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Loop Plant Modeling:

Economic Design of Distribution Cable Networks

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Distribution plant under the Serving Area Concept (SAC) is the plant on the customer side of the Serving Area Interface. The major part of this plant is the cable network connecting each customer to the interface. Sizing of distribution cables involves a trade-off between current construction costs and future costs that may be incurred. Thus, providing more cable pairs initially costs more at the outset, but reduces relief and rearrangement costs in the future. A set of cost models is described which allows these trade-offs to be studied. These models are applied to examples of aerial cable plant to show how the best cable sizing may be determined.

I. INTRODUCTION AND SUMMARY

The overall structure of the loop plant, as well as the relevant terminology, is described by Long.[1] As described there, the complex cable network that makes up the plant is divided into feeder—the large cables emanating from the central office; and distribution—the finer cable branches ending in the customer's premises. Under the Serving Area Concept (SAC), to which attention is directed in this paper, distribution plant is that on the customer side of the Serving Area Interface. In the past, feeder plant has received considerable analytical attention,[2] distribution plant less so.

This paper describes a set of analytical models which are specifically tailored to the distribution plant, and which may be used for economic evaluation. In a later section, these models are applied to an example of aerial distribution plant to show how the best cable sizes may be obtained in that case.

The purpose of the models is to enable an economic trade-off to be made between current construction costs, and future costs, for distribution plant. When new distribution plant is constructed, or existing plant is upgraded, a basic decision that must be made is how much cable to place. The more cable placed now, the less future costs will be, because there will be less future need to relieve (provide more cable) or rearrange the network.

In the past, in some networks employing multiple plant,* rearrangement costs have been high, giving a continuing operating cost which was burdensome. In contrast, new SAC plant as currently installed is sized so that *no* future relief or rearrangement should be necessary. The models described in this paper enable comparison of these and other alternatives, so that the optimum trade-off between current and future costs may be determined. The optimum situation is one in which the present worth of all costs is minimized .

Economic sizing of plant in this way is currently practiced in the feeder network.[2,3] This paper extends the concept to the distribution plant.

The paper is organized as follows. The remainder of this section describes in more detail the problem under study and the approach taken: Section 1.1 details the sources of costs and the resulting cost models, Section 1.2 tells more about SAC design, and Section 1.3 describes the standard serving area used in all subsequent analysis. Details of the cost models then follow in Section II. In Section III these models are applied to examples of aerial cable plant.

1.1 Sources of costs; cost models

The most obvious cost for distribution plant is the *cost of current construction.* For a new serving area, or major upgrading of an existing one (such as conversion from multiple plant to SAC), this will be the major cost. What is modeled is the cost of material and installation for cables, terminals, and interface for a serving area. The interface connects the feeder and distribution cables, and the terminal connects the distribution cable to the service wire entering the customer's premises (see Fig. 1).

The size of distribution cabling is conventionally specified by the number of pairs provided per customer living unit† (pr/l.u.). As an example, SAC design generally specifies two pr/l.u. This means that for each living unit, two cable pairs are provided from the distribution terminal (or, in some cases, the customer's premises) back to the interface.

It is assumed that a primary line pair should be provided to each living unit. Extra pairs above this 1.0 pr/l.u. are available for *additional lines* (such as teenage or alarm lines) going to a living unit that already has a primary line. Additional line *penetration* is the number of additional

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* See Ref. 1 for a description of the various types of distribution plant.

† A living unit is one customer's address—a house, apartment, etc. A living unit may require more than one telephone line (e.g., a primary and a teenage line).

