THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 57, No. 8, October 1978
Printed in U.S.A.

The Construction for Symmetrical Zone-Balanced Networks

By F. K. HWANG and T. C. LIANG

(Manuscript received January 27, 1978)

For many real networks, the input and output switches can often be partitioned into subsets called zones such that switches in the same zone have a certain community of interest and are more likely to request a connection. Such a network will be called a zone-balanced network if, among other regularity conditions, the channel graph between an input switch u and an output switch v is either isomorphic to a graph G_1 if u and v are in the same zone, or isomorphic to a graph G_2 if otherwise. In this paper, we continue the study of using balanced incomplete block design for the construction of zone-balanced networks. We introduce some new methods to construct a wide class of such networks, which include some previous constructions as special cases.

I. INTRODUCTION

The topology of a switching network can often be represented by a graph by taking switches as vertices and links as edges. By this representation, a *multistage* (*switching*) *network* is a graph the vertex-set of which can be naturally partitioned into s subsets $V_1,\ldots,V_s$ and the edge-set into $s-1$ subsets $E_1,\ldots,E_{s-1}$, for some number s, so that E_i connects V_i to V_{i+1}. (We do not allow multiple edges between two vertices.) Vertices in V_1 correspond to the input switches of the network and vertices in V_s correspond to the output switches. Let $u \epsilon V_1$ and $v \epsilon V_s$. Then the *channel graph* $G(u,v)$ is the union of all paths connecting u to v in the network. A multistage network is said to be *regular* if every vertex in V_i has the same number of edges in E_{i-1} and the same number of edges in E_i. A regular multistage network is *balanced* if the channel graphs $G(u,v)$ over all $u \epsilon V_1$ and all $v \epsilon V_s$ are isomorphic.

For many real networks, the input and output switches can often be partitioned into subsets called *zones* such that switches in the same zone have a certain community of interest and are more likely to request a connection. (In this paper, we are concerned only with connection between an input switch and an output switch.) Such a network will be called a *zone-balanced network* if it is regular and there exists two graphs G_1 and G_2 so that $G(u,v)$ is isomorphic to G_1 if u and v are in the same zone and $G(u,v)$ is isomorphic to G_2 if not. G_1 and G_2 will be referred to as the *intrazone* and the *interzone* channel graphs, respectively. A zone-balanced network is said to be *symmetrical* if it is symmetrical with respect to the center stage or the two stages in the middle.

A *balanced incomplete block design* (abbreviated as BIBD) with parameters (v,b,r,k,λ) is a family of *blocks*, with each block being a k-subset of the set $\{1,2,\ldots,v\}$, satisfying the following properties:

(*i*) Every element in the set $\{1,2,\ldots,v\}$ appears in exactly r blocks.
(*ii*) Every pair of elements in the set $\{1,2,\ldots,v\}$ appears together in exactly λ blocks.

BIBDs have long been a favorite subject for mathematicians and statisticians. The reader is referred to Ref. 1 for the existence and construction for many BIBDs. The use of BIBDs for constructing zone-balanced networks was first studied in Ref. 2. Some further constructions were given in Ref. 3. In this paper, we give some methods for such constructions. The zone-balanced networks constructed previously, as well as in this paper, are all symmetrical.

II. SOME PRELIMINARY RESULTS

A zone-balanced network is called *canonical* if each zone consists of a single input switch and a single output switch. Therefore, a CZBN (canonical zone-balanced network) can be viewed as a prototype for a full-fledged network with the same interzone and intrazone channel graphs. The mechanism for expanding a CZBN into a full-fledged network is the operation of "parallel expansion," which was first introduced by Takagi[4] and by Timperi and Grillo.[5] For an s-stage network N, a (k,j) left (right) parallel expansion means taking k copies of N and identifying their subgraphs from stage j to stage s (from stage 1 to stage j). Figure 1 gives some examples of parallel expansion. It is clear that parallel expansion preserves the isomorphisms of the interzone and intrazone channel graphs.

