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Jitter Comparison of Tones Generated by Squaring and by Fourth-Power Circuits

By J. E. MAZO

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The tone-to-jitter power ratio is calculated for some conventional methods of generating a tone at the pulse repetition frequency of a PAM data signal by operating on the latter by an appropriate nonlinearity. Attention is focused on this ratio for a fourth-order nonlinearity, which will produce a tone even in the absence of excess bandwidth, and on this ratio for a square-law nonlinearity for small excess bandwidth. If the excess bandwidth is less than about 50 percent, the fourth power is superior. In particular, it yields a 10-dB improvement for 12 percent roll-off and binary data.

I. INTRODUCTION

Successful detection of the symbols in a pulse-modulated waveform requires a knowledge of the pulse repetition period T. Specifically, if the signal $s(t)$ is of the form

$$s(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT) \tag{1}$$

where a_n are independent equiprobable binary symbols having values ± 1, knowledge of T is required for proper sampling of $s(t)$ to recover the a_n. Due to small differences in transmitter and receiver oscillators, *a priori* information concerning T is not usually sufficient, and constant updating of the precise current value of T is required. Often one prefers to deduce such information directly from eq. (1), rather than directly transmitting a tone at frequency $1/T$ Hz.

A very popular method is to pass $s(t)$ through an appropriate nonlinear circuit (e.g., a square-law) so that a tone is generated at frequency $1/T$.[1] For example, using the square-law operation we have the following identity:

$$s^2(t) = \left(\sum_{-\infty}^{\infty} a_n g(t - nT)\right)^2 = \sum_{n=-\infty}^{\infty} g^2(t - nT) + \sum_{n,m}^{n \neq m} a_n a_m g(t - nT) g(t - mT) \tag{2}$$

where in writing (2) we have used $a_n^2 = 1$.† The essential features of (2) are made clear on noting the Poisson sum formula‡

$$\sum_{n=-\infty}^{\infty} f(t - nT) = \frac{1}{T} \sum_m e^{2\pi imt/T} F\left(\frac{2\pi}{T} m\right). \tag{3}$$

Thus the first term of the last member of (2) consists of a series of sine and cosine terms given by the right member of (3), where in (3) $f(t)$ is replaced by $g^2(t)$. In our particular case, we are especially interested in the terms corresponding to frequency $\omega = 2\pi/T$ rad/sec, i.e., the terms

$$\frac{1}{T}\left[e^{2\pi i/T} F\left(\frac{2\pi}{T}\right) + e^{-2\pi i/T} F\left(-\frac{2\pi}{T}\right)\right] \tag{4}$$

where, explicitly

$$F(\omega) = \int_{-\infty}^{\infty} g^2(t) e^{-i\omega t} dt. \tag{5}$$

In the hardware, these terms are isolated by a very narrow postfilter (of bandwidth B Hz, say) approximately centered about this frequency.

In order that $F[\pm 2\pi/T] \neq 0$, "excess bandwidth" is required for the pulse $g(t)$, that is, its frequency spectrum must extend beyond the Nyquist frequency $\omega = \pi/T$. The percent of excess is usually referred to as the "rolloff."

Once $g(t)$ is given, the tone power is easily evaluated using (4) and (5). Thus, for simplicity, assume the $g(t)$ is an ideal Nyquist pulse having the real Fourier transform $G(\omega)$ shown in Fig. 1, where percent of rolloff equals $100 \times \alpha$.

Using the general formula

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega') G(\omega - \omega') d\omega' \tag{6}$$

we note how $F(2\pi/T)$ depends on $G(\omega)$ only in the rolloff region

$$\frac{\pi}{T}(1 - \alpha) \leq |\omega| \leq \frac{\pi}{T}(1 + \alpha).$$

For the special case of $G(\omega)$ given in Fig. 1 we calculate

$$F\left(\pm \frac{2\pi}{T}\right) = \frac{\alpha T}{4}, \tag{7}$$

† For a multilevel situation we would replace $a_n^2 = 1$ with $E(a_n)^2$, where E denotes statistical expectation.

‡ In (3)

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-i\omega t] f(t) dt$$

is the Fourier transform of $f(t)$.

