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On Blocking Probabilities for a Class of Linear Graphs

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In a t-stage linear graph, all vertices are arranged in a sequence of stages such that any edge goes between a vertex in stage i and a vertex in stage i + 1 for some i. Lee first proposed the use of t-stage linear graphs for studying the blocking performance of t-stage switching networks. In his model, each edge is in either of two states, busy or idle, and the states of the edges are independent. Furthermore, an edge connecting a vertex in stage i to a vertex in stage i + 1 has the constant probability p_i of being busy. In the current paper, we use Lee's model to compare the blocking probabilities of different linear graphs. In particular, a t-stage linear graph is said to be superior to another t-stage linear graph if the blocking probability of the former never exceeds that of the latter for any choice of the p_i. For a class of linear graphs known as SP-canopies, we give simple necessary and sufficient conditions that one t-stage linear graph is superior to another.

I. INTRODUCTION

We consider a t-stage linear graph with a source (the vertex of the first stage) and a sink (the vertex of the last stage). All vertices are lined up in a sequence of stages such that any edge goes from a vertex in stage i to a vertex in stage $i + 1$ for some i, while each edge can be in either of the two states, *busy or idle.* The linear graph is said to be *blocked* if every path joining the source and the sink contains a busy edge. Lee[1] first proposed the concept of linear graphs in his study of the blocking performance of switching networks. We use Lee's model and follow his independence assumptions, namely, that the probabilities of occupancy for edges being busy in successive stages are independent. Thus, we may assume that any edge connecting a vertex in stage i with a vertex in stage $i + 1$ has some probability p_i of being busy for $1 \le i \le t - 1$. The sequence $(p_1,p_2,\cdots,p_{t-1})$ will be called *link occupancies* of the t-stage linear graph. A linear graph is said to be *superior* to another if, for any given link occupancies, the blocking probability of the first graph does not exceed that of the second graph.

In this paper, we restrict ourselves to a special kind of linear graph, called an *SP-canopy*. By definition, the smallest SP-canopy is a series combination of two edges. Any other *SP-canopy* is either a *parallel* combination of two smaller SP-canopies or a *series* combination of a smaller SP-canopy and an edge. For readers familiar with graph-theoretic terminology, an SP-canopy can be viewed as the union of two rooted trees with identified sets of terminal nodes such that the union is a planar graph (see Fig. 1a for an example).

Let e be an edge from a vertex **a** in stage i to a vertex **b** in stage $i + 1$. We define $\lambda(e)$ to be the ratio of the outdegree of **a** to the indegree of **b**. If all $\lambda(e)$, where e ranges over all edges between stage i and stage $i + 1$ (for a fixed i), have the same value λ_i for $1 \leq i \leq t - 1$, then this linear graph is said to be a *regular* linear graph (see Fig. 1b). Thus, a regular linear graph is associated with a unique degree sequence $(\lambda_1,\lambda_2,\cdots,\lambda_{t-1})$. In the case in which the regular linear graph is an SP-canopy, it can be uniquely represented by the degree sequence. Define $\lambda^* = \max_{1 \leq i \leq t-1} \{\lambda_1\lambda_2\cdots\lambda_i\}$. It is easy to verify that λ^* is just the number of distinct paths from the source to the sink. A regular SP-canopy is said to be a *symmetric* SP-canopy if $\lambda_i\lambda_{t-i} = 1$ for $1 \leq i \leq t - 1$. Thus, the degree sequence of a symmetric SP-canopy can be written, abbreviated as $(\lambda_1,\cdots,\lambda_{\lfloor (t-1)/2 \rfloor})$, when $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. The linear graph

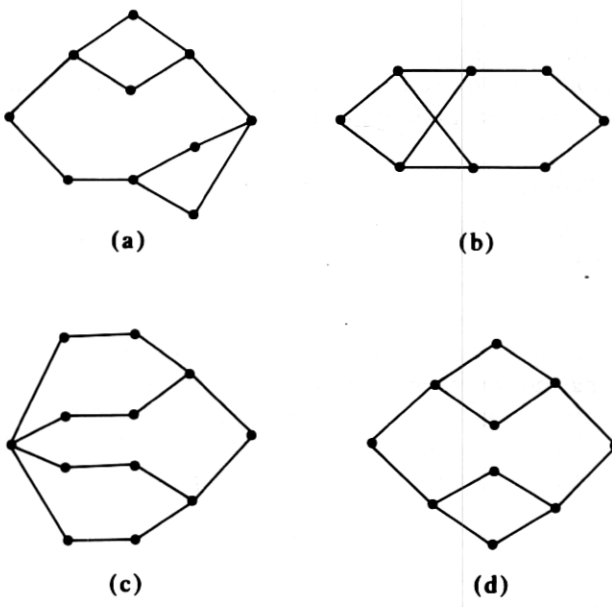


Fig. 1—(a) An SP-canopy. (b) A regular linear graph. (c) A regular SP-canopy. (d) A symmetric SP-canopy.

in Fig. 1c is a regular SP-canopy, and the linear graph in Fig. 1d is a symmetric SP-canopy.

