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Zone-Balanced Networks and Block Designs

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A switching network can be viewed as a collection of interconnected crosspoints that provides connection between input terminals and output terminals. Many networks have the property that the set of input terminals and output terminals can be partitioned into a number of zones such that the requests for connection between an input terminal and an output terminal in the same zone are more likely than those connecting terminals in different zones. In this paper, we study the structure of switching networks of this type by the use of block designs.

I. INTRODUCTION

We shall consider multistage switching networks composed of rectangular switches. For an input terminal u and an output terminal u', the *channel graph for u and u'*, denoted by $G(u,u')$, is defined to be the union of all paths that can be used to connect u and u'. (A channel graph is also called a linear graph.) A network is said to be *balanced* if all channel graphs $G(u,u')$, where u is in the set I of input terminals and u' is in the set Ω of output terminals, are isomorphic.[1,2] A network is said to be *zone-balanced* if it has two nonisomorphic channel graphs, say G_1 and G_2, so that the channel graph $G(u,u')$ is isomorphic to G_1 if u and u' are in the same zone, and $G(u,u')$ is isomorphic to G_2 if u and u' are in different zones. (See Fig. 1. Note that the switches in the network can be viewed as nodes of the corresponding graph.) G_1 is called the *internal graph* and G_2 is called the *external graph* of the switching network.

A zone-balanced network usually consists of three parts. The primary part consists of a few stages where traffic distribution takes place within each zone so that each input terminal has sufficient access to the central stage. The secondary part is the central stage which provides interconnections between different zones. The tertiary part plays the same role for the output terminals as the primary part does for the input terminals.

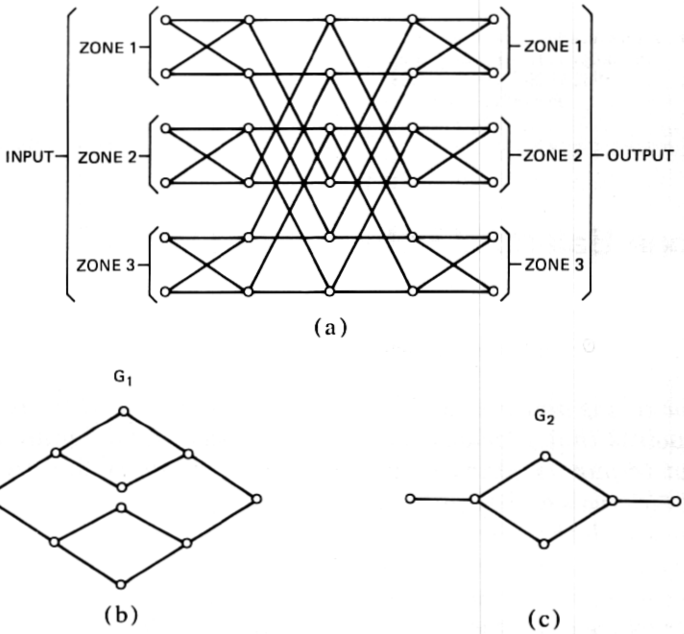

Fig. 1—(a) A zone-balanced network. (b) The internal graph. (c) The external graph.

The primary part is, in fact, composed of a number of *distribution networks,* each of which provides traffic distribution within each zone. We first investigate distribution networks in Section II. In Section III, we introduce the concept of block designs, which will then be used for connecting the distribution networks in the primary part and the switches in the central stage (Sections IV and V). In Section VI, we study zone-balanced networks of more general types.

II. DISTRIBUTION NETWORKS

Figure 2a illustrates an example of a distribution network. The *distribution graph* for an input terminal u is defined to be the union of all paths containing u. In a distribution network, the distribution graphs for any two input terminals are isomorphic. The distribution graph of the distribution network in 2a is shown in 2b. The labeling function f is explained later in this paper.

We consider a distribution network M_s, which is an s-stage network with switches in stage i having size $n_i \times m_i$ for $1 \leq i \leq s$. The distribution graph is then as shown in Fig. 3.

The switch sizes ($n_i \times m_i$, $1 \leq i \leq s$) and the number s of stages are generally dependent upon the traffic loads and the number of input terminals in each zone in order to reduce the "cost" (the number of