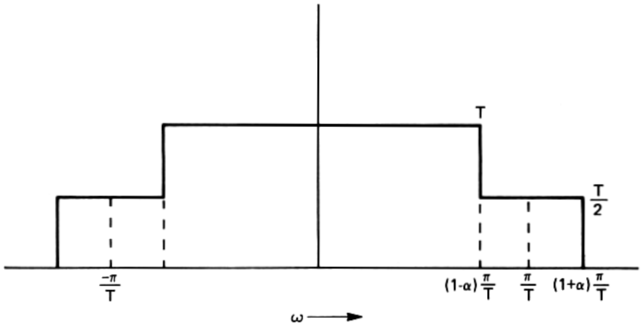

Fig. 1—Fourier transform $G(\omega)$ of the Nyquist pulse $g(t)$ used for the calculations.

allowing (4) to be rewritten as

$$\frac{\alpha}{2} \cos \frac{2\pi t}{T} \tag{8}$$

Thus from (8),

$$\text{square-law tone power} = \frac{\alpha^2}{8}. \tag{9}$$

Tone power is not a sufficient measure of performance, of course. This power must be compared to the power in the background noise. One component of this background would be additive noise on the channel. This is usually negligible, however, and the background to which we refer is self-generated. Mathematically it is given by the last term in (2). This added background will cause the zero-crossings of (8) to be perturbed, or "jitter," about their nominal values. Computing the tone power/background power ratio for various situations is the purpose of our work.† In addition to the state of affairs just described, two other proposals are of considerable practical interest. These may be conveniently and descriptively described as

(*i*) Prefiltering.
(*ii*) Fourth power law.

To motivate the first, recall that the tone power is determined by the "overlap" of the excess bandwidth regions in the integral (6). This power will not be changed if we filter out the remaining central portion of the pulse (the region $|\omega| < (\pi/T)(1-\alpha)$ in Fig. 1) before we do the squaring operation. The elimination of this portion of $G(\omega)$ does decrease the *total* power in the background term of (2). Will it improve the tone-to-jitter ratio in the neighborhood of $\omega = 2\pi/T$ as well?

† Since the tone will be "picked-off" out of the background by a narrow-band filter or phase-lock loop, only the value of the background power spectrum at $\omega = 2\pi/T$ will be needed.

The motivation of the second situation also begins by mentioning the overlap contribution to the integral in (6). We note that the overlap, and hence the tone power (9) vanish as α vanishes. Thus for a squaring circuit no tone is produced if there is no excess bandwidth. We shall see later that this is not true for all nonlinearities, and in fact a fourth-power law will produce a strong tone even when $\alpha = 0$. Again, what about the jitter? Our calculation of the latter for the fourth power is a completely new result. To appreciate the difficulties involved here note that the time averaged autocorrelation function $R(\tau)$ for the output of the fourth-power law is given by

$$R(\tau) = E \frac{1}{T} \int_0^T s^4(t) s^4(t - \tau) dt \tag{10}$$

with $s(t)$ given by (1). A straightforward evaluation of the matematical expectation in (10) would involve dealing with the eighth-order terms

$$E a_{n_1} a_{n_2} a_{n_3} \cdots a_{n_8}. \tag{11}$$

The bookkeeping involved with (11) would be unmanageable. One novelty of our method is the introduction of a technique from the algebra of symmetric polynomials[2] for skirting the direct evaluation implied by (11).

Since we are mainly interested in knowing if prefiltering or the fourth-power law can produce large improvements in tone-to-jitter ratio, our explicit evaluations will be based on simple pulse shapes. For prefiltering, overlap is important and we stick to the pulse shape with frequency characteristic given in Fig. 1. For fourth-order we assume that small excess bandwidth produces only higher-order corrections to the effect present when $\alpha = 0$. Consequently in this case we use only the $\alpha = 0$ pulse.

We shall show that prefiltering offers no improvement at all. With or without prefiltering the output tone-to-jitter ratio is about 12 dB if an output filter 10 Hz wide is used or 22 dB for one 1 Hz wide. This assumes a 12 percent excess bandwidth as in the Bell System 209 data set. Numbers for the fourth-power law are 10 dB better than this, which is a significant improvement.