Next we introduce a method which we will use later to describe the connection between switches in two adjacent stages. To use this method, every switch in the two adjacent stages should be labeled by a subset of a given set. Then two switches in the adjacent stages should be connected if the label of one is contained in the label of the other. This type of connection will be called a *labeled-subset connection.*

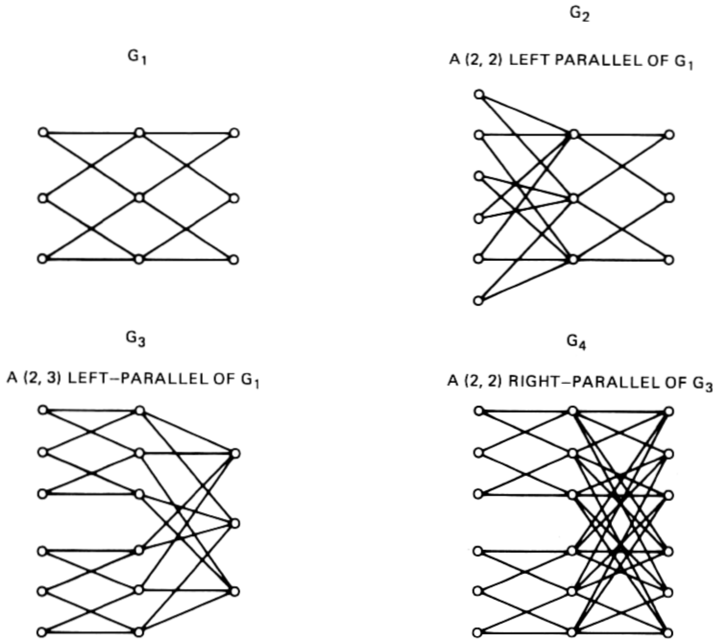

Fig. 1—Examples of parallel expansion.

III. A RECURSIVE CONSTRUCTION FOR CZBN

A 2-stage CZBN with v zones is necessarily a $v \times v$ complete bipartite graph; hence, its construction is trivial. We now give a construction for a 3-stage CZBN (noting that a 3-stage channel graph is uniquely determined by its number of paths).

Theorem 1: Suppose that a (v,b,r,k,λ)-BIBD exists. Then we can construct a 3-stage CZBN with v zones which has r paths in its intrazone channel graph and λ paths in its interzone channel graph.

Proof: Take b switches of V_2 and label each of them by a distinct block of the given BIBD. Take v switches of $V_1(V_3)$ and label each of them by a distinct element of $Z = \{1,2,\ldots,v\}$. Apply a labeled-subset connection between V_2 and $V_1(V_3)$. It is easy to verify that the resulting network is the one specified in Theorem 1.

Example 1: Let the given BIBD have parameters (7,7,3,3,1) and have blocks (1,2,4), (2,3,5), (3,4,6), (4,5,7), (5,6,1), (6,7,2), and (7,1,3). Figure 2 gives a 3-stage CZBN with 7 zones.

We now give a recursive construction for a symmetrical s-stage CZBN for $s \geq 4$.

Theorem 2: Suppose that an s-stage CZBN with k zones exists which has G_1 and G_2 as its intrazone and interzone channel graphs. Fur-

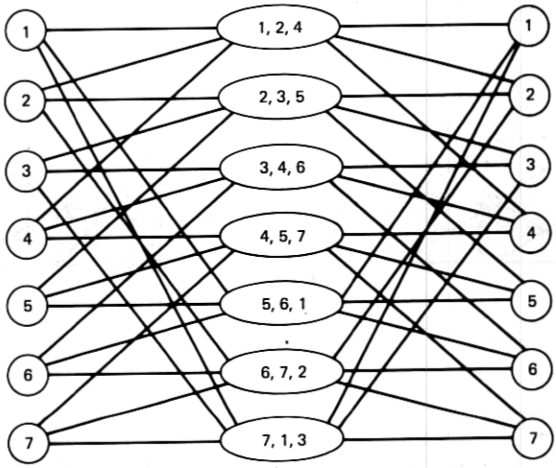

Fig. 2—A 3-stage CZBN.

thermore, suppose that a (v,b,r,k,λ) BIBD exists. Then there exists an $(s + 2)$-stage CZBN with v zones which has the channel graphs as shown in Fig. 3.

Proof: Let N be the given s-stage CZBN and assume that every input (output) switch of N is labeled by the zone it belongs to. Take b copies of N and let N_i denote the ith copy. Replace the k zones in N_i by the k elements in the ith block of the given BIBD. Take v switches of $V_1(V_3)$ and label each switch by a distinct element of the set $Z = \{1,2,\ldots,v\}$. Apply a labeled-subset connection between $V_1(V_3)$ and the input (output) switches of the b copies of N. It is easy to verify that the resulting network is indeed the one specified in Theorem 2.