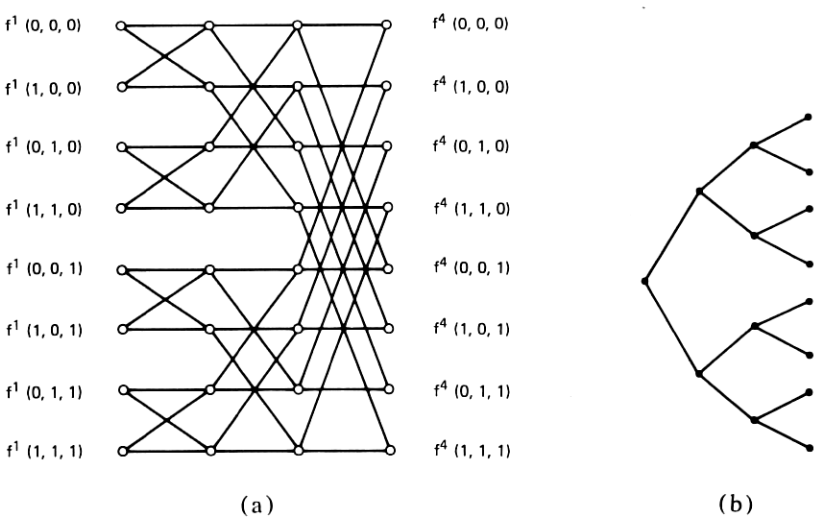

Fig. 2—(a) A distribution network. (b) The distribution graph.

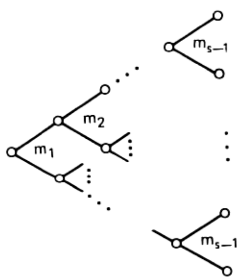

Fig. 3—A general distribution graph.

crosspoints) of the network.[3,4] Here we will only study the structure of distribution networks and will not be concerned with the complicated problem of determining s and n_i, m_i, $1 \leq i \leq s$.

A distribution network can usually be constructed recursively. Figure 4 illustrates a *complete* distribution network in which the number of inlet lines is the product of n_i, $1 \leq i \leq s$, and the number of outlet lines is the product of m_i. We note that, in a complete distribution network, we have $p = p'$ in Fig. 4 and exactly one link exists between a copy of M_{i-1} and a switch in the last stage of M_i for $1 \leq i \leq s$.

Here is an explicit method for the interconnection in the complete distribution network M_s. While the notation may appear at first to be somewhat complicated, it will turn out to be very useful and precise in specifying the link connections of the network.

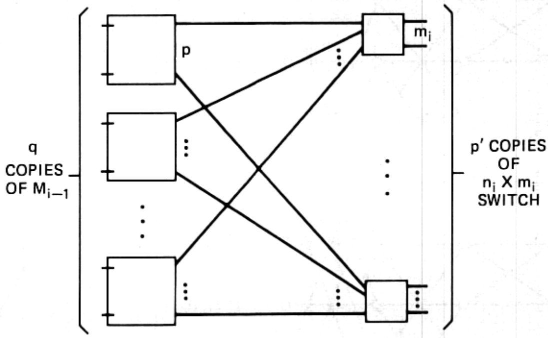

Fig. 4—A complete distribution network.

Let $f^j(i_1,i_2,\ldots,i_{s-1})$ denote the $(i_1 + i_2m_1 + \ldots + i_{s-1}m_1\ldots m_{s-2} + 1)$th switch in stage j where $0 \le i_q \le m_q$, $1 \le q \le s-1$.

Let $f_q^j(i_1,i_2,\ldots,i_{s-1})$ denote the $(q+1)$th outlet line of the switch $f^j(i_1,\ldots,i_{s-1})$ where $0 \le q < m_j$. Then we have:

$$f_q^j(i_1,\ldots,i_{s-1}) \text{ is connected to}$$

$$f^{j+1}(i_1,\ldots,i_{j-1},q,i_{j+1},\ldots,i_{s-1})$$

for all j, $1 \le j < s$.

Sometimes the values of n_i, m_i, $1 \le i \le s$, do not allow complete connections between consecutive stages, i.e., $p < p'$ and the number of inlet lines of the zone is not equal to the product of n_i, $1 \le i \le s$ (see Fig. 5). These distribution networks are said to be incomplete distribution networks. In this case, an appropriate connection, according to some rules to assign links cyclically, usually can result in a distribution network of this type (called cyclic distribution network). Here we give a scheme of constructing a cyclic distribution network in which copies of M_{i-1} and

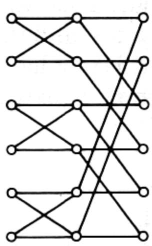

Fig. 5—An incomplete distribution network.

the switches of the last stage of M_i are connected according to the connection pattern of a regular bipartite graph as follows.

Consider a bipartite graph with sets of nodes A and B. A link connects a node in A to a node in B. A bipartite graph is said to be *regular* if the degree of each node in A is equal to some integer x and the degree of each node in B is equal to some integer y. It then follows that $ax = by$ where $a = |A|$, $b = |B|$. We restrict ourselves to the case that $a \geq y$, $b \geq x$. We will show that the condition $ax = by$ is sufficient for the existence of such a regular bipartite graph. We consider the following two possibilities.