Before proceeding, a final comment is in order. This concerns a recent publication of Franks and Bubrowski[3] concerning prefiltering and the square-law nonlinearity. Their claim is that if prefiltering has a symmetrical result about π/T and the post-filter is symmetrical about $2\pi/T$, there will be no jitter about the zero-crossings of (8). This is true, but if T were known exactly so that the required symmetrization could be done exactly, then there would be no need to measure T. If, however, we symmetrize about a $T' \neq T$, then the background, using a standard representation of passband signals, would have the form

$$y(t)\cos\frac{2\pi}{T'}t \tag{12}$$

with no quadrature component relative to $2\pi/T'$. If $T' = T$ we see why the zeros of (8) are unchanged. If $T' \neq T$ the quadrature component of (12) relative to $2\pi/T$ will come in, with a strength *independent* of how small $T' - T$ is. Only the beat frequency depends on $T' - T$. Thus the Franks-Bubrowski result might be termed unstable and not applicable.

II. BACKGROUND SPECTRUM FOR SQUARING CIRCUIT

In this section we compute $S_c(2\pi/T)$, the value of the spectrum $S_c(\omega)$ of the jitter term which appears at the output of the squaring circuit at angle frequency $\omega = 2\pi/T$.† We do this for the special pulse of Fig. 1 with and without prefiltering. In terms of this quantity the tone-jitter ratio will be, for a final filter of bandwidth B, using (9),

$$\frac{\text{tone power}}{\text{background power}} = \frac{\alpha^2}{8}\bigg/ 2S\left(\frac{2\pi}{T}\right)B. \tag{13}$$

The quantity $S_c(\omega)$ is the Fourier transform of the autocorrelation function

$$R(\tau) = \frac{1}{T}\int_0^T E[s^2(t)s^2(t-\tau)]dt \tag{14}$$

$$S_c(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau}R(\tau)d\tau - \begin{pmatrix}\text{spectral}\\ \text{lines}\end{pmatrix} \tag{15}$$

where in (14) $s^2(t)$ is given by (2). Denoting $g(t - nT)$ by g_n and $g(t - \tau - nT)$ by h_n, the first item to evaluate is

$$Es^2(t)s^2(t-\tau) = E[(\Sigma a_n g_n)^2(\Sigma a_n h_n)^2], \tag{16}$$

the expectation being taken over the i.i.d. binary variables a_n, having values ± 1 with equal probability. The expectation in the right side of (16) can be done directly, using

$$Ea_pa_qa_ra_s = \delta_{pq}\delta_{rs} + \delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr} - 2\delta_{pq}\delta_{pr}\delta_{ps} \tag{17}$$

to yield

$$E[(\Sigma a_n g_n)^2(\Sigma a_n h_n)^2] = (\Sigma g_n^2)(\Sigma h_n^2) + 2(\Sigma g_n h_n)^2 - 2\Sigma g_n^2 h_n^2. \tag{18}$$

Looking ahead to the fourth-order case when we will need the average of eight order terms, we will not be able to write the analog of (17) in any

† The subscript c on $S_c(\omega)$ emphasizes that only the continuous portion of the complete spectrum is being considered.

Table I — Summary of the evaluation of background power for a squaring circuit

Coefficient	Term	Contribution to $S(2\pi/T)$ (without coefficient)
1	$(11)(aa)$	Tone term
2	$(1a)^2$	$\frac{\alpha T}{16}$
−2	$(11aa)$	$\frac{\alpha^2 T}{16}$

manageable way. Thus it will be pedagogically useful to introduce the new method here first, and evaluate (18) again. We first notice from the structure of the left side of (18) that only terms of the type $(\Sigma g^2)(\Sigma h^2)$, $(\Sigma gh)^2$ and $\Sigma g^2 h^2$ can occur on the right-hand side, i.e., the answer must be of the form

