, a class of optical fiber guides for which appropriate approximations can be made so that the modulation-transfer-function $G(\omega)$ of a guide can be written in the form

$$G(\omega) = \int_T e^{-i\omega\tau} da(\tau), \tag{1}$$

in which T denotes a closed, finite, real interval whose end points depend on the refractive indices of the core and cladding, and $a(\tau)$ is a real-valued nondecreasing function.† It is assumed throughout that $a(\tau)$ satisfies the normalization condition

$$\int_T da(\tau) = 1.$$

As mentioned previously, in Ref. 1 attention is focused on the class of fiber guides that can propagate n discrete modes without mode mixing.‡ In that case, $G(\omega)$ can be written as

$$\sum_{j=1}^{n} d_j e^{-i\omega\tau_j}, \tag{2}$$

in which each d_j is a positive number that represents the initial excitation of the jth mode, and $\tau_1 < \tau_2 < \ldots < \tau_n$.

In this note we suppose that the right side of (1) can be expressed as

$$\sum_{j=1}^{k_F} d_j e^{-i\omega\tau_j} + \int_T e^{-i\omega\tau} b(\tau) d\tau, \tag{3}$$

in which: k_F is a positive integer (the motivation for using the subscript F will become clear shortly), $d_j > 0$ for $1 \le j \le k_F$, $\tau_1 < \tau_2 < \ldots < \tau_{k_F}$, and $b(\tau)$ is a bounded piecewise-continuous§ nonnegative function which takes into account mode mixing. It is also assumed that τ_1 has the following property: $b(\tau) = 0$ for $\tau \in T$ with $\tau < \tau_1$ (which, of course, allows the possibility, but does not require, that τ_1 is equal to the lower endpoint

* The general problem of determining when the Hilbert-transform approach (i.e., when the so-called Kramers-Kronig transformation) is valid is of interest in many fields (see, for example, Ref. 5).

† Thus, roughly speaking, $da(\tau)$ in (1) can be replaced with $f(\tau)d\tau$ in which the function $f(\tau)$ is nonnegative and may contain impulses corresponding to discrete modes. See Refs. 2 and 3 for the relevant background material. We are assuming that material dispersion can be neglected.

‡ Typically, $n > 100$.

§ The piecewise-continuity assumption appears to be adequate for applications. For basically a somewhat more general version of the theorem stated in Section II, see Section 2.2.

of T). Since (3) is a specialization of the right side of (1), we also have $\tau_j \varepsilon T$ for each j, as well as

$$\sum_{j=1}^{k_F} d_j + \int_T b(\tau)d\tau = 1.$$

In physical terms, b is the relative-power density function associated with the nondiscrete modes, and the idealized impulse response of the guide is the inverse Fourier transform of (3). The observable impulse response of the guide (i.e., the impulse response of the guide-detector combination) is a somewhat smoothed version of the idealized response, with the smoothing provided by the detector (see Ref. 3).

The important assumption that $d_1 > 0$ and that $b(\tau) = 0$ for $\tau \varepsilon T$ with $\tau < \tau_1$ means that a discrete mode corresponding to the smallest modal delay is propagated. This assumption appears to be reasonable for at least some interesting classes of guides. For example, if a guide with a step-index profile is short enough to neglect mode conversion phenomena, then it is not unreasonable to assume that $G(\omega)$ has the form given in (2) with τ_1 the modal delay corresponding to the fundamental mode. In a real fiber, geometrical perturbations couple energy among the modes so that the distribution of modal delays changes continuously from a discrete set to a continuum as the fiber length L increases. Experimental evidence indicates that the assumption is reasonable at least if the guide is not too long.* (For a particular fiber, there is a characteristic coupling length L_c such that for $L > L_c$, it is difficult in the time domain to isolate discrete modes with appreciable energy.)

II. THE RESULT

In this section, z denotes a complex variable and $F(z)$ is defined by

$$F(z) = \sum_{j=1}^{k_F} d_j e^{-z\tau_j} + \int_T e^{-z\tau} b(\tau)d\tau \tag{4}$$

for each z. (Of course, if $G(\omega)$ denotes (3), then $G(\omega) = F(i\omega)$, and $F(z)$ is simply the Laplace transform of the generalized function whose Fourier transform is $G(\omega)$.)