Case 1: a and y are relatively prime. In this case, y divides x. Let $\alpha_1, \ldots, \alpha_a$ denote nodes in A and $\beta_1, \ldots, \beta_b$ denote nodes in B. We will then connect α_i to β_j where $j \equiv i\, x/y + k \pmod b$ where $1 \leq k \leq x$, and $1 \leq j \leq b$. It is easy to see that the resulting graph is a regular bipartite graph (see Fig. 6a).

Case 2: Let d be the greatest common divisor of a and y. Then we can construct, by Case 1, a regular bipartite graph G' on sets of nodes A' and B' where $|A'| = a' = a/d$ and $|B'| = b$. The degrees of nodes in A' are all equal to $x' = x$ and the degrees of nodes in B' are all equal to $y' = y/d$. Now, we construct the regular bipartite graph on A and B as follows: The set A can be viewed as d copies of A'. The connection between each copy of A' and B is the same as G'. It is easily verified that the resulting graph is regular and a node in A or B has degree x or y, respectively (see Fig. 6b).

Now we can construct the incomplete distribution network M_i by connecting a copies of M_{i-1} and b copies of $n_i \times m_i$ switches if $ap = bn_i$ (p is the number of outlet lines of M_{i-1}), according to the regular bipartite graph we described above by taking each copy of M_{i-1} as a node in A and each switch $n_i \times m_i$ as a node in B.

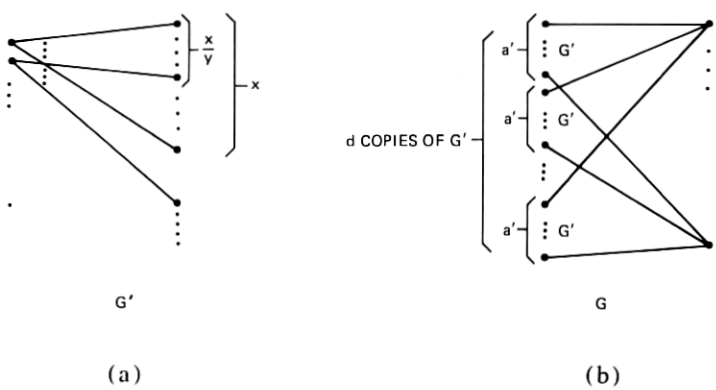

Fig. 6—Regular bipartite graphs.

We note that there are many nonisomorphic regular bipartite graphs of the given sizes. Therefore, there are many different incomplete distribution networks of the same "cost" (number of crosspoints). In Section IV and V we see some incomplete distribution networks obtained by some methods other than the cyclic connections described here. Those methods are based on a basic concept in combinatorics called a *block design,* which we next describe.

III. BLOCK DESIGNS

A (v,b,r,k,λ) block design is a family of subsets $X_1,X_2,\ldots,X_b$ of a v-element set X, satisfying the following conditions:

(*i*) Each X_i has k elements, $1 \le i \le b$.

(*ii*) Each 2-element subset of X is a subset of exactly $\lambda > 0$ of the sets $X_1,\ldots,X_b$.

Properties (*iii*) and (*iv*) follow immediately from (*i*) and (*ii*).

(*iii*) Each element of X is in exactly r of the sets $X_1,\ldots,X_b$.

(*iv*) $r(k-1) = \lambda(v-1)$ and $bk = vr$.

For example, the following is a (7,7,3,3,1) block design.

$$X_i = \{i, i+1, i+3\} \pmod 7 \text{ for } 1 \le i \le 7.$$

The reader is referred to Refs. 5 and 6 for the existence and construction of various classes of block designs.

IV. THREE-STAGE ZONE-BALANCED NETWORKS

Let $X_1,\ldots,X_b$, be a (v,b,r,k,λ) block design. We will construct a three-stage zone-balanced network having internal graphs containing r paths and external graphs containing λ paths (see Fig. 7a and b), and having v switches in the first or third stage. The input terminals of the same zone go to the same switch. Let $y_1,\ldots,y_v$ be switches in the first

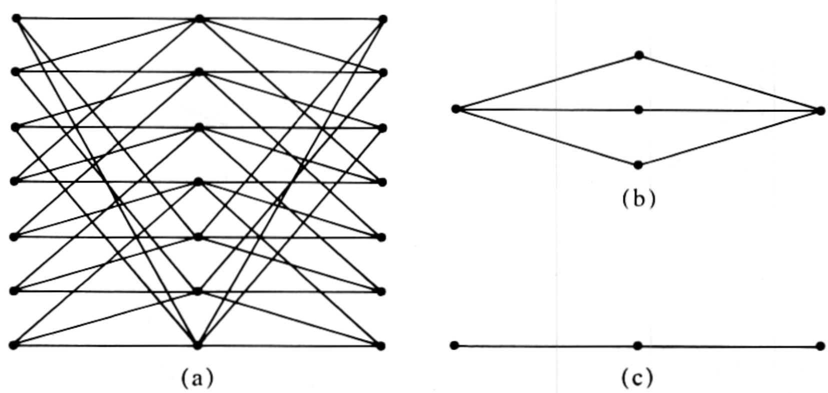

Fig. 7—(a) A zone-balanced network constructed by using (7,7,3,3,1) block design. (b) The internal graph. (c) The external graph.

stage (the primary part), each of which has r output lines. Let $z_1, \ldots, z_b$ be switches of size $k \times k$ in the second stage (the secondary part). Then we connect y_i to z_j if and only if i is an element of X_j (see Fig. 7 for the (7,7,3,3,1) block design). The connection between the second and the third stage is a mirror image of that between the first and the second stage. It is easily verified that the resulting network is zone-balanced and has internal and external graphs as shown in Fig. 7b and c, respectively. This is the basic model of zone-balanced networks.