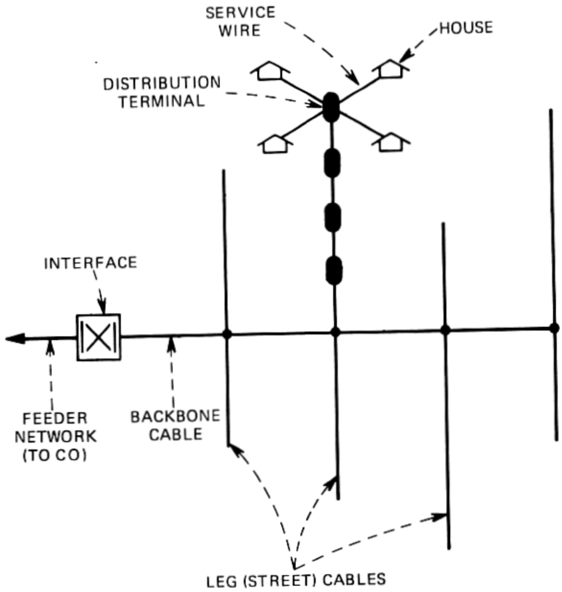

Fig. 1—Standard serving area configuration used in models.

lines per living unit in an area (usually expressed in percent). If fewer than two pr/l.u. are provided, each living unit cannot have a unique additional line pair. The available additional line pairs must then be shared among several living units.

Although additional line penetrations of only a few percent are common today, it is possible that they may increase considerably in the future. Also, unforeseen* growth in living units (due, for example, to subdivision of one-family houses into multifamily) has to be provided from the additional line pairs. To take care of these uncertainties, SAC design specifies a minimum of two pr/l.u. More are sometimes provided in localized areas.

If two or more pr/l.u. are provided, it is very unlikely that future relief cable will be required. On the other hand, if smaller distribution cables (i.e., fewer than two pr/l.u.) are provided initially, the possibility of future relief cannot be ignored. The *cable relief model* makes probabilistic calculations, based on additional line and living unit growth, to obtain the expected cost of future cable relief for a serving area.

If fewer than two pr/l.u. are provided in the distribution network, additional line pairs must be shared among the living units. Under this condition, there may be insufficient additional line pairs in a given terminal to serve the total additional line demand among the living units

* Attempts are, of course, made to forecast such increases and make provision for them.

served by that terminal. Such an event is known as a *blockage* and results in a cost penalty due to the construction or rearrangement activity required to provide the desired additional line. Models of blockage cost are discussed in this issue by Koontz[4] and Freedman.[5] A *terminal blockage* cost model, tailored specifically to a SAC distribution network, is developed in this paper.

Even if blockages do not occur, additional line demand can result in extra cost when fewer than two pr/l.u. are provided. This extra cost is the cost of disconnecting and reconnecting additional line pairs to living units as additional line demand moves from one house to another. The components of this cost are known as the *break connect-through* (BCT) cost and the *reterminate connection* (RTC) cost. These costs are modeled in a general context by Koontz[4] and Freedman[5] and in the SAC context here. Henceforth, we shall refer to the combined BCT/RTC costs as break connect-through costs.

The need to share additional pairs and change assignment of them can lead to complications in the plant assignment office, where the pair records are kept. The increased time of assignment leads to another cost model, for *assignment cost*.

To sum up, the five cost models are current construction cost, cable relief cost, terminal blockage cost, break connect-through cost, and assignment cost. Each of these models is described in detail in Section II and the appendices.

1.2 SAC design

Distribution design under the Serving Area Concept (SAC) is based on *ultimate* living units in a serving area. This is the maximum number of living units ever expected to exist in the area, taking account of future growth. Standard SAC design requires a minimum of two pairs for each ultimate living unit—one primary pair and one additional line pair. Since additional line penetration is unlikely to reach 100 percent, this design avoids future cable relief. Since every living unit can be given a specific additional line pair, terminal blockage and breaking of connect-throughs are also unlikely.* Assignment costs are low because the additional line pairs are always assigned to the same residence.

Under SAC design, each primary pair is *dedicated* to its living unit. That is, once assigned at an address, the pair cannot be reassigned elsewhere. Additional line pairs may be either dedicated or reassignable, depending on the local situation.

