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A Combinatorial Lemma and Its Application to Concentrating Trees of Discrete-Time Queues

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Concentrating rooted tree networks of discrete-time single server queues, all with unit service time, are considered. Such networks occur as subnetworks connecting remote access terminals to a node in a data communications network. It is shown that the network of queues may be replaced by a single queue, with prescribed input, which has the same output as the queue at the root of the tree. The result is applied, in particular, to the case of several queues in tandem, and it is shown how this problem may be reduced to that of just two queues in tandem. The latter problem was analyzed earlier by the author.

I. INTRODUCTION

In this paper we consider concentrating rooted tree networks of discrete-time single server queues, all with unit service time. Such networks occur as subnetworks connecting remote access terminals to a node in a data communications network.[1] Our purpose is to show that the rooted tree network of queues may be replaced by a single queue, with prescribed input, which has the same output as the queue at the root of the tree. In particular, the result is applied to the case of queues in tandem.

In Section II we consider the pooling of data from M buffers into a single buffer, which also receives data from another source, as depicted in Fig. 1. We establish a combinatorial lemma which shows that there is a single equivalent buffer, with prescribed input, and the same output as the buffer in which the data is pooled. It is then pointed out how this result may be applied to a concentrating rooted tree network of queues, such as the one depicted in Fig. 3. A related observation was made by Kaspi and Rubinovitch[2] in connection with networks of continuous time queues involving the pooling of data from inputs with idle periods that are exponentially distributed.

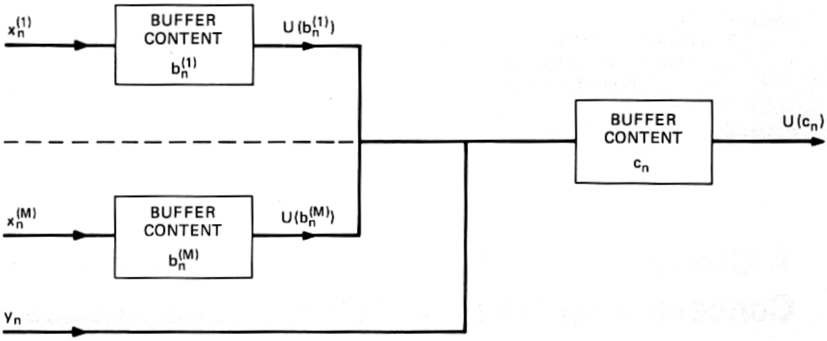

Fig. 1—Schematic of pooling of data from several buffers into a single buffer.

In Section III we consider the repeated application of the lemma to the case of several queues in tandem, as depicted in Fig. 4. It is shown how this problem may be reduced to that of just two queues in tandem, so that the results obtained for that problem[3] may be applied. Specifically, in the case that the input processes $z_n^{(i)}, i = 1, \ldots, I$, are mutually independent, and each process is a sequence of independent identically distributed nonnegative integer valued random variables, the generating function of the steady state distribution of the content of each buffer in Fig. 4 may be determined. Also, under the assumption that all arrivals take place at the end of a unit time interval, the average waiting time in each queue may be obtained.

II. COMBINATORIAL LEMMA

We first consider the pooling of data from M buffers into a single buffer, which also receives data from another source, as depicted in Fig. 1. It is assumed that a buffer transmits one packet, the basic unit of data, in a unit time interval, provided that it is not empty, and that the buffers are of unlimited size. Let $b_n^{(j)}, j = 1, \ldots, M$, denote the contents of the M buffers at time n, and let $x_n^{(j)}$ denote the corresponding number of packets entering the buffers in the time interval $(n,n + 1]$. We define

$$U(\ell) = \begin{cases} 1, & \ell = 1,2,\ldots, \\ 0, & \ell = 0. \end{cases} \tag{1}$$

Then the contents of the buffers at time $(n + 1)$ are given by the equations

$$b_{n+1}^{(j)} = b_n^{(j)} - U(b_n^{(j)}) + x_n^{(j)}, \quad j = 1, \ldots, M, \tag{2}$$

for $n = 0,1,2,\ldots$. It is assumed that the initial contents $b_0^{(j)}$, as well as the inputs $x_n^{(j)}$, are nonnegative integers.