V. MULTISTAGE ZONE-BALANCED NETWORKS

In this section, we give explicit constructions for various types of multistage zone-balanced networks. Roughly speaking, we construct these multistage zone-balanced networks by replacing each switch in the first or the last stage of the three-stage zone-balanced network described in Section III by a distribution network. The internal graphs in these zone-balanced networks depend on the switch sizes in the distribution networks. Suppose a distribution network has s stages and has switch of size $n_i \times m_i$ in stage i, $1 \leq i \leq s$. Then the internal graph in the zone-balanced network has $m_1 m_2 \ldots m_s$ paths as shown in Fig. 8a. The external graph depends heavily upon the linking pattern between the distribution networks in the primary part (or tertiary part) and the switches in the central stage. In general, it would be desirable to have the external graph as "spread-out" as possible. We will not define "spread-out" rigorously here. For example, the graph in Fig. 10c is more spread-out than the graph in Fig. 1c, although they contain the same number of paths. It can be shown[7,8] that the more spread-out the channel graph is, the less the traffic congestion will be. The external graph in the zone-balanced network we construct will be as shown in Fig. 8b.

In this section, we restrict ourselves to the case in which the number of input terminals is the same as the number of output terminals and the zone sizes are equal, i.e., $|I| = |\Omega|$, $I = I_1 \cup \ldots \cup I_v$, $\Omega = \Omega_1 \cup \ldots \cup \Omega_v$

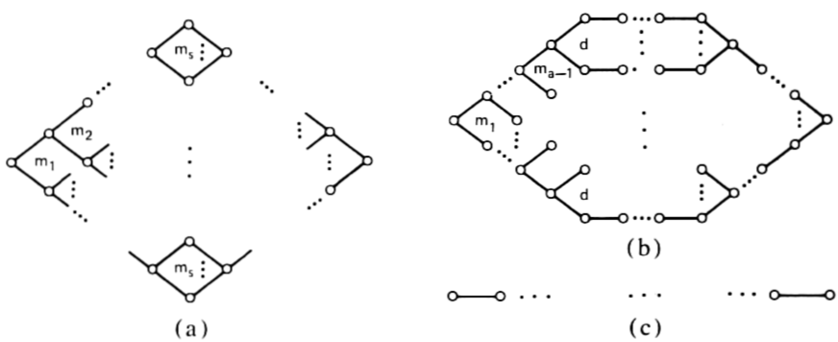

Fig. 8—(a) A general multistage internal graph. (b) A general multistage external graph. (c) A path.

and $|I_i| = |\Omega_j|$, $1 \le i,j \le v$. An input terminal u, $u \epsilon I_i$, and an output terminal u', $u' \epsilon \Omega_j$, are said to be in the same zone if and only if $i = j$.

We first present a simple technique, called a Type 1 construction, for construcing zone-balanced networks. The distribution network M_s in the primary part has the property that every switch in the last stage has r output lines, i.e., $m_s = r$. Let us assume M_s has q switches in the last stage. Thus, M_s has qr output lines. We will construct a zone-balanced network with internal graphs containing qr paths and external graphs containing $q\lambda$ paths (see Fig. 9a and b) as follows.

Let the central stage consist of bq switches of size $k \times k$. Then the q'th switch of the v'th copy of M_s in the primary part is connected to the $((b'-1)q + q')$th switch in the central stage for any b' with $v' \epsilon X_{b'}$. We connect the central part and the tertiary part in the same way (symmetrically) that the primary and central parts are connected.

For example, using the (3,3,2,2,1) block design $X_1 = \{1,3\}$, $X_2 = \{1,2\}$, $X_3 = \{2,3\}$, we obtain the network shown in Fig. 10. We note that the networks in Fig. 1a and Fig. 10a have the same number of crosspoints, but the channel graph in Fig. 10c is more spread-out than the channel graphs containing r paths, provided a (v,b,r,k,λ) block design exists in Fig. 1a, though their "cost" (number of crosspoints) are the same.