Pairs which are reassignable will often appear for use in several distribution terminals, in contrast to dedicated pairs, which appear in one terminal only. The purpose of this practice, called *multipling* (see Ref.

* They are not impossible, though, as one living unit may require *more* than one additional line.

1), is to make the pairs more widely available. SAC allows multipling, but does not rigidly specify the method.

Use of SAC design results in a network which needs relatively little attention once constructed. This was the intention of its creators, who were responding to operating problems encountered in multiple and dedicated plant, and the use or misuse of some particular terminal hardware. The price paid for this simplicity is increased initial construction cost. The models in this paper enable the initial construction cost to be balanced against future costs in a rational manner. In particular, the overall effect of installing fewer than two pairs per living unit can be evaluated.

1.3 The standard serving area

Actual configurations of distribution cable in serving areas vary widely according to geographical requirements and local practice. However, a few parameters serve to describe the salient features for purposes of these cost models. All but the most unusual areas can adequately be depicted as follows: a single *backbone cable* runs out from the interface, connected to a number of *street cables,* or *legs.* At each leg connection point two legs branch out, and the spacing of connection points is uniform along the backbone. Tapers (reductions in cable size) are allowed at various points along the backbone. Figure 1 gives an example of this configuration.

The leg cables may be various sizes and lengths. Spacing of terminals on the legs is uniform, and the same number of houses is served by each terminal (multifamily houses may contain various numbers of living units). If terminals are placed on the backbone, they, too, are assumed uniformly spaced. All terminals are assumed to be re-enterable, so that customer service wire connections in them can be changed.

Use of a standardized serving area of this form reduces the number of descriptors to a manageable level. Essentially, the serving area is specified by the number of legs, the leg cable sizes,* placement of backbone taper points, and terminal and leg spacing distances. Backbone cable size is not an independent variable, but a function of the leg cable sizes, since the backbone is assumed to be sized to connect all leg pairs back to the interface. This policy assumes at most one future relief of the backbone, to be done at the same time as leg relief.

II. COST MODELS

The cost models are of varying form and complexity. The initial construction model is straightforward, involving principally tallying and costing of the plant placed. On the other hand, the future cost models

* Leg cable sizes are determined by living units per leg times desired pairs per living unit.

involve probabilistic processes which occur over time, requiring integration to determine the overall expected costs. All the models have been computer implemented, so that a complete set of costs can be calculated for a given set of input parameters.

One set of inputs consists of the serving area parameters described in Section 1.3. Other principal sets include the parameters for living unit and line demand growth, and the component costs. The component costs are the actual installed first costs of the network components. These include both labor and material costs and are calculated from a number of sources. Hardware prices are combined with operating company estimates of labor times and costs and compared with so-called broad gauge costs* for verification.

Living units per house and line demand per living unit are both modeled as a class of saturating functions (this includes linear functions) whose parameters are program inputs. That is, the quantities tend to increase less rapidly as time goes on, or at most linearly. This ensures that the resulting integrals are bounded.

The output of each model is a cost represented as a present worth of annual charges (PWAC) (often shortened to "present worth"). In this way, all costs are referred to a common base at time zero and so may be compared. The present worth calculation can include inflation, if desired.

Section 2.1 describes the initial construction cost model. The future cost models follow. Section 2.2 covers the cable relief and terminal blockage models; Section 2.3, the break connect-through model; and Section 2.4, the assignment model. Further details of the future cost models appear in the appendices.

2.1 Initial construction model

The construction model calculates the cost of installing the serving area plant necessary to achieve a specified number of pairs per ultimate living unit. It incorporates models of living unit growth and of the hardware and connection costs for building the network. Costs are calculated for the backbone cable, leg cables, interface, and terminals. Additional costs for poles in aerial plant, or trenching in buried plant, are not included, as these are assumed to be the same whatever the cable sizes.