We say a degree sequence $(\lambda_1,\lambda_2,\cdots,\lambda_{t-1})$ *majorizes* another degree sequence $(\lambda_1', \lambda_2',\cdots,\lambda_{t-1}')$ if and only if $\lambda_1\lambda_2\cdots\lambda_i \geq \lambda_1'\lambda_2'\cdots\lambda_i'$ for every i, $1 \leq i \leq t-1$. If we consider the set S_{t,λ^*} of all regular canopies with a fixed number t of stages and a fixed number λ^* of distinct paths, we see that $(\lambda^*,1,\cdots,1,(\lambda^*)^{-1})$ majorizes the degree sequence of any other regular SP-canopy in S_{t,λ^*}. In Ref. 2, comparisons are made involving all symmetric SP-canopies with fixed t and λ^* for t odd, with the conclusion that the symmetric SP-canopy with the degree sequence $\lambda^*,1,\cdots,1((t-1)/2$ 1*s*) is superior to all the others and the symmetric SP-canopy with degree sequence $1,\cdots,1,\lambda^*$ $((t-1)/2$ 1*s*) is inferior to all the others. In this paper, we prove a stronger and more general result which says that one regular SP-canopy is superior to another one if and only if the degree sequence of the first one majorizes the degree sequence of the second one.

II. SYMMETRIC SP-CANOPIES

In this section, we study symmetric SP-canopies with degree sequences of the form $(\lambda_1,\lambda_2,\cdots,\lambda_{\lfloor(t-1)/2\rfloor})$.

First, we prove a few auxiliary lemmas needed in the proof of the main result.

Lemma 1: Define

$$F(x) = (1 - a(1 - b^x))^{k/x}, \text{ where } 0 \leq a, b \leq 1.$$

Then $F(x)$ is monotone nondecreasing for $1 \leq x \leq k$.

Proof: We consider several cases:

Case 1: $0 < a, b < 1$.

$$\frac{dF}{dx}(x) = (1 - a(1 - b^x))^{k/x} \cdot \left(-\frac{k}{x^2}\ln(1 - a(1 - b^x)) + \frac{kab^x \ln b}{x(1 - a(1 - b^x))}\right).$$

We define

$$G(a) = -\frac{1}{x}\ln(1 - a(1 - b^x)) + \frac{ab^x \ln b}{1 - a(1 - b^x)}.$$

It is easy to verify that $G(0) = G(1) = 0$. Furthermore, by setting $dG/da = 0$, we obtain the unique solution a_0 which satisfies

$$a_0 = \frac{1}{1 - b^x} + \frac{xb^x \ln b}{(1 - b^x)^2}.$$

Since $d^2G/da^2\,(a_0) < 0$, a_0 is indeed a maximum. Therefore, $G(a) > 0$ for all $0 < a < 1$. Thus dF/dx is positive for $0 < a, b < 1$.

Case 2: $a = 0$ or $b = 1$.

We have $F(x) = 1$ and $dF/dx = 0$.

Case 3: $a = 1$, then

$$F(x) = b^k, \; dF/dx = 0.$$

Case 4: $a \neq 0, 1$ and $b = 0$

$$F(x) = (1-a)^{k/x}.$$

$$\frac{dF}{dx}(x) = -\frac{k}{x^2}(1-a)^{k/x}\ln(1-a) > 0.$$

In any case, dF/dx is nonnegative. Therefore, Lemma 1 is proved.

For a given vector $\alpha = (\alpha_1,\alpha_2,\cdots,\alpha_{t-1})$, $1 \geq \alpha_i \geq 0$, and a sequence of positive real numbers $(\beta_1,\cdots,\beta_{\lfloor (t-1)/2 \rfloor})$, we define the following function:

$$f(\beta_{\lfloor (t-1)/2 \rfloor}) = \begin{cases} (1-\alpha_{(t-1)/2}\cdot\alpha_{(t+1)/2})^{\beta_{\lfloor (t-1)/2 \rfloor}} & \text{if } t \text{ is odd} \\ (1-\alpha_{t/2})^{\beta_{\lfloor (t-1)/2 \rfloor}} & \text{if } t \text{ is even.} \end{cases}$$

and

$$f(\beta_i,\beta_{i+1},\cdots,\beta_{\lfloor (t-1)/2 \rfloor}) = (1-\alpha_i\alpha_{t-i}(1-f(\beta_{i+1},\cdots,\beta_{\lfloor (t-1)/2 \rfloor})))^{\beta_i}$$

for $i = 1,2,\cdots,\lfloor (t-1)/2 \rfloor - 1$.