Now we give a construction (Type 2) of another class of zone-balanced networks which has external graphs containing a path and internal graphs containing r paths, provided a (v,b,r,k,λ) block design exists where $\lambda = 1$. The primary part consists of v copies of a complete distribution network which has r output lines. Thus, r is the product $m_1 m_2 \ldots m_s$. The central stage consists of b switches of size $k \times k$. We will connect the r output lines of the distribution network to the switches in the central stage as follows: One of the output lines of the v'th copy of the distribution network will be connected to the b'th switch in the central stage for any b' such that v' is contained in the block $X_{b'}$. In Fig. 11, we have an example which is constructed by using a (9,12,4,3,1) block design. It can be easily verified that the internal graph is as shown in Fig. 8a and the external graph is as shown in Fig. 8c.

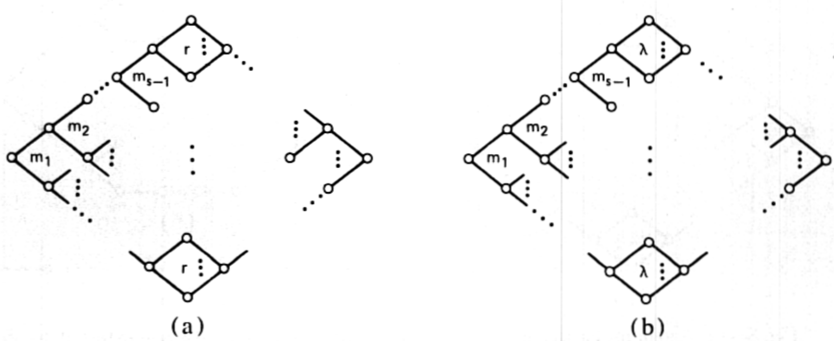

Fig. 9—(a) An internal graph. (b) An external graph.

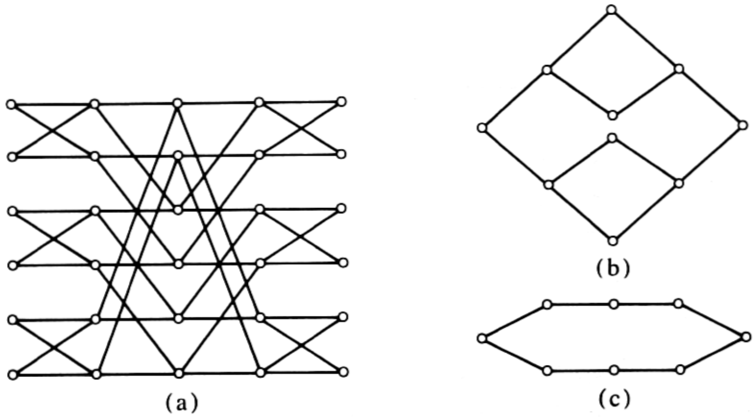

Fig. 10—(a) A type-1 zone-balanced network constructed by using (3,3,2,2,1) block design. (b) The internal graph. (c) The external graph.

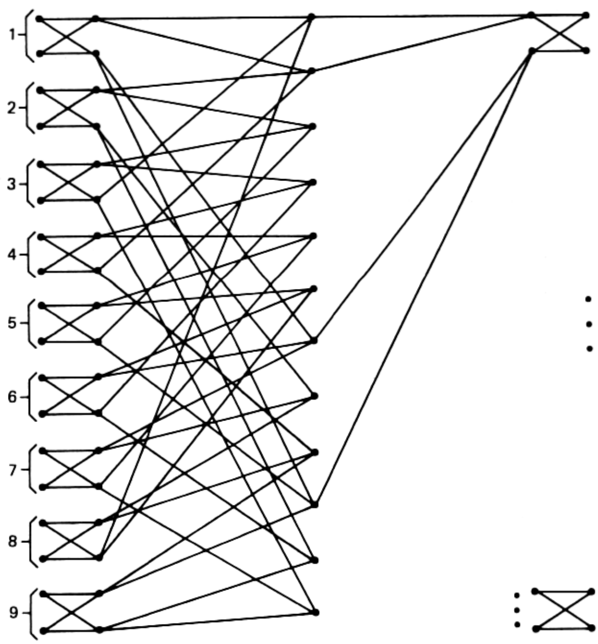

Fig. 11—A type-2 zone-balanced network constructed by using (9,12,4,3,1) block design.