The outputs of the M buffers enter another buffer, the content of which at time n is denoted by c_n. Also, the number of packets entering this other buffer in the time interval $(n,n+1]$ from another source is denoted by y_n. Then the content of this buffer at time $(n+1)$ is given by the equation

$$c_{n+1} = c_n - U(c_n) + \sum_{j=1}^{M} U(b_n^{(j)}) + y_n, \tag{3}$$

for $n = 0,1,2,\ldots$. It is assumed that the initial content c_0, as well as the inputs y_n, are nonnegative integers. We now show that there is a single equivalent buffer, with prescribed input, which has the same output.

Let e_n denote the content of the equivalent buffer at time n, and define

$$\begin{aligned} e_0 &= c_0, \\ e_n &= \sum_{j=1}^{M} [b_n^{(j)} - x_{n-1}^{(j)}] + c_n, \quad n = 1,2,\ldots. \end{aligned} \tag{4}$$

Further, we define the inputs

$$\begin{aligned} w_0 &= \sum_{j=1}^{M} b_0^{(j)} + y_0, \\ w_n &= \sum_{j=1}^{M} x_{n-1}^{(j)} + y_n, \quad n = 1,2,\ldots. \end{aligned} \tag{5}$$

Then we have the following

Lemma 1. Subject to (1)–(5), e_n is a nonnegative integer, and

$$\begin{aligned} U(e_n) &= U(c_n), \\ e_{n+1} &= e_n - U(e_n) + w_n, \quad n = 0,1,2,\ldots. \end{aligned} \tag{6}$$

Proof: It follows from (2)–(4) that

$$e_{n+1} = \sum_{j=1}^{M} b_n^{(j)} + c_n - U(c_n) + y_n, \quad n = 0,1,2,\ldots. \tag{7}$$

But, from (1), $\ell - U(\ell) \geq 0$. Hence e_{n+1} is a nonnegative integer for $n = 0,1,2,\ldots$, and so is $e_0 = c_0$, by assumption. Moreover, $e_{n+1} = 0$ implies that $b_n^{(j)} = 0$, $j = 1,\ldots,M$, $c_n = U(c_n)$ and $y_n = 0$, and hence, from (3), that $c_{n+1} = 0$. On the other hand, $c_{n+1} = 0$ also implies that $b_n^{(j)} = 0$, $j = 1,\ldots,M$, $c_n = U(c_n)$ and $y_n = 0$, and hence, from (7), that $e_{n+1} = 0$. Therefore $U(e_{n+1}) = U(c_{n+1})$, $n = 0,1,2,\ldots$, and $U(e_0) = U(c_0)$ since $e_0 = c_0$. Finally, from (7), with the help of (4) and (5),

$$e_{n+1} = e_n - U(c_n) + w_n, \quad n = 0,1,2,\ldots. \tag{8}$$

Since we have just shown that $U(e_n) = U(c_n)$, $n = 0,1,2,\ldots$, this completes the proof of the lemma.

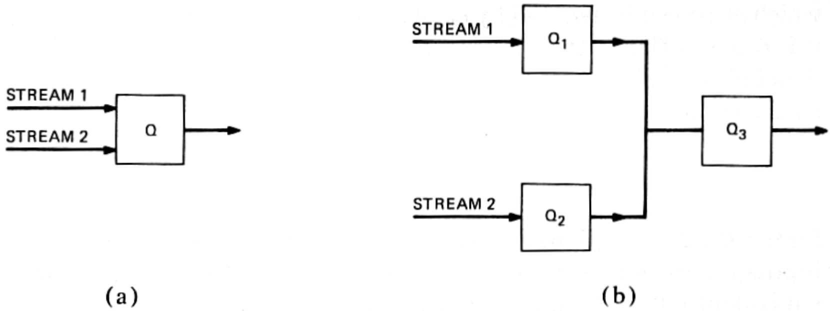

Fig. 2—(a) Queue, Q, fed by two independent input streams, and (b) Queue, Q_3, fed by the outputs of two queues, Q_1 and Q_2, which are fed separately by the two input streams.