Lemma 2: If the sequence $(\beta_1,\beta_2,\cdots,\beta_{\lfloor (t-1)/2 \rfloor})$ *majorizes the sequence* $(\beta'_1,\beta'_2,\cdots,\beta'_{\lfloor (t-1)/2 \rfloor})$, *then we have*

$$f(\beta_1,\beta_2,\cdots,\beta_{\lfloor (t-1)/2 \rfloor}) \leq f(\beta'_1,\beta'_2,\cdots,\beta'_{\lfloor t-1/2 \rfloor}).$$

for any vector $(\alpha_1,\cdots,\alpha_{t-1})$ *satisfying* $1 \geq \alpha_i \geq 0$ *for all* i.

Proof: It is easily checked that Lemma 2 is true for $t = 2$ and 3. By the induction assumption, we have

$$f(\beta_1\beta_2/\beta'_1,\beta_3,\cdots,\beta_{\lfloor (t-1)/2 \rfloor}) \leq f(\beta'_2,\beta'_3,\cdots,\beta'_{\lfloor (t-1)/2 \rfloor}),$$

since $(\beta_1\beta_2/\beta'_1,\beta_3,\cdots,\beta_{\lfloor (t-1)/2 \rfloor})$ majorizes the sequence

$$(\beta'_2,\beta'_3,\cdots,\beta'_{\lfloor (t-1)/2 \rfloor}).$$

It is also clear that the following holds:

$$f(\beta'_1,\beta_1\beta_2/\beta'_1,\beta_3,\cdots,\beta_{\lfloor (t-1)/2 \rfloor}) \leq f(\beta'_1,\beta'_2,\cdots,\beta'_{\lfloor (t-1)/2 \rfloor}).$$

Therefore, it suffices to show

$$f(\beta_1,\beta_2,\cdots,\beta_{\lfloor (t-1)/2 \rfloor}) \leq f(\beta'_1,\beta_1\beta_2/\beta'_1,\beta_3,\cdots,\beta_{\lfloor (t-1)/2 \rfloor}).$$

Now we have

$$f(\beta_1,\beta_2,\cdots,\beta_{\lfloor (t-1) \rfloor}) = (1-\alpha_1\alpha_{t-1}(1-f(\beta_2,\cdots,\beta_{\lfloor (t-1)/2 \rfloor})))^{\beta_1}$$
$$= (1-\alpha_1\alpha_{t-1}(1-(1-\alpha_2\alpha_{t-2}(1-f(\beta_3,\cdots,\beta_{\lfloor (t-1)/2 \rfloor})))^{\beta_2}))^{\beta_1}.$$

We set

$$a = \alpha_1\alpha_{t-1},$$
$$b = 1-\alpha_2\alpha_{t-2}(1-f(\beta_3,\cdots,\beta_{\lfloor (t-1)/2 \rfloor})),$$
$$k = \beta_1\beta_2.$$

Then we have

$$f(\beta_1,\beta_2,\cdots\beta_{\lfloor (t-1)/2 \rfloor}) = (1 - a(1 - b^{\beta_2}))^{k/\beta_2}.$$

Similarly,

$$f(\beta_1',\beta_1\beta_2/\beta_1',\beta_3,\cdots,\beta_{\lfloor (t-1)/2 \rfloor}) = (1 - a(1 - b^{\beta_1\beta_2/\beta_1'}))^{\beta_1' k/\beta_1\beta_2}.$$

Since $\beta_2 \leq \beta_1\beta_2/\beta_1'$, by Lemma 1 we have the following:

$$f(\beta_1,\beta_2,\cdots,\beta_{\lfloor (t-1)/2 \rfloor}) \leq f(\beta_1',\beta_1\beta_2/\beta_1',\beta_3,\cdots,\beta_{\lfloor (t-1)/2 \rfloor}) \leq f(\beta_1',\beta_2',\cdots,\beta'_{\lfloor (t-1)/2 \rfloor}),$$

and Lemma 2 is proved.

Theorem 1: Consider two t-stage symmetric SP*-canopies with degree sequences* $(\lambda_1,\cdots,\lambda_{\lfloor (t-1)/2 \rfloor})$ *and* $(\lambda_1',\cdots,\lambda'_{\lfloor (t-1)/2 \rfloor})$ *respectively. Then the first linear graph is superior to the second if and only if the sequence* $(\lambda_1,\cdots,\lambda_{\lfloor (t-1)/2 \rfloor})$ *majorizes the sequence* $(\lambda_1',\cdots,\lambda'_{\lfloor (t-1)/2 \rfloor})$.