In order to state our result, consider an arbitrary function $H(z)$ of the same type as $F(z)$. More explicitly, let $H(z)$ be given for all z by

$$H(z) = \sum_{j=1}^{k_H} \delta_j e^{-zt_j} + \int_T e^{-z\tau} \beta(\tau)d\tau, \tag{5}$$

in which k_H is a positive integer (not necessarily equal to k_F), and the δ_j, the t_j, and $\beta(\tau)$ satisfy the restrictions imposed on the corresponding terms in (4). Let S denote the set of all such functions $H(z)$.

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* The writer is indebted to his colleague I. P. Kaminow for a helpful discussion concerning the significance of the assumption described above.

For the purpose of defining a "distance" between F and an arbitrary element H of S, let J_F, J_H, and J, respectively, denote the sets of numbers $\{1,2,\ldots,k_F\}$, $\{1,2,\ldots,k_H\}$, and $\{1,2,\ldots,\min(k_F,k_H)\}$, and, with y a real variable, let $q(y)$ denote any continuous nondecreasing function of y such that $q(0) = 0$.

Let the "distance" $\rho(F,H)$ between F and any H in S be defined by*

$$\rho(F,H) = \sum_{j \varepsilon J} |d_j - \delta_j| + \sum_{j \varepsilon (J_F - J)} d_j + \sum_{j \varepsilon (J_H - J)} \delta_j$$
$$+ q\left(\max_{j \varepsilon J} |\tau_j - t_j|\right) + \max_{u,v \varepsilon T} \left| \int_u^v [b(\tau) - \beta(\tau)] d\tau \right|. \quad (6)$$

Each term on the right side of (6) has a direct interpretation. In particular,

$$\sum_{j \varepsilon (J_F - J)} d_j + \sum_{j \varepsilon (J_H - J)} \delta_j,$$

in which at most one sum is nonzero, reflects the extent to which terms in one of the two finite sums in (4) and (5) do not have counterparts in the other. Also,

$$\int_u^v [b(\tau) - \beta(\tau)] d\tau$$

is an integral of the difference of two power-density functions, and, roughly speaking, if

$$\max_{u,v \varepsilon T} \left| \int_u^v [b(\tau) - \beta(\tau)] d\tau \right|$$

is sufficiently small, then, for practical purposes, the functions b and β are indistinguishable in the sense that the observable impulse response of the guide is essentially unchanged if b is replaced with β. (The portion of the idealized impulse response that does not contain impulses is g defined by $g(t) = b(t)$ for $t \varepsilon T$ and $g(t) = 0$ for $t \not\varepsilon\ T$. If, for example, the smoothing introduced by the detector is modeled by a filter with impulse response r given by $r(t) = \rho^{-1}$ for $t \varepsilon [0, \rho]$ and $r(t) = 0$ otherwise, in which ρ is a small positive constant, then the observable version of $g(t)$ is $\rho^{-1} \int_{t-\rho}^{t} g(\tau) d\tau$ for each t. Similarly, if instead $r(t) = 0$ for $t < 0, r(0)$ is finite, and the derivative of r is absolutely integrable on $[0,\infty)$, then an integration by parts shows that the observable version of g is essentially unchanged when b is replaced with a sufficiently nearby β in the sense indicated above.)

Our result, the theorem given below, provides an answer to the following question: When is it true that $F(z) \neq 0$ for $\mathrm{Re}(z) \geq 0$ *and* that

* We adopt the convention that a sum over the empty set is zero.

property of F is *structurally stable* in the sense that there is a positive constant ϵ (which can be thought of as a "tolerance") such that $H(z) \neq 0$ for $\mathrm{Re}(z) \geq 0$ for every H in S such that $\rho(F,H) \leq \epsilon$.