The living unit growth model assumes that the number of houses is fixed, but that one-family houses may divide into two- or three-family houses at specified rates. Each terminal is assumed to serve a fixed number of houses. Hence, the average number of living units in twenty years (the "ultimate") may be calculated for a street (leg) with N_T terminals. Multiplication by a specified number of pairs per living unit (say,

* Costs derived from the average costs of actual construction projects.

1.5) then gives the ultimate number of pairs needed on the street, and the leg cable is sized for this number. All legs with the same number of terminals, N_T, are sized identically by the algorithm.

An important aspect of leg cable sizing is that cable is available only in discrete sizes. For some N_T, the available leg cable size may be only slightly larger than the required numbers of pairs; for others, much larger. In the latter case, the actual number of pairs per living unit for the leg is greater than the specified value, and for such "oversized" legs the future costs (due to relief, etc.) will be less.

Example: Suppose a leg cable contains six terminals, each supplying four one-family houses. If a minimum of 1.5 pairs per living unit (pr/l.u.) were specified, the pairs required would be $6 \times 4 \times 1.5 = 36$. The next larger cable size, 50 pairs, would be installed, so the resulting available pr/l.u. would be $50/24 = 2.08$. The *same* cable would be installed for a minimum of 2.0 pr/l.u.

Separate treatment for each N_T also allows exact (pair-by-pair) specification of the terminal multipling method; that is, the way multipled pairs in a cable are shared between the terminals on the cable. The detail is needed by the terminal blocking model. Once the cost of each leg is established, a specified distribution of N_T is used to give the total leg costs for the serving area.

The backbone is then sized according to the aggregated pair demand. To reduce complexity, the backbone size is calculated from the mean aggregate pair demand summed over all legs. Taper points may be specified in a backbone. When a backbone is tapered, each section of cable is sized separately, to serve only the requirements of the legs feeding through it. Terminals may or may not be placed on the backbone. Both options can be evaluated.

2.2 Cable relief and terminal blockage

As line demand increases with time, for some legs the initially installed cable will eventually be too small, and relief will be required. Also, it may turn out that, although there are nonworking pairs in a leg cable, they are not accessible at particular terminals on the cable. These terminals are then said to be blocked,[4] and action must be taken to give them access to the available pairs. Unblocking may involve pair rearrangement or new terminal addition.

Cable relief turns out to be the major future cost incurred in most cases. Terminal blockage, on the other hand, results in rather minor costs unless very small terminals are used. Both phenomena may be treated by the same model, a model that calculates the probability of relief or blocking at each future time, multiplies that by the cost of correction (with appropriate present worth factors), and integrates over time. A mathematical description of the models is given in Appendix A. Here, it will be sufficient to sketch the approach taken.

Some simplifications in the mathematical treatment allow much more tractable models. First, the complex and fluctuating line demand process has been modeled by a simple growth process—the saturating functions referred to earlier.[6] Second, the discrete demand process has been replaced by a continuous analog, and when this is done, it can be shown that line demand can be quite accurately approximated by a normal distribution. The probability of cable relief is then the probability that this normal line demand exceeds the installed cable size.* As time goes on, this probability increases, due to demand growth.

Cost of relief for a leg cable is calculated as that required to install a parallel leg cable complete with terminals, so as to give a designated total number of pairs per living unit (usually two). This method of relief is roughly equivalent in costs to other alternatives (such as throwing existing terminals). The backbone is assumed to be relieved at the same time that the first leg relief occurs. This may be somewhat conservative, as spare pairs existing in the backbone could sometimes accommodate initial leg relief.

Terminal blockage only occurs when some cable pairs are inaccessible in some terminals. Since we have assumed that each ultimate living unit is provided with a dedicated primary line, the blockage problem only applies to additional line pairs. It turns out that fewer than two pairs per living unit and 25-pair terminals allow the use of multipling schemes which reduce blockage to quite low levels. In fact, for one family houses, the probability of blockage can often be reduced to zero, because all additional line pairs can be made available to all terminals on a leg.