Proof: If we let $\alpha_i = 1 - p_i$, for $1 \leq i \leq t - 1$, in Lemma 2, it is easy to see that the blocking probability $P(\lambda_1,\cdots,\lambda_{\lfloor (t-1)/2 \rfloor})$ for the symmetric SP-canopy with degree sequence $(\lambda_1,\cdots,\lambda_{\lfloor (t-1)/2 \rfloor})$ has the same value as $f(\lambda_1,\cdots,\lambda_{\lfloor (t-1)/2 \rfloor})$. Thus, the fact that $(\lambda_1,\cdots,\lambda_{\lfloor (t-1)/2 \rfloor})$ majorizes $(\lambda_1',\cdots,\lambda'_{\lfloor (t-1)/2 \rfloor})$ implies the symmetric SP-canopy with degree sequence $(\lambda_1,\cdots,\lambda_{\lfloor (t-1)/2 \rfloor})$ is superior to the symmetric SP-canopy with degree sequence $(\lambda_1',\cdots,\lambda'_{\lfloor (t-1)/2 \rfloor})$.

We also want to show that, if the symmetric SP-canopy with degree sequence $(\lambda_1,\cdots,\lambda_{\lfloor (t-1)/2 \rfloor})$ is superior to the symmetric SP-canopy with degree sequence $(\lambda_1',\cdots,\lambda'_{\lfloor (t-1)/2 \rfloor})$, then it is necessary to have $(\lambda_1, \cdots,\lambda_{\lfloor (t-1)/2 \rfloor})$ majorizing $(\lambda_1',\cdots,\lambda'_{\lfloor (t-1)/2 \rfloor})$. Suppose, on the contrary, that there exists an integer k, $1 \leq k \leq \lfloor (t-1)/2 \rfloor$ such that $\Pi_{i=1}^k \lambda_i < \Pi_{i=1}^k \lambda_1'$. We consider the link occupancies $(p_1,\cdots,p_{t-1})$, where $p_i = p_{t-i} = 1 - \epsilon$ if $1 \leq i \leq k$ and $p_i = p_{t-i} = 0$ if $k \leq i \leq \lfloor (t-1)/2 \rfloor$. Then for ϵ sufficiently small, we have

$$\begin{aligned} P(\lambda_1,\cdots,\lambda_{\lfloor (t-1)/2 \rfloor}) &= P(\lambda_1,\cdots,\lambda_k) \\ &= (1 - \epsilon^2(1 - P(\lambda_2,\cdots,\lambda_k)))^{\lambda_1} \\ &= 1 - (\lambda_1\epsilon^2 + 0(\epsilon^4))(1 - P(\lambda_2,\cdots,\lambda_k)) \\ &= 1 - \epsilon^{2k} \prod_{i=1}^{k} \lambda_i + 0(\epsilon^{2k+2}). \end{aligned}$$

Similarly, $P(\lambda_1',\cdots,\lambda'_{\lfloor (t-1)/2 \rfloor})$ is approximately $1 - \epsilon^{2k} \Pi_{i=1}^k \lambda_i'$. Thus we have $P(\lambda_1',\cdots,\lambda'_{\lfloor (t-1)/2 \rfloor}) < P(\lambda_1,\cdots,\lambda_{\lfloor (t-1)/2 \rfloor})$. This contradicts the fact that the symmetric SP-canopy with degree sequence $(\lambda_1,\cdots,\lambda_{\lfloor (t-1)/2 \rfloor})$ is superior to the symmetric SP-canopy with degree sequence $(\lambda_1',\cdots,\lambda'_{\lfloor (t-1)/2 \rfloor})$, and Theorem 1 is proved.

III. REGULAR SP-CANOPIES

From definitions in Section I, it can be easily seen that a regular SP-canopy is either a parallel combination of copies of a smaller regular SP-canopy or series combination of a smaller regular SP-canopy and an edge. A regular SP-canopy has many special properties, described in the following lemma.