Theorem: We have $F(z) \neq 0$ for $Re(z) \geq 0$ with that property of F structurally stable, if and only if

$$d_1 > \sum_{\substack{j \, \varepsilon \, J_F \\ j \neq 1}} d_j + \int_T b(\tau) d\tau.$$

Note that the theorem* does not rule out the possibility that the condition given in the theorem is violated and $F(z)$ is zero-free in the closed right-half plane. (In fact, an example given in Ref. 1 shows that the possibility can occur. Essentially the same example can be used to show also that if the condition is violated, then it need not be the case that *all* functions in S "sufficiently close" to but different from F have a zero in $\mathrm{Re}(z) \geq 0$.) On the other hand, at present it appears that there are complex, and for practical purposes impossible-to-specify, additional relationships among the τ_j, the d_j, and b that, in particular take into account geometrical perturbations along the length of a real guide. The theorem shows that, when additional information is unavailable, it is not possible to prove that $F(z)$ is zero-free in the closed right-half plane whenever

$$d_1 < \sum_{\substack{j \, \varepsilon \, J_F \\ j \neq 1}} d_j + \int_T b(\tau) d\tau$$

(which, in view of the normalization condition concerning a, is equivalent to the statement that the discrete mode with the smallest delay has relative power less than ½) and, in the sense indicated above, the τ_j, the d_j, and b are known only to within some tolerance, no matter how small the tolerance is.

A proof of the theorem is given in the next section.

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* As indicated earlier, one application of the theorem is that it provides further detailed support for the material reported on in Ref. 1. For the benefit of the reader who has not read Ref. 1, we mention that a much simplified version of essentially the proof given in Section 2.1 can be used to show that if $k_F \geq 2$, if $b(\tau)$ and $\beta(\tau)$ in (4) and (5), respectively, are each replaced by the zero function, if S is further restricted so that $k_H = k_F$ and $\delta_j = d_j$ for $j = 1,2,\ldots, k_F$ for all $H \, \varepsilon \, S$, and if $\rho(F,H)$ is instead $q(max_{j \, \varepsilon \, J} |\tau_j - t_j|)$ [i.e., just the fourth term in the sum on the right side of (6)], then: $F(z) \neq 0$ for $\mathrm{Re}(z) \geq 0$ and there is an $\epsilon > 0$ such that $H(z) \neq 0$ for $\mathrm{Re}(z) \geq 0$ for every H in the corresponding S with $\rho(F,H) \leq \epsilon$, if and only if $d_1 > \sum_{j=2}^{k_F} d_j$. This result is basically a slight generalization of the comparable proposition in Ref. 1.

2.1 Proof of the theorem

For the reader's convenience, we first repeat some of the material described above. We have $T = [T_1, T_2]$ in which T_1 and T_2 are real numbers such that $T_1 < T_2$, and S denotes the set of all functions of the complex variable z of the form (5) where k_H is a positive integer, $\delta_j > 0$ for $1 \leq j \leq k_H$, $T_1 \leq t_1 < t_2 < \ldots < t_{k_H} \leq T_2$, $\beta(\tau)$ is a nonnegative bounded piecewise-continuous* function defined on T such that $\beta(\tau) = 0$ for $\tau \, \varepsilon \, T$ with $\tau < \tau_1$, and

$$\sum_{j=1}^{k_H} \delta_j + \int_T \beta(\tau) d\tau = 1.$$

The "distance" $\rho(F,H)$ between any $H \, \varepsilon \, S$ and the particular element F of S given by (4), is defined by (6).

Proof of the "If" Part: Suppose that

$$d_1 > \sum_{\substack{j \, \varepsilon \, J_F \\ j \neq 1}} d_j + \int_T b(\tau) d\tau.$$

With

$$r = d_1 - \sum_{\substack{j \, \varepsilon \, J_F \\ j \neq 1}} d_j - \int_T b(\tau) d\tau,$$

let ϵ satisfy $0 < \epsilon < (1/4)r$. For each $H \, \varepsilon \, S$ with $\rho(F,H) \leq \epsilon$, we see that $|d_1 - \delta_1| \leq \epsilon$,

$$\sum_{\substack{j \, \varepsilon \, J \\ j \neq 1}} |d_j - \delta_j| \leq \epsilon, \quad \sum_{j \, \varepsilon \, (J_F - J)} d_j \leq \epsilon, \quad \sum_{j \, \varepsilon \, (J_H - J)} \delta_j \leq \epsilon,$$

as well as

$$\left| \int_T b(\tau) d\tau - \int_T \beta(\tau) d\tau \right| \leq \epsilon,$$

and therefore

$$\delta_1 > \sum_{\substack{j \, \varepsilon \, J_H \\ j \neq 1}} \delta_j + \int_T \beta(\tau) d\tau.$$