Example: Suppose a leg contains eight terminals, each supplying four one-family houses, and that a minimum of 1.5 pr/l.u. is specified. Then a total of $8 \times 4 \times 1.5 = 48$ pairs is required, and a 50-pair cable would be used. In each terminal four primary pairs are terminated, leaving $25 - 4 = 21$ binding posts available for additional line pairs. These allow space for all 18 ($= 50 - 8 \times 4$) nonprimary line pairs in the cable to be terminated.

Because terminal blockages are unlikely, a simplified model can be used for them. It is assumed that a terminal blockage is cleared by throwing† pairs, or adding another terminal, rather than by piecemeal

* In practice, relief takes place when the cable is, say, 85 percent full. This is allowed for in the model.

† Connecting the terminal to a different set of pairs in the cable.

rearrangements. In other words, blockage is relieved in a lumped fashion, similar to cable relief, rather than in a continuous fashion. This simplifies the model and, in fact, allows use of the same model as was used for cable relief.

2.3 Break connect-throughs

When fewer than two pairs per living unit are provided, the additional line pairs must be shared between living units, and so must be transferred from one living unit to another as demand moves around. This gives rise to a cost called the break connect-through (BCT) cost. The source of this cost is the need for an installer to disconnect a service wire from a cable pair and reconnect a different service wire, possibly in a different terminal.

Primary line pairs, which are dedicated, are not included in this model. For additional line pairs, a connect-through (CT) policy[4,5] is assumed, so that a service wire, once connected to a cable pair, is left connected (even though idle) until that pair is required elsewhere. At that time an installer changes the service wires.*

The model has to take account of various possible situations. For example, it may not always be necessary to break a connect-through to provide additional line service. There may be a connected-through pair already in place; or there may be pairs available not connected to any customer, which would be used in preference to a BCT. These *spare pairs* will gradually be connected, until all pairs are connected to service wires. It is necessary to model this process. As shown in Appendix B, this can be done by means of a differential equation.

When a connection has to be broken to provide service, that break may take place in the same terminal that provides the new service, or in another terminal. In the latter case, costs are higher because two terminals must be visited and opened. The model calculates the probabilities of these two situations and weights the costs accordingly. Note that costs of initial service wire installation are not included, as these are independent of the cable sizing or pair dedication policies being evaluated.

Appendix B provides the model details. The model is similar to the cable relief model, except that the BCT process is a continuous one, rather than a single event as in the case of cable relief. Hence it is necessary to integrate a product of present-worth-adjusted BCT cost and rate of BCTs. BCT rate, in turn, is the product of inward additional line service order rate and the probability that such an order requires a BCT. BCTs are

* Some connect-through policies specify a "reserve time" during which the connection cannot be broken. No such time is assumed in this model; if the pair is needed, it is used at any time.

assumed to stop if the cable is relieved, because then there are enough additional line pairs for each customer to have one.

2.4 Assignment costs

Sharing of additional line pairs leads to complications in the assignment process. The assigner can no longer look up the customer address and find the relevant additional line pair. Rather, a pair must be found from the available pool, possibly by breaking a connect-through, as described in the previous section. Estimates are available of the time taken to assign in each of these cases. Hence it is possible to ascribe a cost to the difference in the assignment process.

The overall assignment cost model is similar to, but simpler than, the BCT model. It is simpler because the extra cost is assumed to apply to all assignments, whether to spare pairs, connected-through pairs, or BCTs. The model is then an integral of the product of additional line service order rate and differential assignment cost.

III. COST RESULTS AND AERIAL PLANT EXAMPLE

All the models described have been computer implemented, so that it is possible to obtain the total cost (initial plus future) of a serving area constructed and operated with any desired set of input parameters. Input parameters include serving area geometry, growth rates for lines and living units, and minimum number of pairs specified per living unit.