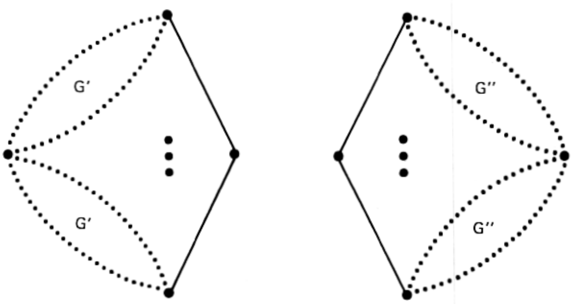
Lemma 3. Let $(\lambda_1,\cdots,\lambda_{t-1})$ *be the degree sequence of a regular SP-canopy G.*

(i) $\lambda_1\lambda_{t-1}$ *is either an integer or the reciprocal of an integer. If* $\lambda_1\lambda_{t-1}$ *is an integer, G is a parallel combination of copies of a smaller regular SP-canopy as shown in Fig. 2a, where G′ has degree sequence* $(\lambda_1\lambda_{t-1},\lambda_2,\cdots,\lambda_{t-2})$. *If* $\lambda_1\lambda_{t-1}$ *is the reciprocal of an integer, G is a parallel combination of copies of a small regular SP-canopy as shown in Fig. 2b where G'' has degree sequence* $(\lambda_2,\cdots,\lambda_{t-2},\lambda_{t-1}\lambda_1)$.

(ii) $\prod_{i=1}^{t-1} \lambda_i = 1.$

(iii) *If* $\lambda_k > 1$ *for some* k, $1 \le k < t-1$, *then* $\lambda_i \ge 1$ *for all* $i < k$. *If* $\lambda_{k'} < 1$ *for some* k', $1 < k' \le t-1$, *then* $\lambda_1 \le 1$ *for all* $i > k'$.

Proof: Since G is a regular SP-canopy, the configuration of G can be easily shown to be either as in Fig. 2a or as in Fig. 2b. If G is as in Fig. 2a, G is a parallel combination of k copies of a regular SP-canopy (for some k) which is a series combination of G' and an edge. Let G' have degree sequence $(\lambda'_1,\lambda'_2,\cdots,\lambda'_{t-2})$. It is clear that the degree sequence of G is $(k\lambda'_1,\lambda'_2,\cdots,\lambda'_{t-2},k^{-1})$ and $\lambda_1\lambda_{t-1} = (k\lambda'_1)k^{-1} = \lambda'_1$ is an integer. If G is as in Fig. 2b, G is a parallel combination of k' copies of a regular SP-canopy (for some k') which is a series combination of an edge and G''. Let G'' have degree sequence $(\lambda''_1,\lambda''_2,\cdots,\lambda''_{t-2})$. It is easy to see that the degree sequence of G is $(k',\lambda''_1,\lambda''_2,\cdots\lambda''_{t-2}(k')^{-1})$ and $\lambda_1\lambda_{t-1} = k'(\lambda''_{t-2}(k')^{-1}) = \lambda''_{t-2}$ is the reciprocal of an integer. Since one of the two cases must occur, (i) is proved.



(a) (b)

Fig. 2—Regular SP-canopy G.

We may assume without loss of generality that G is as in Fig. 2a. By the induction assumption, we have
$\Pi_{i=1}^{t-2} \lambda_1' = 1$. Thus, we have

$$\prod_{i=1}^{t-1} \lambda_i = k\lambda_1' \left(\prod_{i=2}^{t-2} \lambda_i'\right) k^{-1} = \prod_{i=1}^{t-2} \lambda_i' = 1,$$

and (ii) is proved.

Finally, λ_k being an integer implies $\lambda_k' = \lambda_k$ is an integer if $k > 1$. By the induction assumption, $\lambda'_j, j = 1,\cdots,k-1$, is an integer. Therefore, $\lambda_1 = k\lambda_1'$ is an integer and $\lambda_j = \lambda'_j, j = 2,\cdots,k$ integer. The other half of (iii) can be similarly verified. This proves Lemma 3.

We note that (ii) holds for the degree sequence of any regular linear graphs. However, (i) and (iii) are not true for series-parallel, regular linear graphs.

For a given vector, $\alpha = (\alpha_1,\cdots,\alpha_{t-1})$, $0 \le \alpha_i \le 1$ and positive real numbers $(\beta_1,\cdots,\beta_{t-1})$, we define the following function g by

$$g(\beta_1,\cdots,\beta_{t-1};\alpha) = \begin{cases} (1 - \alpha_1(1 - g(\beta_2,\cdots,\beta_{t-1}\beta_1;\alpha_{t-2,2})))^{\beta_1} & \text{if } \beta_1\beta_{t-1} \le 1, \\ (1 - \alpha_{t-1}(1 - g(\beta_1\beta_{t-1},\beta_2,\cdots,\beta_{t-2};\alpha_{t-2,1})))^{1/\beta_{t-1}} & \text{otherwise,} \end{cases}$$

where $\alpha_{k,i}$ is the vector $(\alpha_i,\cdots,\alpha_{i+k-1})$ and $g(\beta_i;\alpha_{1,i}) = 1 - \alpha_i$. Note that