Type 3 zone-balanced networks are variations of Type 2 zone-balanced networks. The primary part consists of copies of a distribution network which is not necessarily a complete distribution network. In Figure 12a, we have an example which is constructed by using (3,3,2,2,1) design. Its internal and external graphs are shown in Figs. 12b and c, respectively. The distribution network usually has rw output lines for some integer

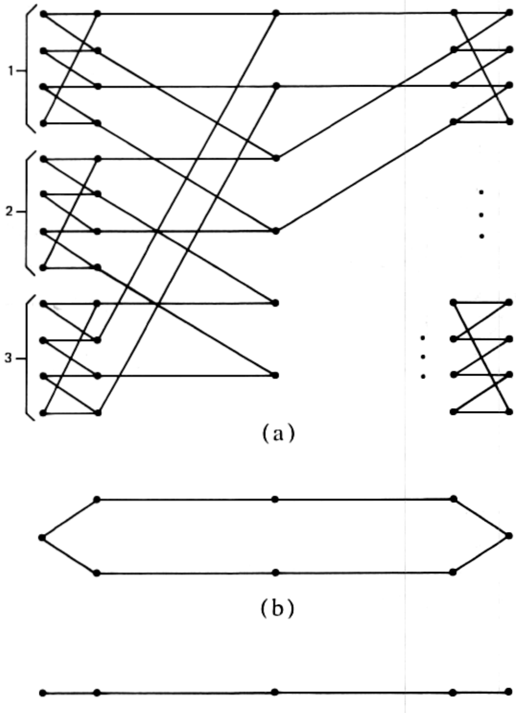

Fig. 12—(a) A type-3 zone-balanced network constructed by using (3,3,2,2,1) block design. (b) The internal graph. (c) The external graph.

w. The central stage consists of bw switches which can be partitioned into b subsets, each of which has w switches. Among the rw output lines of the v'th copy of the distribution network, there are w output lines which are connected to the w switches, respectively, in the b'th subset if X_b' contains v' and any input line of the distribution network does not have access to more than one output line of these w output lines. The example in Fig. 12 more or less reveals the scheme for constructing zone-balanced networks of this type. The detail of the construction is omitted.

Type 4 zone-balanced networks are generalizations of Type 2 zone-balanced networks. A Type 4 zone-balanced networks has external graph containing λw paths and internal graphs containing rw paths, provided a (v,b,r,k,λ) block design exists. The primary part consists of v copies of a complete distribution network which has s stage and has switches of size $n_i \times m_i$ in ith stage. Thus, rw is the product $m_1 m_2 \ldots m_s$. Since we want the external graph as spread-out as possible, we will restrict ourselves to use block designs with $\lambda = 1$. (By using block design with $\lambda > 1$, the subsequent scheme for constructing Type 4 zone-balanced network can be applied with some modification. However, the resultant networks will usually have internal graphs which are not as spread-out as possible.) The internal graph and the external graph are as shown in Fig. 8b. Thus, we will choose w in the form $m_1 m_2 \ldots m_{a-1} d$ where d divides m_a. The connection between the primary part and the central part is more complicated than that in the Type 2 construction. However, it can be explicitly specified by the following simple method.

Let $f_q^s(i_1, \ldots, i_{s-1}; v')$ denote the $(q + 1)$th outlet lines of the $(i_1 + i_2 m_1 + \ldots + i_{s-1} m_1 \ldots m_{s-2} + 1)$th switch in stage s of the v'th copy of M_s.

The central stage of the zone-balanced network consists of bw switches of size $k \times k$. We define

$$f^c(i_1, \ldots, i_a; b') \text{ to be the}$$

$$(i_1 + i_2 m_1 + \ldots + i_a m_1 \ldots m_{a-1} + (b' - 1)\, m_1 \ldots m_{a-1} d + 1)\text{th switch},$$

where

$$0 \le i_q < m_q,\ 1 \le q < a, 0 \le i_a < d \text{ and } 0 < b' \le b.$$

Let $X_1, X_2, \ldots, X_b$ denote the sets of a (v,b,r,k,λ) block design with $\lambda = 1$. For any element $y \epsilon X = \cup_i X_i$, we say the ith y-set is X_j if X_i is the ith set containing y, i.e., $|\{X_q : y \epsilon X_q, 1 \le q \le j\}| = i$.

First, we consider the special case when $d = m_a$.

$$f_q^s(i_1, \ldots, i_{s-1}; v') \text{ is connected to}$$
$$f^c(i_1, \ldots, i_a; b')$$

if $v' \epsilon X_b{}'$ and the

$$(i_{a+1} + i_{a+2} m_{a+1} + \ldots + i_{s-1} m_{a+1} \ldots m_{s-2} + q m_{a+1} \ldots m_{s-1} + 1)\text{th } v'\text{-set}$$

is $X_b{}'$.

Now, if d is a proper divisor of m_a, the above scheme has to be modified slightly. Note that i_a can be written as $i_a' + i_a'' d$ where $0 \le i_a' < d$. Then we have:

$$f_q^s(i_1, \ldots, i_{s-1}; v') \text{ is connected to}$$
$$f^c(i_1, \ldots, i_{a-1}, i_a'; b')$$

if $v' \epsilon X_b{}'$ and the

$$(i_a' + i_{a+1} d + \ldots + i_{s-1} d m_{a+1} \ldots m_{s-2} + q d m_{a+1} \ldots m_{s-1} + 1)\text{th } v'\text{-set}$$

is X_b'.