The results of this section describe the application of the models to some typical cases of aerial plant. An aerial plant example was chosen because there were higher potential savings due to installing fewer than two pairs per living unit, and because it was easier to obtain accurate costs for cable relief. Relief in the case of buried plant would involve retrenching, an operation of high and uncertain costs. For this reason, use of fewer than two pairs per living unit would be expected to be more cost-effective in aerial plant than in buried plant. (Some further remarks on buried plant follow in Section 3.4.)

In the examples, variations are made in serving area size and housing configuration,* in additional line growth rates, and in minimum pairs per living unit (pr/l.u.). Lot size in the serving area is held constant at one-quarter acre, and four houses are assumed served by each distribution terminal. Twenty-five-pair distribution terminals are used.

The cost calculated in each case is the total cost: the sum of the present worths of the initial construction cost and future costs. For each set of input parameters, two such costs have been calculated: one for the designated minimum pr/l.u. (say, 1.3), and the other for 2.0 minimum pr/l.u. (Henceforth all pr/l.u. figures will be understood to be designated *minimum* values, unless otherwise specified.) The difference between

* "Configuration" primarily refers to housing type: one- versus two-family (see Section 3.2 for further explanation).

costs is expressed as a percentage of the costs for two pr/l.u. Hence the results represent percent cost saving (or cost increase) with respect to a network designed at two pr/l.u. Positive numbers mean that the network with the designated pr/l.u. has a lower present worth cost than that for two pr/l.u.

In what follows, Section 3.1 describes the cost results for one-family housing, and Section 3.2 extends the results to other housing situations. Section 3.3 discusses the effect of variations in the line growth rate, and Section 3.4 provides some comments on buried plant.

3.1 One-family houses

The first results to be presented involve serving areas containing only one-family houses. The housing is assumed to be *stable,* which means that the housing type is not changing. In particular, the houses are not subdividing into multifamily houses (this situation will be examined in Section 3.2). A typical set of cost results is shown in Table I. The parameters varied are serving area size (both backbone length and leg length, or number of terminals per leg), additional line growth, and designated pairs per living unit. Initial additional line penetration is five percent. Additional line growth is expressed by the additional line penetration in 20 years. Growth is assumed saturating; that is, pene-

Table I — Cost savings for one-family houses

			Percent savings versus 2 pr/l.u.			
			1.3 pr/l.u.		1.5 pr/l.u.	
b	*t*	LU	*a* = 10	*a* = 20	*a* = 10	*a* = 20
1500	6.5	208	13.3	9.8	10.7	10.3
	7.5	240	10.7	8.6	9.1	8.4
	8.5	272	8.8	6.0	2.8	2.6
2250	4.5	216	10.1	5.3	9.4	9.1
	5.5	264	10.9	9.7	10.9	9.7
	6.5	312	14.9	12.9	13.4	13.0
	7.5	360	13.2	11.2	11.6	11.0
	8.5	408	13.8	11.2	5.5	5.3
3000	4.5	288	7.5	2.8	1.0	0.8
	5.5	352	14.3	13.8	14.3	13.8
	6.5	416	16.6	13.4	7.1	6.7
	7.5	480	8.1	6.5	6.5	5.9
	8.5	544	8.8	6.6	5.4	4.8
3750	4.5	360	12.0	11.3	12.0	11.3
	5.5	440	15.7	9.1	7.9	7.5
	6.6	520	12.0	10.0	10.2	9.2
	7.5	600	9.0	0.8	5.4	4.3
Averages*			11.8	8.8	8.5	7.9
Standard deviations*			2.7	3.5	3.5	3.5

Initial additional line penetration = 5 percent
No terminals on backbone
b = backbone length (feet)
t = average number of terminals per leg (equivalent to leg length)
LU = average number of living units in serving area (at end of 20 years)
a = percent additional line penetration in 20 years

* Both calculated by assuming that all cases are equally likely.

tration increases most rapidly at first and slows down later. The effect of changing this assumption is examined in Section 3.3.