$$A(\Sigma g^2)(\Sigma h^2) + B(\Sigma gh)^2 + C(\Sigma g^2 h^2) \tag{19}$$

where A, B, and C are constants independent of whatever values the g_n and h_n take. Setting $g_n = h_n = \delta_{n0}$, the left-hand side of (18) is obviously unity. Then also using (19) we obtain the result that

$$1 = A + B + C. \tag{20}$$

Likewise setting $g_n = h_{n+1} = \delta_{n0}$ yields (since $g_n h_n = 0$, all n)

$$1 = A \tag{21}$$

while the choice $g_0 = h_0 = 1$, $g_1 = h_1 = 1$, $g_k = h_k = 0$, $k \neq 0,1$ provides

$$8 = 4(A + B) + 2C. \tag{22}$$

The solution of (20)–(22) is $A = 1, B = 2, C = -2$ in complete agreement with (18).

It is convenient to introduce the following shorthand: a sum $\Sigma_{n=-\infty}^{\infty}$ will be denoted by a parenthesis ( ). If the nth term of the sum is $g_n^p h_n^q$ the notation

$$(\underbrace{1\,1 \cdots 1\,1}_{p \text{ times}}\ \underbrace{a\,a \cdots a}_{q \text{ times}})$$

is used. Thus the terms in (19) are of the types $(11)(aa)$, $(1a)^2$, and $(11aa)$ and the right side of (18) is given in the first two columns of Table I.

Having now obtained the sums and coefficients in (18), the next step is to evaluate the sums by the Poisson sum formula (regarding τ as a fixed parameter). Note first that, from (2), the term $(11)(aa)$ in (18) is due to the deterministic part of (2) and hence the background terms are only

$$2(1a)^2 - 2(11aa). \tag{23}$$

After the Poisson sum formulas are evaluated we next perform

$$\frac{1}{T}\int_0^T (\ )dt$$

to eliminate the t-dependence. Finally the Fourier transform with respect to T is taken. For example,

$$(11aa) = \sum_{n=-\infty}^{\infty} g^2(t-nT)g^2(t-\tau-nT)$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty} e^{2\pi imt/T}F\left(\frac{2\pi}{T}m\right)$$

where $F(\omega)$ is the Fourier transform of $g^2(t)g^2(t-\tau)$. If $G_2(\omega)$ is the Fourier transform of $g^2(t)$ then

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega'\tau}G_2(\omega')G_2(\omega-\omega')d\omega'.$$

When we time-average $(11aa)$ only the $m=0$ term survives, giving

$$\frac{1}{T}\int_0^T (11aa)dt = \frac{1}{T}F(0) = \frac{1}{2\pi T}\int_{-\infty}^{\infty} e^{i\omega'\tau}G_2(\omega')G_2(-\omega')d\omega'.$$

The Fourier transform of this with respect to τ is simply $(1/T)G_2(\omega)$-$G_2(-\omega)$ which is to be evaluated at $\omega = 2\pi/T$.

The actual contribution of the terms in (23) is listed in the third column of Table I, the same values being obtained with or without prefiltering. Thus from (9), (13), (23) and Table I we obtain for the squaring loop, with or without a prefilter,

$$\frac{\text{tone power}}{\text{background power}} = \frac{\alpha^2/8}{2B[2(1a)^2 - 2(11aa)]}$$

$$= \frac{1}{2BT}\frac{\alpha}{1-\alpha}, \quad \alpha < 1. \tag{24}$$

The fact that (24) becomes infinite for $\alpha = 1$ is due to the fact that the spectrum has a zero then, and higher-order terms in B would be required to estimate the background power.

Applying these results to the 209 data set where $\alpha = 0.12$, $1/T = 2400$ sec^{-1}, (24) evaluates to about a 12 dB tone/filter ratio if B = 10 Hz, or 22 dB if $B = 1$ Hz.