In the particular case $M = 2$ and $y_n \equiv 0$, Ziegler and Schilling[4] obtained a related result, not restricted to discrete-time queues. They considered single server queues with identical constant service times, but assumed that the interarrival times between packets for each of the two independent input streams were governed by some general probability distribution. They compared a queue, Q, fed directly by the two input streams, and a queue, Q_3, fed by the outputs of two queues, Q_1 and Q_2, which are fed separately by the two input streams, as depicted in Fig. 2a and b. They established that the number of packets serviced at Q during its jth busy period is equal to the number serviced at Q_3 during its jth busy period, and hence that the jth idle periods at Q and Q_3 have the same duration. Note that in the discrete-time case we have shown that $U(e_n) = U(c_n)$, so that the corresponding buffers are empty at the same times.

Returning to our lemma, the result may be applied to concentrating rooted tree networks of discrete-time single server queues with unit service time, such as the network depicted in Fig. 3. The queues Q_1, Q_2 and Q_3 may be replaced by a single equivalent queue, $\hat{Q}_3$ say, which has a prescribed input sequence, $\hat{z}_n^{(3)}$ say, and the same output as Q_3. Then, by a second application of the lemma, the queues $\hat{Q}_3$, Q_4, Q_5 and Q_6 may be replaced by a single equivalent queue, $\hat{Q}_6$ say, which has a prescribed input sequence, $\hat{z}_n^{(6)}$ say, and the same output as Q_6. Thus the rooted tree network of Fig. 3 may be replaced by a single queue with the same output and prescribed input. In the next section we consider the repeated application of the lemma to several queues in tandem.

III. TANDEM QUEUES

We now consider I discrete-time single server queues, with unit service times, in tandem, as depicted in Fig. 4. The output of buffer i enters buffer $i + 1$, for $i = 1, \ldots, I - 1$. Let $d_n^{(i)}$, $i = 1, \ldots, I$, denote the content

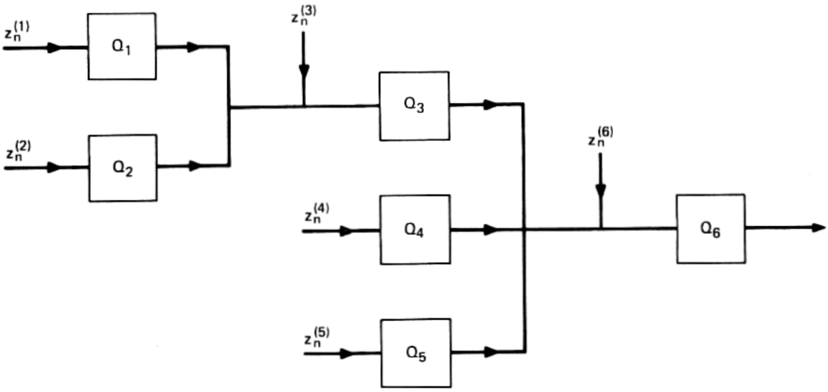

Fig. 3—Example of a concentrating rooted tree network of queues.

of buffer i at time n, and let $z_n^{(i)}$ denote the corresponding number of packets entering the buffer from a source in the time interval $(n,n+1]$. For convenience, we define $d_n^{(0)} \equiv 0$. Then the content of buffer i at time $n+1$ is given by the equation

$$d_{n+1}^{(i)} = d_n^{(i)} - U(d_n^{(i)}) + U(d_n^{(i-1)}) + z_n^{(i)}, \tag{9}$$

for $n = 0,1,2,\ldots$, and $i = 1,\ldots,I$. It is assumed that the initial contents $d_0^{(i)}$, as well as the inputs $z_n^{(i)}$, are nonnegative integers.