The most obvious feature of Table I is the considerable scatter of the results, also shown in Fig. 2 for the first column of results. This scatter is due principally to the fact that cables come in discrete sizes, so a small change in serving area parameters can cause a large change in cable sizing (this effect is most prominent in backbones). No obvious trend of the results with serving area size is evident, and this is generally true. Hence it is natural to express the results in terms of averages over the serving area size. These averages are shown in Table I, along with the associated standard deviations. Both the averages and the standard deviations were calculated by simply assuming that all cases were equally probable. The deviations show that while large fluctuations from the averages are possible, in the cases shown the savings will rarely become negative (more than two standard deviations).

Figure 3 extends the average values of Table I to a larger range of additional line growths. The average savings for 1.3 and 1.5 pr/l.u. are shown, together with a one standard deviation band for 1.3 pr/l.u. (to simplify the figure, the similar band for 1.5 pr/l.u. is omitted). It can be seen that, in this case, 1.3 pr/l.u. provides savings, even in the worst cases, up to about 20 percent penetration, but that 1.5 pr/l.u. is on the average better for penetrations greater than about 20 percent. Averaged results of this kind would allow establishment of a pair per living unit policy over a large geographical region containing diverse serving areas, when the appropriate line growth parameters were known.

To give an idea of the contributions of the various future costs to total cost, Fig. 4 shows the average* present worth of future costs as a percentage of total costs (initial construction plus future costs) for the case

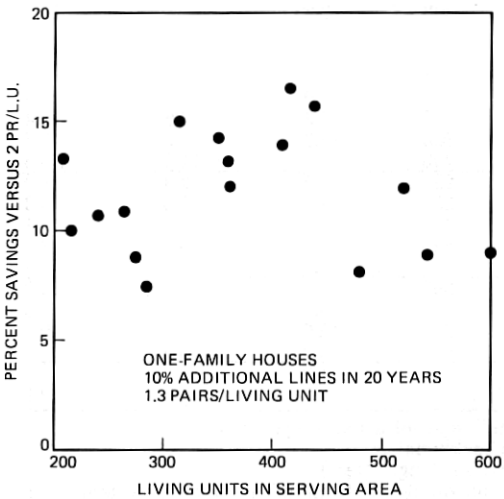

Fig. 2—Cost savings versus serving area size.

* Averaged over serving area size, as in Fig. 3.

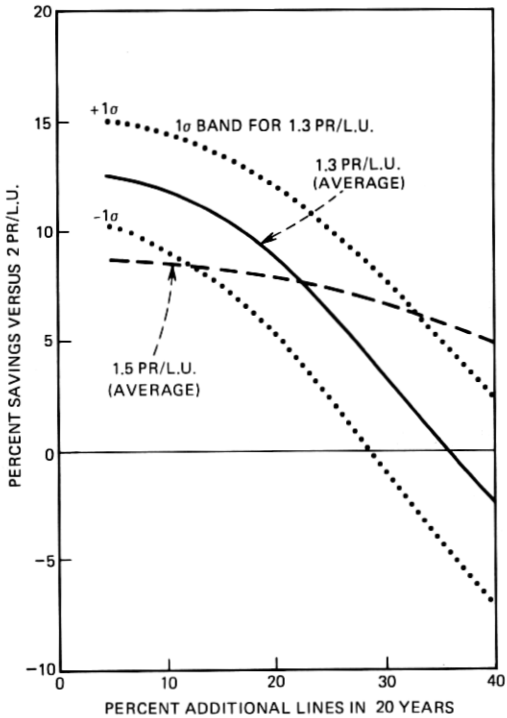

Fig. 3—Average cost savings for one-family houses.

of Table I, and 1.3 pr/l.u. The average percentage cost of cable relief only is also shown. It can be seen that the other future costs (terminal blockage, break-connect throughs, and assignment) never contribute more than a few percent to total cost in this example.