Corollary: *Suppose that a (v,b,r,k,λ) BIBD exists. Then we can construct a 4-stage CZBN with v zones such that its intrazone channel graph consists of r disjoint paths and its interzone channel graph consists of λ disjoint paths.*

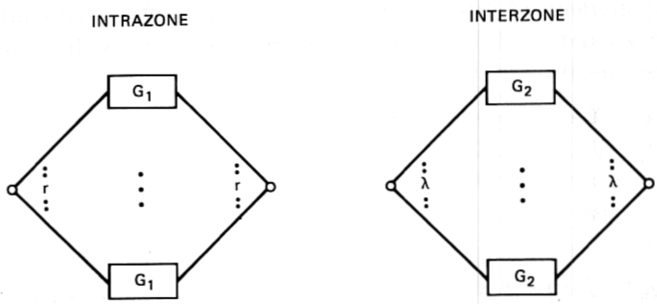

Fig. 3—Channel graphs for Theorem 2.

Example 2: Using the same BIBD as in Example 1, we obtain the 4-stage CZBN as shown in Fig. 4.

IV. A CONSTRUCTION FOR CZBNS USING A BALANCED PARTITION OF BLOCKS

Consider a (v,b,r,k,λ) design, and let F_i denote the subfamily of blocks containing element i. A partition of F_i is said to be *balanced* with parameters (p,d) if the following conditions are satisfied:

(i) F_i is divided into p disjoint parts such that each part consists of r/p blocks.

(ii) Exactly $d+1$ distinct elements appear in each part.

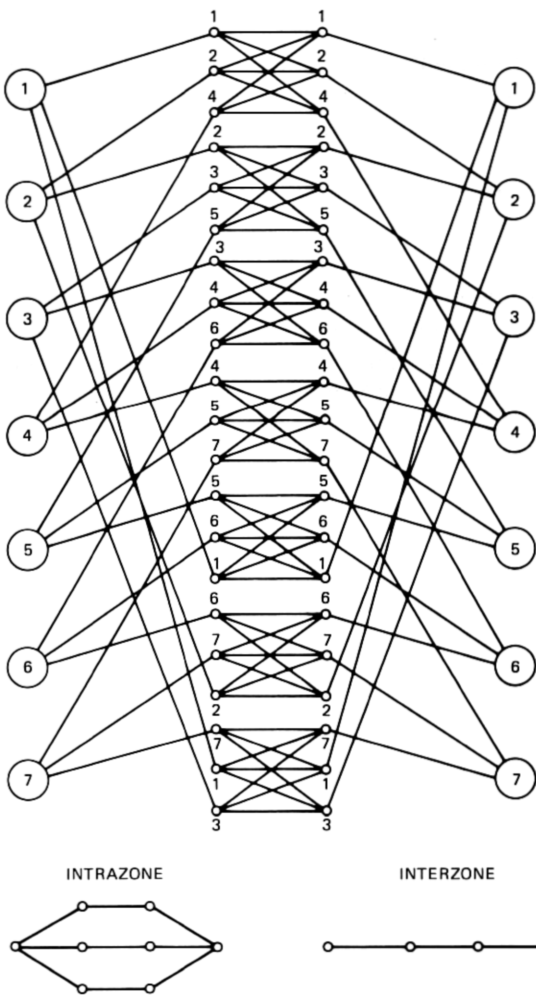

Fig. 4—A 4-stage CZBN.

(*iii*) For any element $i' \neq i$ contained in part j, the number of blocks in part j containing i' is a constant. (From (*i*) and (*ii*), this constant must also be independent of j.)

We say F_i has a 2-step balanced partition with parameters (p_1,d_1,p_2,d_2) if F_i has a balanced partition $P_1,\ldots,P_{p_1}$, with parameters (p_1,d_1) and each P_j has a balanced partition with parameters (p_2,d_2). Similarly, we can define a t-step balanced partition of F_i with parameters $(p_1,d_1,p_2,d_2,\ldots,p_t,d_t)$.

Note that any t-step nested partition of a set N induces a partial ordering which can be represented by a $(t+2)$-level rooted tree. Suppose that N has n elements. Then the first level of the tree corresponds to the crudest partition, namely, a single node representing the set N itself, and the $(t+2)$-level of the tree corresponds to the finest partition, namely, n nodes each representing a single element of N. The t intermediate levels of the tree correspond to the t partitions sequentially. By taking two copies of this tree and identifying their nodes at the $(t+2)$-level, we obtain a $(2t+3)$-stage symmetrical network. This mapping from a nested partition to a multistage network is critically used in the following theorem.