Let

$$e_0^{(i+1)} = d_0^{(i+1)}, \quad i = 1,\ldots,I-1, \tag{10}$$

and

$$e_n^{(1)} = d_n^{(1)}, \quad n = 0,1,2,\ldots. \tag{11}$$

Moreover, define

$$v_n^{(i)} = \begin{cases} \sum_{k=1}^{i} z_{n-i+k}^{(k)}, & n = i-1,i,\ldots, \\ d_0^{(i-n-1)} + \sum_{k=i-n}^{i} z_{n-i+k}^{(k)}, & n = 0,\ldots,i-2, \end{cases} \tag{12}$$

for $i = 1,\ldots,I$, and let

$$e_n^{(i+1)} = e_n^{(i)} + d_n^{(i+1)} - v_{n-1}^{(i)}, \tag{13}$$

for $n = 1,2,\ldots$, and $i = 1,\ldots,I-1$. Then we have the following

Lemma 2. Subject to (9)–(13), $e_n^{(i)}$ is a nonnegative integer, and

$$\begin{aligned} U(e_n^{(i)}) &= U(d_n^{(i)}), \\ e_{n+1}^{(i)} &= e_n^{(i)} - U(e_n^{(i)}) + v_n^{(i)}, \end{aligned} \tag{14}$$

for $n = 0,1,2,\ldots,$ and $i = 1,\ldots,I$.

Proof: Since $d_n^{(0)} \equiv 0$, it follows by definition, from (9), (11), and (12), that the lemma holds for $i = 1$. We proceed by induction on i. We assume that the lemma holds for some $i < I$, and will show that it holds for $i + 1$. We identify $e_n^{(i)}$, $d_n^{(i+1)}$, $e_n^{(i+1)}$, $v_n^{(i)}$ and $z_n^{(i+1)}$ with $b_n^{(1)}$, c_n, e_n, $x_n^{(1)}$ and y_n, respectively, for $n = 0,1,2,\ldots$. The induction hypothesis then implies (2), with $M = 1$, and, from (9),

$$d_{n+1}^{(i+1)} = d_n^{(i+1)} - U(d_n^{(i+1)}) + U(e_n^{(i)}) + z_n^{(i+1)}, \tag{15}$$

and hence (3), with $M = 1$. Moreover, (10) and (13) imply that (4) holds, with $M = 1$. Hence, from Lemma 1, with $M = 1$, it follows that $e_n^{(i+1)}$ is a nonnegative integer, and

$$U(e_n^{(i+1)}) = U(d_n^{(i+1)}), \quad n = 0,1,2,\ldots. \tag{16}$$

Also, using (5),

$$e_1^{(i+1)} = e_0^{(i+1)} - U(e_0^{(i+1)}) + e_0^{(i)} + z_0^{(i+1)}, \tag{17}$$

and

$$e_{n+1}^{(i+1)} = e_n^{(i+1)} - U(e_n^{(i+1)}) + v_{n-1}^{(i)} + z_n^{(i+1)}, \tag{18}$$

for $n = 1,2,\ldots$.

But, from (10)–(12),

$$e_0^{(i)} + z_0^{(i+1)} = d_0^{(i)} + z_0^{(i+1)} = v_0^{(i+1)}. \tag{19}$$

Also, for $i \geq 2$ and $n = 1,\ldots,i-1$,

$$v_{n-1}^{(i)} + z_n^{(i+1)} = d_0^{(i-n)} + \sum_{k=i-n+1}^{i+1} z_{n-1-i+k}^{(k)} = v_n^{(i+1)}. \tag{20}$$

Finally, for $n = i, i+1,\ldots,$

$$v_{n-1}^{(i)} + z_n^{(i+1)} = \sum_{k=1}^{i+1} z_{n-1-i+k}^{(k)} = v_n^{(i+1)}. \tag{21}$$

Hence, from (17)–(21),

$$e_{n+1}^{(i+1)} = e_n^{(i+1)} - U(e_n^{(i+1)}) + v_n^{(i+1)}, \tag{22}$$

for $n = 0,1,2,\ldots$. In view of (16), this completes the proof by induction.