We note that the f_q^s (or f^c) is just a digital expression for address assignments of the output lines in stage s. The last $s - a$ digits (i.e., $i_a, \ldots, i_{s-1}$ of $f_q^s(i_1, \ldots, i_s)$) are used to find the location of the "block" of switches in the central stage and the first a digits are used to specify the location of the switch in that block to which the output lines of $f^s(i_1, \ldots, i_{s-1})$ should be connected. Figure 13a gives an example using the (13,13,4,4,1) design where $X_1 = \{i, i+1, i+3, i+9\}$ (mod 13). Its internal and external graphs are also shown in Fig. 13b and c.

We note that, by modifying the construction mentioned above, we could easily obtain zone-balanced networks having external graphs not necessarily as spread-out as possible. For example, let us modify the definition of f_q^s, the digital expression for address assignments of output lines of the switches in stage s. Instead of using the last a digits, we use the first a digits to assign the location of the "block" of switches in the central stage and use the last $s - a$ digits to specify the location of the switch in the "block" to which the output line should be connected. The resultant zone-balanced network then has the external graph which is the least spread-out channel graph among all possible graphs having the same number of paths. In general, we can use arbitrary x digits to specify the location of the block (as long as the necessary condition on divisibility is satisfied) to obtain zone-balanced networks having various external graphs.

We also note that it is possible to derive a generalized version of Type 3 zone-balanced networks using incomplete distribution networks. However, it is more complicated and has more constraints than Type 4 construction. We shall not discuss it here.

VI. ZONE-BALANCED NETWORKS OF MORE GENERAL TYPES

We note that the right half of a zone-balanced network is itself an incomplete distribution network. These incomplete distribution networks, called BD-distribution networks, seem to distribute traffic more evenly and have richer combinatorial properties than the cyclic incomplete distribution networks described in Section II. One of the obvious reasons is that a BD-distribution network together with its mirror image gives a zone-balanced network, whereas a cyclic incomplete distribution network does not.

Suppose the set of input and output terminals can be partitioned into a number of zones which can themselves then be partitioned into several areas such that requests for connecting terminals in the same area (zone) are more likely than those for connecting terminals in different areas (zones). In such a network, there usually are four channel graphs, say G_1, G_2, G_3, G_4, such that $G(u,u')$ is isomorphic to G_1, if u and u' are in the same zone and area; $G(u,u')$ is isomorphic to G_4 if u and u' are in different zones and areas; $G(u,u')$ is isomorphic to G_2 if u and u' are in the same zone but different areas; $G(u,u')$ is isomorphic to G_3 if u and