Thus, for $\mathrm{Re}(z) \geq 0$ and $H \, \varepsilon \, S$ with $\rho(F,H) \leq \epsilon$, we have

$$|e^{zt_1} H(z)| = \left| \delta_1 + \sum_{\substack{j \, \varepsilon \, J_H \\ j \neq 1}} \delta_j e^{-z(t_j - t_1)} + \int_T e^{-z(\tau - t_1)} \beta(\tau) d\tau \right|$$

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* It will become evident that the theorem holds also if "piecewise continuous" is replaced with either "Riemann integrable" or "Lebesgue integrable." In this connection, see Section 2.2.

$$\geq \delta_1 - \left| \sum_{\substack{j \,\varepsilon\, J_H \\ j \neq 1}} \delta_j e^{-z(t_j - t_1)} + \int_T e^{-z(\tau - t_1)} \beta(\tau) d\tau \right|$$

$$\geq \delta_1 - \sum_{\substack{j \,\varepsilon\, J_H \\ j \neq 1}} \delta_j - \int_T \beta(\tau) d\tau. \tag{7}$$

Since the right side of (7) is positive, it is clear that $H(z) \neq 0$, which completes the proof of the "if" part.

Proof of the "Only If" Part: Suppose now that

$$d_1 \leq \sum_{\substack{j \,\varepsilon\, J_F \\ j \neq 1}} d_j + \int_T b(\tau) d\tau,$$

and let $\epsilon > 0$ be given.

Let $k_G = \max(k_F, 2)$, let $\eta = \min((1/6)\,\epsilon, (1/2)\,d_1)$, and let $\delta_1 = d_1 - \eta$. If $k_F > 1$, let $\delta_2 = d_2 + \eta$ and $\delta_j = d_j$ for $j \,\varepsilon\, \{j{:}j \,\varepsilon\, J_F; j \neq 1,2\}$, and if $k_F = 1$, let $\delta_2 = \eta$ and $\tau_2 = T_2$. Then the function G defined (for all z) by

$$G(z) = \sum_{j=1}^{K_G} \delta_j e^{-z\tau_j} + \int_T e^{-z\tau} b(\tau) d\tau$$

belongs to S, and we have (if we set $H = G$):

$$\sum_{j \,\varepsilon\, J} |d_j - \delta_j| + \sum_{j \,\varepsilon\, (J_F - J)} d_j + \sum_{j \,\varepsilon\, (J_G - J)} \delta_j \leq \frac{1}{3}\,\epsilon \tag{8}$$

and the strict inequality

$$\delta_1 < \sum_{\substack{j \,\varepsilon\, F_G \\ j \neq 1}} \delta_j + \int_T b(\tau) d\tau. \tag{9}$$

Let δ_τ denote $\min\{(\tau_{j+1} - \tau_j){:}\ 1 \leq j, j+1 \leq k_G\}$, and let $\Delta = \sup_{t \,\varepsilon\, T} b(\tau)$. With $B = \{\tau \,\varepsilon\, T{:}b(\tau) > 0\}$, let s_1 and s_2 denote inf B and sup B, respectively, when B has nonzero measure, and T_1 and T_2, respectively, otherwise.

Choose any $\delta_\epsilon > 0$ such that $q(\delta_\epsilon) \leq (1/3)\,\epsilon$, and let δ be any positive number such that

$$\delta < \min\left(\frac{1}{2}\,\delta_\epsilon, \frac{1}{4}\,\delta_\tau, \frac{1}{3}\,\epsilon, \epsilon(18\Delta)^{-1}, \frac{1}{4}\,(s_2 - s_1)\right). \tag{10}$$

Let $\omega = \pi\delta^{-1}$, and let K_δ denote the set of numbers of the form $\tau_1 + k\delta$, with k an odd positive integer. Clearly, $\exp[-i\omega(t - \tau_1)] = -1$ for $t \,\varepsilon\, K_\delta$.