In order to provide some contrast between the cases which do or do not involve prefiltering, the output spectrum in a neighborhood of $2\pi/T$

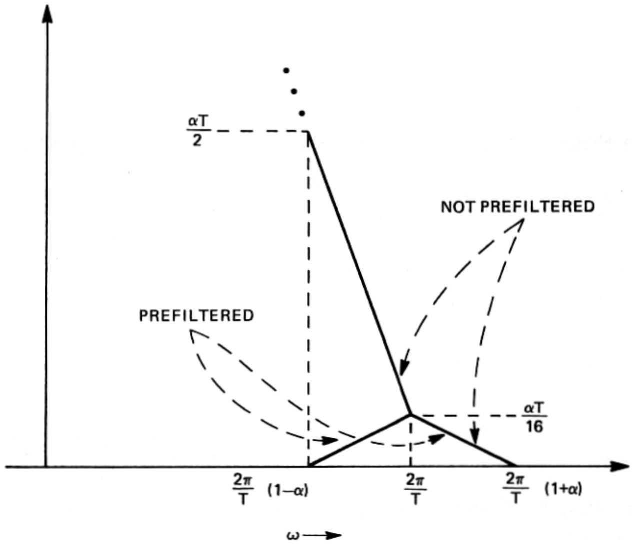

Fig. 2—Background power spectra for squaring circuit, with and without prefilter.

is given in Fig. 2 for small α, i.e. neglecting the $(11aa)$ contribution which is proportional to α^2.

The two spectra coincide above $2\pi/T$, but the divergence between the two on the lower side of the tone frequency is apparent from the curve.

III. BACKGROUND SPECTRUM FOR FOURTH-POWER CIRCUIT

The lines in the spectrum of $s^4(t)$ come from the deterministic terms, namely from $Es^4(t)$. Setting $g_n = h_n$ in (18) provides us with the evaluation

$$Es^4(t) = 3[\Sigma g^2(t - nT)]^2 - 2\Sigma g^4(t - nT). \tag{25}$$

If $\alpha = 0$ only the second term in (25) contributes a tone at $2\pi/T$. Since, when $\alpha = 0$,

$$2\Sigma g^4(t - nT) = \frac{4}{3} + \frac{2}{3}\cos\frac{2\pi t}{T},$$

the power in the tone is

$$\frac{1}{2}\left(\frac{2}{3}\right)^2 = \frac{2}{9}.$$

As before, the first step in the evaluation of the background spectrum is the calculation of

$$Es^4(t)s^4(t - \tau). \tag{26}$$

Table II — Summary of the evaluation of the background power for a fourth-power circuit

Coefficient	Term	Contribution to $S(2\pi/T)$ (without coefficient)
96	$(1111aa)(aa)$	0
256	$(111aaa)(1a)$	$T/20$
96	$(11aaaa)(11)$	0
24	$(1a)^4$	$T/6$
9	$(11)^2(aa)^2$	Tone term
72	$(11)(aa)(1a)^2$	0
4	$(1111)(aaaa)$	Tone term
64	$(111a)(1aaa)$	$T/120$
72	$(11aa)^2$	$T/30$
−6	$(1111)(aa)^2$	Tone term
−6	$(aaaa)(11)^2$	Tone term
−96	$(111a)(1a)(aa)$	0
−96	$(aaa1)(1a)(11)$	0
−72	$(11aa)(11)(aa)$	0
−144	$(11aa)(1a)^2$	$T/12$
−272	$(1111aaaa)$	$T/36$

Here the second technique introduced in Section II becomes decisive. Just as the results of the evaluation of (18) are summarized in Table I, the first two columns of Table II give the evaluation of (26).

We must next evaluate the contribution of the terms to $S_c(2\pi/T)$. Simplifications occur when $\alpha = 0$, since then

$$(11) = (aa) = 1 \text{ and } (1a) = \frac{\sin \dfrac{\pi\tau}{T}}{\dfrac{\pi\tau}{T}}.$$

The final result of applying the Poisson sum formula, averaging over t, and Fourier-transforming with respect to τ gives the results in the third column, Table II.

Collecting this we have, for a final filter of bandwidth B at $2\pi/T$, that

$$\frac{\text{signal power}}{\text{background power}} = \frac{5}{8BT}. \tag{27}$$

This is a significant improvement over the result (24) for the squaring-loop for small α. In fact, using $\alpha = 0.12$ again we calculate an improvement factor of 9.16 or close to 10 dB.

IV. ACKNOWLEDGMENT

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