Theorem 3: Consider a (v,b,r,k,λ) *BIBD and let* F_i *be the subfamily of blocks containing the element i. Suppose that for each* F_i, $i = 1,2,\ldots,v$, *there exists a t-step balanced partition with the parameters* $(p_1,d_1,p_2,d_2,\ldots,p_t,d_t)$. *Then there exists a* $(2t+3)$*-stage* ($(2t+4)$*-stage*) *CZBN which has channel graphs as shown in Fig. 5:* ($q = r/\Pi_{l=1}^{t}p_l$). (To obtain the channel graphs for the $(2t+4)$-stage CZBN, replace each vertex in the center stage by the graph O–O.)

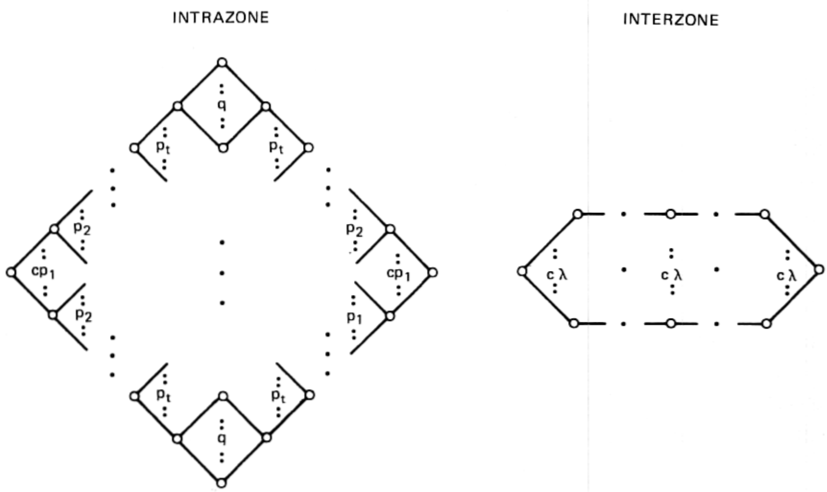

Fig. 5—Channel graphs for Theorem 3.

Proof (only for the case that n is odd): For each F_i, $i = 1,\ldots,v$, construct a $(2t + 3)$-stage symmetrical network by using the given t-step balanced partition. The union of these v networks (they overlap at the center stage, since the F_i's overlap) yields a CZBN with channel graphs as specified in Fig. 5 with $c = 1$. Taking c copies of these networks and identifying their first stages and last stages, we obtain the desired CZBN. That the constructed network is "balanced" is a consequence of the partition being balanced.

Corollary: Suppose a $(v,b,r,k,1)$ BIBD exists and $\Pi_{l=1}^{t} p_l$ divides r. Then a CZBN with channel graphs as specified in Fig. 5 exists.

Proof: With $\lambda = 1$, any partition which satisfies condition (i) of a balanced partition is a balanced partition. The same is true for a t-step partition. Therefore, when $\Pi_{l=1}^{t} p_l$ divides r, then a t-step balanced partition with parameters $(p_1,\ldots,p_t)$ always exists (the parameters d_ls are determined by p_ls).

Note that by applying Theorem 2 several times to the network constructed in Theorem 3, we can obtain CZBNs with various types of channel graphs. In particular, we obtain the following:

Theorem 4: Suppose that a sequence of BIBDs with parameters $(v_j,b_j,r_j,k_j,\lambda_j)$, $j = 1,2,\ldots,m$ exists. Furthermore, suppose $k_j = v_{j+1}$ for $j = 1,2,\ldots,m-1$, $\lambda_m = 1$, and $\Pi_{l=1}^{t} p_l$ divides r_m. Then there exists a $(2t + 2m + 1)$-stage ($(2t + 2m + 2)$-stage) CZBN which has channel graphs as shown in Fig. 6: ($q = r_m/\Pi_{l=1}^{t} p_l$).

Proof: Use the $(v_m,b_m,r_m,k_m,\lambda_m)$ BIBD to construct a $(2t + 3)$-stage CZBN from Theorem 3. Then apply Theorem 2 $m - 1$ times.