$$g(\beta_1,\cdots,\beta_{t-1};\alpha) = g(\beta_{t-1}^{-1},\beta_{t-2}^{-1},\cdots,\beta_2^{-1},\beta_1^{-1};\overline{\alpha})$$

where

$$\overline{\alpha} = (\alpha_{t-1},\alpha_{t-2},\cdots,\alpha_2,\alpha_1).$$

Lemma 4: For any vector $\alpha = (\alpha_1,\cdots,\alpha_{t-1})$, if the sequence $(\beta_1,\beta_2,\cdots,\beta_{t-1})$ majorizes the sequence $(\beta_1',\beta_2',\cdots\beta_{t-1}')$, then we have

$$g(\beta_1,\beta_2,\cdots,\beta_{t-1};\alpha) \le g(\beta_1',\beta_2',\cdots,\beta_{t-1}';\alpha).$$

Proof: It is easy to see that Lemma 4 is true for $t = 2$. When $t = 3$, we have

$$g(\beta_1,\beta_2,;\alpha) = (1 - \alpha_1\alpha_2)^{\beta_1} \le (1 - \alpha_1\alpha_2)^{\beta_1'} = g(\beta_1',\beta_2';\alpha).$$

It suffices to consider $t \ge 4$. We note that
$(\beta_1,\beta_2,\cdots,\beta_{t-1})$ majorizes $(\beta_1',\cdots,\beta_{t-1}')$ if and only if $(\beta_{t-1}^{-1},\cdots,\lambda_2^{-1},\beta_1^{-1})$ majorizes $((\beta_{t-1}')^{-1},\cdots,(\beta_2')^{-1},(\beta_1')^{-1})$.
Therefore we may assume, without loss of generality, that $\beta_1\beta_{t-1} \le 1$ (we may consider the inverse sequence otherwise). Then we have

$$g(\beta_1,\cdots,\beta_{t-1};\alpha) = (1 - \alpha_1(1 - g(\beta_2,\cdots,\beta_{t-2},\beta_{t-1}\beta_1;\alpha_{t-2,2})))^{\beta_1}.$$

Let us consider several possibilities for $\beta_1'\beta_{t-1}'$.

Case (i): $\beta_1'\beta_{t-1}' \le 1$.

Since $\beta_1' \le \beta_1$, and $\beta_1'\beta_{t-1} \le 1$, we have

$$g(\beta_1',\beta_1\beta_2/\beta_1',\beta_3,\cdots,\beta_{t-1};\alpha)$$
$$= (1-\alpha_1(1-g(\beta_1\beta_2/\beta_1',\beta_3,\cdots,\beta_{t-1}\beta_1';\alpha_{t-2,2})))^{\beta_1'}.$$

We want to show that

$$g(\beta_1,\cdots,\beta_{t-1};\alpha) \le g(\beta_1',\beta_1\beta_2/\beta_1',\beta_3,\cdots,\beta_{t-1};\alpha).$$

We note that $\beta_2(\beta_{t-1}\beta_1) = (\beta_1\beta_2/\beta_1')(\beta_{t-1}\beta_1')$. It suffices to consider the following two cases.

(a) $\beta_1\beta_2\beta_{t-1} \le 1$.

$$g(\beta_1,\cdots,\beta_{t-1};\alpha)$$
$$= (1-\alpha_1\,(1-(1-\alpha_2(1-g(\beta_3,\cdots,\beta_{t-2},\beta_{t-1}\beta_1\beta_2;\alpha_{t-3,3})))^{\beta_2}))^{\beta_1}$$

and

$$g(\beta_1',\beta_1\beta_2/\beta_1',\beta_3,\cdots,\beta_{t-1};\alpha)$$
$$= (1-\alpha_1(1-(1-\alpha_2(1-g(\beta_3,\cdots,\beta_{t-2},\beta_{t-1}\beta_1\beta_2;\alpha_{t-3,3})))^{\beta_1\beta_2/\beta_1'}))^{\beta'_1}$$

By using Lemma 1 and the fact that $\beta_2 \le \beta_1\beta_2/\beta_1'$, it can be easily seen that

$$g(\beta_1,\cdots,\beta_{t-1};\alpha) \le g(\beta_1',\beta_1\beta_2/\beta_1',\beta_3,\cdots,\beta_{t-1};\alpha).$$

(b) $\beta_1\beta_2\beta_{t-1} > 1$.

The proof is similar to that of Case (a), and the proof is omitted.