Choose $t_1 = \tau_1$, and (using $k_G \geq 2$ and $2\,\delta < (\frac{1}{2})\,\delta_\tau$) for each $j = 2,3,\ldots,k_G$ choose a t_j in $K_\delta \cap [\tau_1,T_2]$ such that $|\tau_j - t_j| \leq 2\delta$. Since $2\delta < (\frac{1}{2})\,\delta_\tau$ and $2\delta < \delta_\epsilon$, we have $t_1 < t_2 < \ldots < t_{k_G}$ and $q(\max_{j\,\varepsilon\,J}|\tau_j - t_j|) \leq (\frac{1}{3})\epsilon$.

We see that the "distance" between F and the element E of S given by

$$E(z) = \sum_{j=1}^{k_G} \delta_j e^{-zt_j} + \int_T e^{-z\tau} b(\tau)d\tau$$

is at most $(\frac{2}{3})\,\epsilon$. It therefore suffices to show that there is an H in S defined by

$$H(z) = \sum_{j=1}^{k_G} \delta_j e^{-zt_j} + \int_T e^{-z\tau} \beta(\tau)d\tau$$

with

$$\max_{u,v\,\varepsilon\,T}\left|\int_u^v [b(\tau) - \beta(\tau)]d\tau\right| \leq \frac{1}{3}\epsilon$$

such that $H(z) = 0$ for some z with $\mathrm{Re}(z) > 0$.

Let $L = K_\delta \cap (s_1,s_2)$. Since $\delta < \frac{1}{4}(s_2 - s_1)$,† L contains at least two points. Let the points in L be $p_1,p_2,\ldots,p_n$, ordered so that $p_1 < p_2 < \ldots < p_n$. Let σ be a positive number such that $\sigma < \delta$, $p_1 - s_1 > \sigma$, and $s_2 - p_n > \sigma$. With $I(u,v)$ denoting $\int_u^v b(\tau)d\tau$ for $u,v\;\varepsilon\;[s_1,s_2]$, let $\beta_\sigma(\tau)$ be defined for $\tau\;\varepsilon\;T$ by $\beta_\sigma(\tau) = f(t - p_1)I(s_1,p_1 + \delta) + f(t - p_2)I(p_2 - \delta, p_2 + \delta) + \ldots + f(t - p_n)I(p_n - \delta,s_2)$, in which $f(t) = (2\sigma)^{-1}$ for $|t| \leq \sigma$ and $f(t) = 0$ otherwise. Since

$$\int_T \beta_\sigma(\tau)d\tau = I(s_1,p_1 + \delta) + I(p_2 - \delta,p_2 + \delta) + \ldots + I(p_n - \delta,s_2) = \int_T b(\tau)d\tau, \quad (11)$$

we see that the function H_σ defined by

$$H_\sigma(z) = \sum_{j=1}^{k_G} \delta_j e^{-zt_j} + \int_T \beta_\sigma(\tau)e^{-z\tau}d\tau$$

belongs to S.

Using $I(s_1,p_1 + \delta) \leq 3\delta\Delta$,‡ $I(p_j - \delta,p_j + \delta) \leq 2\delta\Delta$ for $j = 2,\ldots,(n - 1)$, $I(p_n - \delta,s_2) \leq 3\delta\Delta$,

$$\int_{T_1}^t b(\tau)d\tau = \int_{T_1}^t \beta_\sigma(\tau)d\tau$$

for $t = s_1,p_1 + \delta,p_2 + \delta,\ldots,p_{n-1} + \delta,s_2$, and the fact that $b(\tau)$ and $\beta_\sigma(\tau)$ are nonnegative, we have

$$\left|\int_{T_1}^t b(\tau)d\tau - \int_{T_1}^t \beta_\sigma(\tau)d\tau\right| \leq 3\delta\Delta$$

† See (10).
‡ Notice that $p_1 - s_1 \leq 2\delta$.

for all $t \varepsilon T$. It follows at once that for $u,v \varepsilon T$,

$$\left| \int_u^v [b(\tau) - \beta_\sigma(\tau)]d\tau \right| \leq 6\delta\Delta$$

$$< \frac{1}{3} \epsilon.^\dagger$$

Therefore $\rho(F,H_\sigma) \leq \epsilon$, uniformly for σ as described.