From (9) and (14) we have the following

Corollary. For $n = 0,1,2,\ldots,$ and $i = 1,\ldots,I-1$,

$$\left.\begin{aligned} e_{n+1}^{(i)} &= e_n^{(i)} - U(e_n^{(i)}) + v_n^{(i)}, \\ d_{n+1}^{(i+1)} &= d_n^{(i+1)} - U(d_n^{(i+1)}) + U(e_n^{(i)}) + z_n^{(i+1)}. \end{aligned}\right\} \tag{23}$$

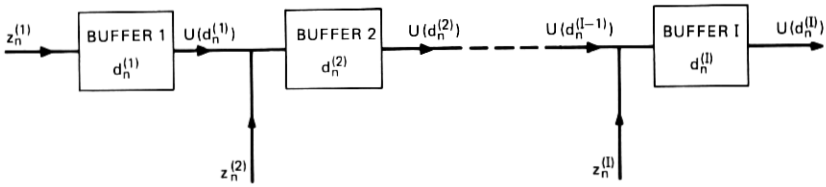

Fig. 4—Schematic of several queues in tandem.

Thus, by replacing the first i queues in Fig. 4 by a single equivalent queue, with the same output as the ith queue, we have reduced the problem of several queues in tandem to that of just two in tandem.

Suppose now that the input processes $z_n^{(i)}$, $i = 1, \ldots, I$, are mutually independent, and that each is independently and identically distributed (i.i.d.), with

$$E(s^{z_n^{(i)}}) = \phi_i(s), \quad i = 1, \ldots, I. \tag{24}$$

Then, from (12),

$$E(s^{v_n^{(i)}}) = \prod_{k=1}^{i} \phi_k(s), \quad n = i-1, i, \ldots, \tag{25}$$

and the input processes $v_n^{(i)}$ and $z_n^{(i+1)}$ are mutually independent, and each is i.i.d. The problem of two queues in tandem was investigated recently,[3] and the results are applicable to (23). The generating function of the steady state distribution of the contents of the two buffers was calculated, under the assumption that the mean combined input rate from the two sources is less than unity. Accordingly, we assume that

$$\sum_{i=1}^{I} E(z_n^{(i)}) < 1. \tag{26}$$

Then we may use (23) to calculate the generating function of the steady state distribution of the content of each buffer in Fig. 4. The initial values $v_0^{(i)}, \ldots, v_{i-2}^{(i)}$, for $i \geq 2$, do not affect the steady state distributions.

A particular example was considered,[3] in which the input to the first queue is geometrically distributed, while the input from the source into the second queue is either 0 or 1, with fixed probabilities. The steady state probability that the content of the second buffer exceeds m was calculated, and asymptotic results were derived for $m \gg 1$. It would be of interest to carry out an analogous derivation for the case of Poisson inputs to both queues. The results would be applicable to the case of Poisson inputs into I queues in tandem, corresponding to $\phi_i(s) = \exp[\lambda_i(s-1)]$ in (24). Then, from (25), the input process $v_n^{(i)}$ is also Poisson, for $n = i-1, i, \ldots$, with parameter $\sum_{k=1}^{i} \lambda_k$.

Formulas were derived[3] for the average waiting times in two queues in tandem, under the assumption that all arrivals take place at the end of a unit time interval. The average waiting time in the second queue was taken over all arrivals to that queue, both from the source and from the first queue. The results may be applied to (23), to obtain the average waiting times in each of the I queues in Fig. 4. The averages are over all arrivals to each queue.

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