The next step is to show that

$$g(\beta_1',\beta_1\beta_2/\beta_1',\beta_3,\cdots,\beta_{t-1};\alpha) \le g(\beta_1',\beta_2',\cdots,\beta_{t-1}';\alpha).$$

We note that $\beta_1'\beta_{t-1}' \le 1$ and

$$g(\beta_1',\cdots,\beta_{t-1}';\alpha) = (1-\alpha_1(1-g(\beta_2',\cdots,\beta_{t-1}'\beta_1';\alpha_{t-2,2})))^{\beta_1'}.$$

Because $(\beta_1\beta_2/\beta_1',\beta_3,\cdots,\beta_{t-1}\beta_1')$ majorizes $(\beta_2',\beta_3',\cdots,\beta_{t-2}',\beta_{t-1}'\beta_1')$, we have

$$g(\beta_1,\cdots,\beta_{t-1};\alpha) \le g(\beta_1',\beta_1\beta_2/\beta_1',\beta_3,\cdots,\beta_{t-1};\alpha)$$
$$\le g(\beta_1',\beta_2',\cdots,\beta_{t-1}';\alpha).$$

Case (ii): $\beta_1'\beta_{t-1}' > 1$.

Since $(\beta_1,\beta_2,\cdots,\beta_{t-1})$ majorizes $(\beta_1,\beta_2,\cdots,\beta_{t-2}\beta_{t-1}\beta_1,\beta_1^{-1})$ and $\beta_1\beta_1^{-1} = 1$, we have, from Case (i), that

$$g(\beta_1,\beta_2,\cdots,\beta_{t-1};\alpha) \le g(\beta_1,\beta_2,\cdots,\beta_{t-2}\beta_{t-1}\beta_1,\beta_1{}^{-1};\alpha)).$$

It suffices to show that

$$g(\beta_1,\beta_2,\cdots,\beta_{t-2}\beta_{t-1}\beta_1,\beta_1^{-1};\alpha) \le g(\beta_1',\beta_2',\cdots,\beta_{t-1}';\alpha).$$

It is easy to see that

$$(\beta_1,(\beta_{t-2}\beta_{t-1}\beta_1)^{-1},\beta_{t-3}^{-1},\cdots,\beta_2^{-1},\beta_1^{-1}) \text{ majorizes}$$
$$((\beta_{t-1}')^{-1},\cdots,(\beta_2')^{-1},(\beta_1')^{-1})$$

and $\beta_1\beta_1^{-1} \le 1$, $(\beta'_{t-1})^{-1}(\beta'_1)^{-1} \le 1$.

From Case (i) we have

$$g(\beta_1,(\beta_{t-1}\beta_{t-1}\beta_1)^{-1},\beta_{t-3}^{-1},\cdots,\beta_2^{-1},\beta_1^{-1};\overline{\alpha}) \le g((\beta'_{t-1})^{-1},\cdots,(\beta'_2)^{-1},(\beta'_1)^{-1};\overline{\alpha})$$

where $\overline{\alpha} = (\alpha_{t-1}, \alpha_{t-2},\cdots,\alpha_2,\alpha_1)$.

Therefore

$$g(\beta_1,\beta_2,\cdots,\beta_{t-2}\beta_{t-1}\beta_1,\beta_1^{-1};\alpha) \le g(\beta'_1,\beta'_2,\cdots,\beta'_{t-1};\alpha)$$

and Lemma 4 is proved.

It is easy to see that the blocking probability $P(\lambda_1,\cdots,\lambda_{t-1})$ of a regular SP-canopy with degree sequence $(\lambda_1,\cdots,\lambda_{t-1})$ for any link occupancy $\alpha = (\alpha_1,\cdots,\alpha_{t-1})$ has the same value as $g(\lambda_1,\cdots,\lambda_{t-1};\alpha)$. By using Lemma 4, the following theorem can be proved by techniques similar to those used in the proof of Theorem 1.

Theorem 2: Consider two t-stage regular SP*-canopies with degree sequences* $(\lambda_1,\cdots,\lambda_{t-1})$ *and* $(\lambda'_1,\cdots,\lambda'_{t-1})$*, respectively. Then the first linear graph is superior to the second if and only if the sequence* $(\lambda_1,\cdots,\lambda_{t-1})$ *majorizes* $(\lambda'_1,\cdots,\lambda'_{t-1})$.