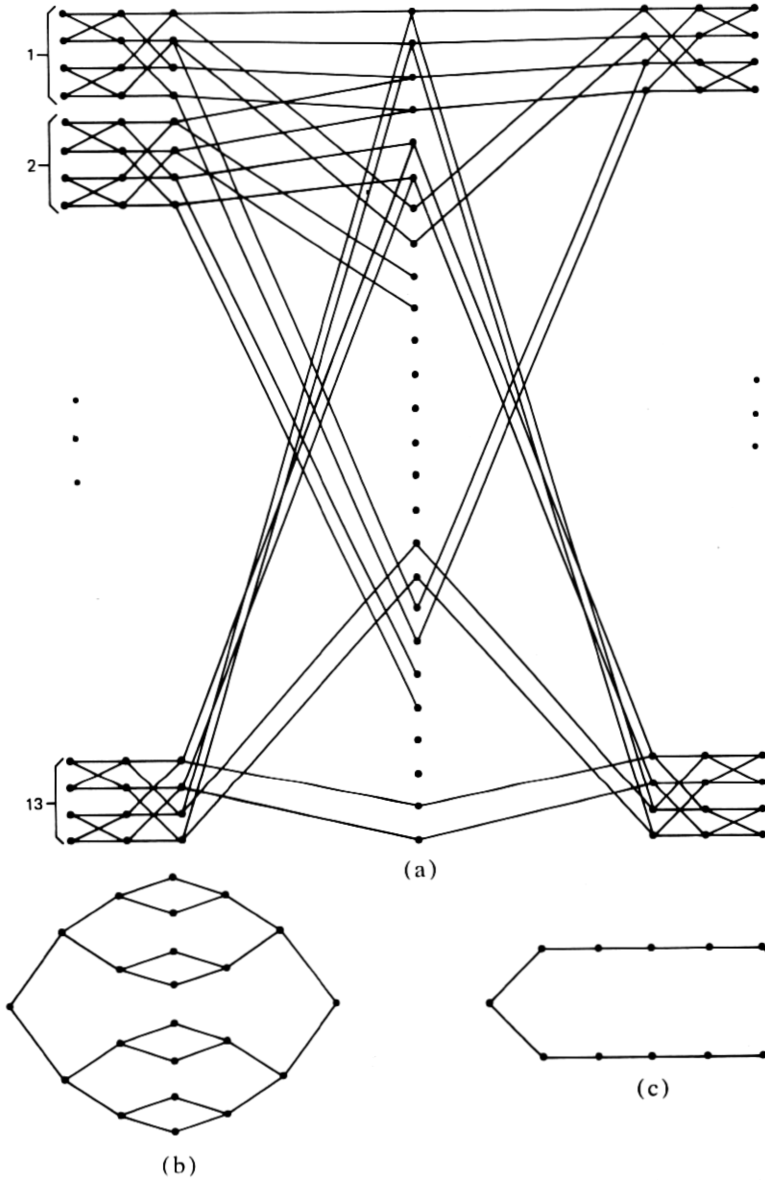

Fig. 13—(a) A type-4 zone-balanced network constructed by using (13,13,4,4,1) block design. (b) The internal graph. (c) The external graph.

u' are in the same area but different zones. These networks will be called multizone-balanced networks, and can be built by schemes similar to those described in Section V. A multizone-balanced network can also be viewed as a combination of three parts. However, the primary part usually consists of copies of a BD-distribution network.

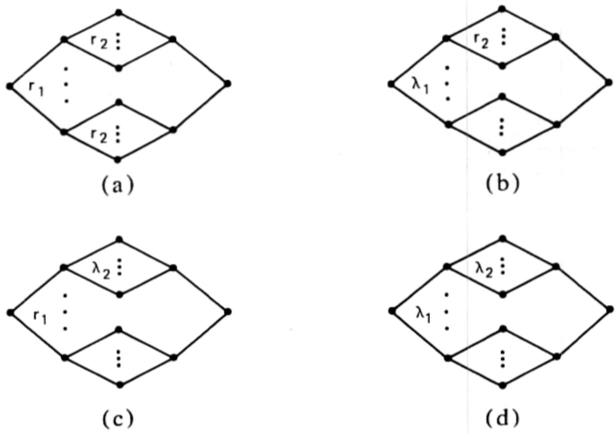

Fig. 14—Some channel graphs for multizone-balanced networks.

A simple model of a five-stage multizone-balanced network can be described as follows

Let $X_1, \ldots, X_{b_1}$ be a $(v_1,b_1,r_1,k_1,\lambda_1)$ block design and $Y_1, \ldots, Y_{b_2}$ be a $(v_2,b_2,r_2,k_2,\lambda_2)$ block design. The distribution network M_s for each zone is exactly the right half of the three-stage zone-balanced network described in Section IV, and the number of output lines of a switch in the last stage of M_s is r_2. The primary part consists of v_2 copies of M_s, and the central part consists of switches of size $k_2 \times k_2$. The ith switch of the jth copy of M_s is connected to the $((b'-1)b_1+i)$th switch if j is an element of Y_b'. Again, the tertiary part is connected to the central part in the same way (symmetrically) that the primary and central parts are connected.

It is easy to verify that the channel graphs G_1,G_2,G_3,G_4 are as shown in Fig. 14.

We could modify the above multizone-balanced network by replacing the switches in the primary part or the tertiary part by distribution networks using the same method used in Section V. However, we will not discuss this here.

Suppose the number of input terminals and the number of output terminals are not equal, say $|\Omega|$ is a multiple of $|I|$. We can still construct zone-balanced networks by combining copies of zone-balanced networks which have the same number of input and output terminals.

Hagelbarger[9] first proposed the use of block designs for constructing switching networks in 1973. Some other techniques for constructing various classes of networks have been investigated in Refs. 10 and 11. Hopefully, more structures in combinatorics and graph theory can be employed as useful techniques to construct interesting classes of switching networks.

REFERENCES

1. F. R. K. Chung and F. K. Hwang, "A Problem on Blocking Probabilities in Connecting Networks," Networks, 7 (1977), pp. 185–192.
2. F. K. Hwang, "Balanced Networks," Conference Record of 1976 International Conference on Communications, pp. 7-13–7-16.
3. F. R. K. Chung, "Optimal Multistage Networks," IEEE Transactions on Communications, to appear.
4. A. Lotze, "Optimum Link Systems," 5th ITC, New York, 1967, pp. 242–251.
5. R. J. Collens, private communication.
6. M. Hall, Jr., *Combinatorial Theory,* Waltham, Mass: Blaisdell, 1967.
7. F. R. K. Chung and F. K. Hwang, "On Blocking Probabilities for Switching Networks," B.S.T.J., *56,* No. 8 (October 1977), pp. 1431–1446.
9. D. W. Hagelbarger, unpublished work.
10. F. R. K. Chung, "On Switching Networks and Block Designs," Conference Records of the Tenth Asilomar Conference on Circuits, Systems, and Computers, 1976, pp. 212–218.
11. F. K. Hwang and S. Lin, "Construction of Balanced Switching Networks," to appear in Networks.