Note that, if we take c copies of each k out of v combination, we obtain a (v,b,r,k,λ) BIBD with $b = c\binom{v}{k}$, $r = c\binom{v-1}{k-1}$ and $\lambda = c\binom{v-2}{k-2}$. By setting $k = v$, it is clear that a (v,r,r,v,r) BIBD always exists. The zone-balanced networks constructed in Ref. 3 are thus seen to be special cases of the networks specified in Theorem 4 by setting $\lambda_j = r_j$ for $j = 1,2,\ldots,m-1$. (The conditions that $\lambda = 1$ and $\Pi_{l=1}^{t} p_l$ divides r_m are not explicitly stated in Ref. 3, but a check with the author of Ref. 3 has verified their necessity.)

V. A GENERALIZATION

We can generalize the definition of zone-balanced network to *partially zone-balanced network* in which every pair of zones is classified into one of the *k associate classes*. The channel graphs of all intrazone pairs of the ith associate are isomorphic to a graph G_i regardless of which pair is chosen. The number of the ith associates of a given zone should be independent of which zone is chosen. Just as balanced incomplete block designs are a natural tool for the construction of zone-balanced networks,

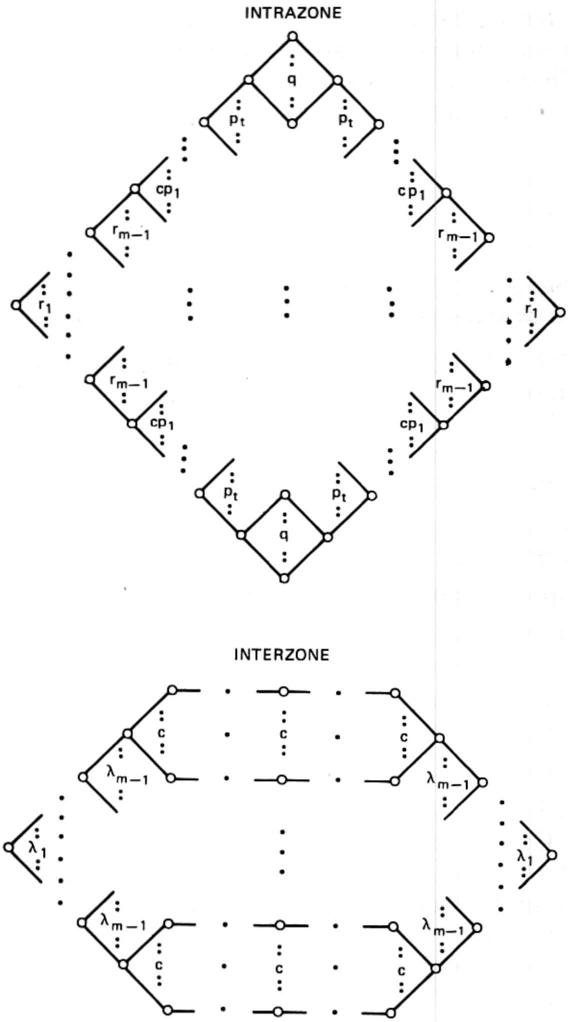

Fig. 6—Channel graphs for Theorem 4.

we can use partially balanced incomplete block designs to construct partially zone-balanced networks, and results similar to those given in this paper can be obtained. However, the partially balanced incomplete block design is really too strong for our construction, since we do not require that for every pair of zones X and Y of the ith associate, the number of zones which are the jth associate of X and the kth associate of Y should be independent of X and Y. This suggests that some design weaker than the partially balanced incomplete block design should be studied for this purpose.

After the completion of this paper, we learned that the author of

Ref. 3 had just revised her paper into a more complete and general account.[6] However, the main difference between our construction and her construction remain as follows: (*i*) Her construction uses only one BIBD, while ours uses many BIBDs sequentially. (*ii*) The method of using balanced partition of blocks is unique in our construction.

REFERENCES

1. M. Hall, Jr., *Combinatorial Theory,* Blaisdell, 1967.
2. F. K. Hwang, "Link Designs and Probability Analysis for a Class of Connecting Networks," to appear in Networks.
3. F. R. K. Chung, "On Switching Networks and Block Designs," Conference Records of the Twentieth Annual Asilomar Conference on Circuits, Systems, and Computers, Pacific Grove, California, 1976, pp. 212–218.
4. K. Takagi, "Optimal Channel Graph of Link System," Electronics and Comm. in Japan, *54-A* (1971), pp. 1–10.
5. G. Timperi and D. Grillo, "Structural Properties of a Class of Link Systems," Alta Frequenza, *XLI* (1972), pp. 279–289.
6. F. R. K. Chung, "Zone-Balanced Networks and Block Designs," B.S.T.J., this issue, pp. 2957–2971.