IV. REMARKS AND EXAMPLES

In Fig. 3 we list a few examples. The degree sequence of the symmetric SP-canopy G_1 in Fig. 3a is $(3,2,1,2^{-1},3^{-1})$. The degree sequence of the symmetric SP-canopy G_2 in Fig. 3b is $(2,3,1,3^{-1},2^{-1})$. The degree sequence of the regular SP-canopy G_3 in Fig. 3c is $(2,3,3^{-1},1,2^{-1})$, and the

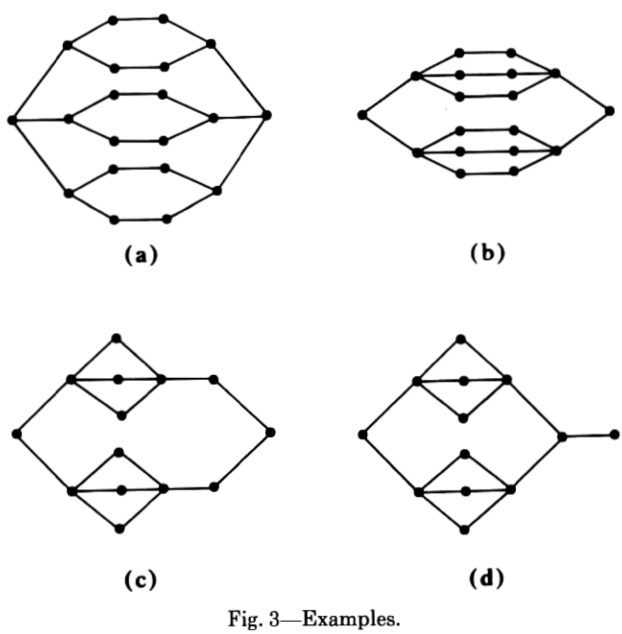


Fig. 3—Examples.

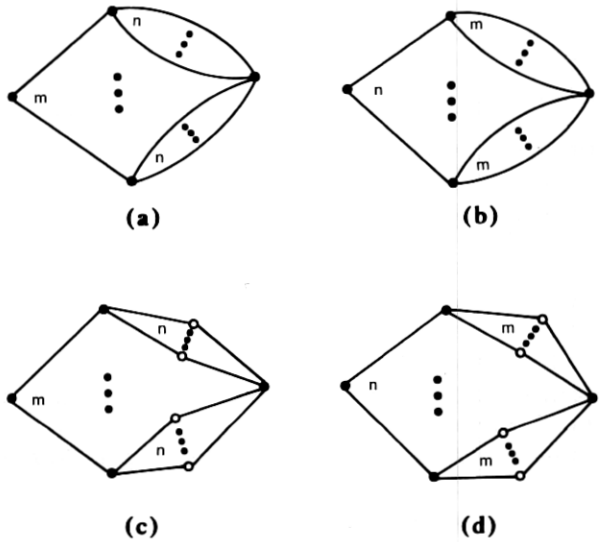


Fig. 4—Linear graphs.

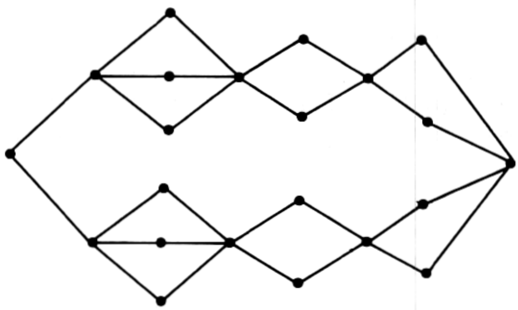


Fig. 5—Series-parallel, regular linear graph.

degree sequence of the regular SP-canopy in Fig. 3d is $(2,3,3^{-1},2^{-1},1)$. By Theorems 1 and 2, we note that, since $(3,2,1,2^{-1},3^{-1})$ majorizes $(2,3,1,3^{-1},2^{-1})$ and so forth, then G_1 is superior to G_2, which is superior to G_3, which is superior to G_4.

Although the result in this paper only involves SP-canopies, they can easily be generalized in the following ways:

(*i*) Every edge between stage i and state $i+1$ can be interpreted as a linear graph G_i.

(*ii*) Some linear graphs with multiple edges can be viewed as SP-canopies by adding imaginary stages or vertices so that we could then apply our results. For example, in order to compare linear graphs in Figs. 4a and 4b, we put an imaginary stage between the second and third stages. The resulting linear graphs are shown in Figs. 4c and 4d, respectively.

Suppose $m \geq n$ for the linear graphs in Fig. 4. It is easily seen that the degree sequence $(m,n,(mn)^{-1})$ of the linear graph in Fig. 4c majorizes the degree sequence $(n,m,(mn)^{-1})$ of the linear graph in Fig. 4d. Thus, we know that the linear graph in Fig. 4c is superior to that in Fig. 4d, and we may therefore conclude that the linear graph in Fig. 4a is superior to that in Fig. 4b.

More generally, we may consider the class of all series-parallel, regular linear graphs. For example, the linear graph in Fig. 5 is a series-parallel, regular linear graph but not a regular SP-canopy.

Is it true that a series-parallel regular linear graph is superior to another if its degree sequence majorizes the degree sequence of the other? We conjecture this is true. However, it seems that it cannot be proved by the methods we used in this paper.

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