Let $P(z)$ be defined for all z by

$$P(z) = \sum_{j=1}^{k_G} \delta_j e^{-zt_j} + e^{-zp_1} I(s_1,p_1 + \delta)$$

$$+ e^{-zp_2} I(p_2 - \delta, p_2 + \delta) + \ldots + e^{-zp_n} I(p_n - \delta, s_2).$$

Let α be a real variable. Since $t_j \varepsilon K_\delta$ for $j = 2,3,\ldots,k_G$, and $p_j \varepsilon K_\delta$ for $j = 1,2,\ldots,n$, and (9) and (11) hold, $P(\alpha + i\omega) \exp[(\alpha + i\omega)t_1]$‡ is real and negative when $\alpha = 0$. On the other hand, $P(\alpha + i\omega) \exp[(\alpha + i\omega)t_1]$ is positive for all sufficiently large α. Thus, $P(z_1) = 0$ for some z_1 with $\mathrm{Re}(z_1) > 0$.

The function $P(z)$ is analytic in z throughout the complex plane. Since it is not identically zero and is analytic at $z = z_1$, its zero at $z = z_1$ is isolated. Therefore, there exists a constant $r > 0$ such that $r < \mathrm{Re}(z_1)$ and, with $\Gamma = \{z:|z - z_1| = r\}$, $P(z) \neq 0$ for $z \varepsilon \Gamma$. It follows that min $\{|P(z)|:z \varepsilon \Gamma\}$ is positive.

Using the fact that

$$(2\sigma)^{-1} \int_{t-\sigma}^{t+\sigma} e^{-z\tau} d\tau = e^{-zt} w(\sigma z),$$

in which $w(\sigma z) = (2\sigma z)^{-1}(e^{z\sigma} - e^{-z\sigma})$, we have

$$H_\sigma(z) = P(z) + [W(\sigma z) - 1]y(z), \tag{12}$$

where

$$y(z) = e^{-zp_1} I(s_1,p_1 + \delta) + e^{-zp_2} I(p_2 - \delta, p_2 + \delta)$$

$$+ \ldots + e^{-zp_n} I(p_n - \delta, s_2).$$

The function $|y(z)|$ is bounded on Γ, and max $\{|w(\sigma z) - 1|:z \varepsilon \Gamma\}$ (and hence max $\{|[w(\sigma z) - 1]y(z)|:z \varepsilon \Gamma\}$) can be made arbitrarily small by choosing σ to be sufficiently small. By Rouché's theorem, for sufficiently small σ, $H_\sigma(z)$ has a zero inside Γ (and hence in $\mathrm{Re}(z) > 0$). This completes the proof.

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† See (10).
‡ Recall that $\omega = \pi\delta^{-1}$.

2.2 Comment

It is apparent that we have proved a somewhat stronger result than the theorem stated. Suppose that the definition of S is changed to the extent that $\beta(\tau)$ need not be piecewise continuous, but merely Lebesgue integrable. Then it it is clear that the "if" part of the theorem remains true. More importantly, the proof shows that if

$$d_1 < \sum_{\substack{j \in J_F \\ j \neq 1}} d_j + \int_T b(\tau)d\tau,$$

then, for any $\epsilon > 0$, there is an H in S of the form (5) with the following properties: $\rho(F,H) \leq \epsilon$, $H(z)$ has a zero in the open right-half of the complex plane, $t_1 = \tau_1, k_H = k_F, \delta_j = d_j$ for $j = 1,2,\ldots,k_F$, $\beta(\tau)$ is piecewise constant, and the smallest closed real interval containing the support of $b(\tau)$ (which might possibly be the "empty interval") also contains the support of $\beta(\tau)$.

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