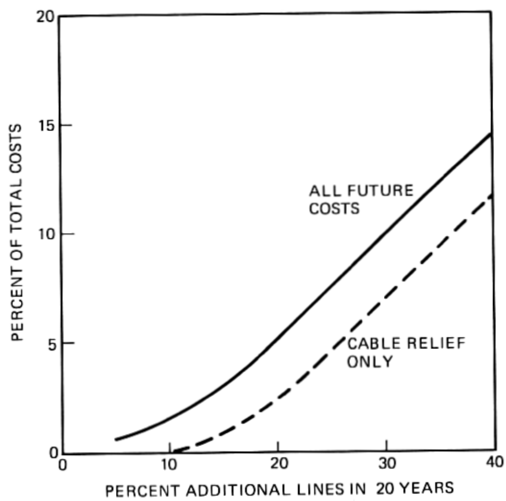

Fig. 4—Future costs as percent of total for one-family houses, 1.3 pr/l.u.

3.2 Other housing configurations

Although one-family stable neighborhoods are a common type, aerial plant tends to be used quite extensively in older, multifamily areas. Consequently, it is worthwhile examining the cost savings for different housing situations. Four configurations have been studied, as shown in Table II.

The first was the one-family stable configuration of Table I. Next, the effect of placing terminals on the backbone cable in the one-family stable case was investigated. The other two cases involved multifamily houses: the first with one-family houses progressively subdividing into two-family houses, and the second with a stable two-family situation.

Table II presents total cost savings averaged over serving area size, as in Table I. As can be seen, adding terminals to the backbone in the Table I situation does not change the results much. However, if the houses in the area are subdividing into two-family (Configuration III), *greater* savings are obtained by using less than two pairs per living unit. This apparently paradoxical result occurs because it is assumed that the growth is accurately predicted and, by designing for the ultimate living units, is allowed for. Thus the network, sized for the ultimate living units, is considerably oversized initially, reducing the probability of cable relief.

Also, fewer terminals are used to serve the ultimate living units. Thus a higher proportion of the total costs are cable costs, which is where savings are principally obtained by reducing pairs per living unit. This is also the reason for the higher savings with the two-family houses.

In all cases of Table II, savings for 1.3 pr/l.u. fall more rapidly than for 1.5 pr/l.u. as the 20-year additional line penetration increases. This is the same trend that was observed in Fig. 3.

If the plant costs were actually proportional to the minimum installed

Table II — Average savings for various housing configurations

Configuration	Initial housing	Division rate	Terminals on backbone
I	1-family	Zero	No
II	1-family	Zero	Yes
III	1-family	5% per year*	No
IV	2-family	Zero	No

Configuration	Average percent savings versus 2 pr/l.u.: 1.3 pr/l.u., $a = 10$†	1.3 pr/l.u., $a = 20$	1.5 pr/l.u., $a = 10$	1.5 pr/l.u., $a = 20$
I	11.8	8.8	8.5	7.9
II	10.2	6.7	7.4	6.7
III	15.0	14.1	12.0	11.8
IV	14.9	12.6	11.1	10.6

* Every year, 5 percent of the 1-family houses divide into 2-family. After 20 years, this increases the number of l.u. about 50 percent.

† a = percent additional line penetration in 20 years

pairs per living unit, much higher savings would be expected than have been evident in the examples so far. Thus, the percent savings in using 1.3 pr/l.u. instead of 2.0 would be 100 × (2.0 − 1.3)/2.0, or 35 percent. In fact, maximum values of 10–15 percent are observed. The reason for this is partly that there are fixed costs of construction (terminals, cable placement, etc.), and partly that discrete cable sizing causes larger average pair per living unit values than the minimum. Thus, instead of comparing 1.3 with 2.0, we should compare (say) 1.8 with 2.5.

3.3 Effect of line growth variations

Although ultimate (20 year) additional line penetration is the most significant line growth parameter, the initial rate of line growth is also important. This is shown in Fig. 5, which is drawn for a particular serving area configuration with high savings at 1.3 pr/l.u. (these are not averaged values, as in Table II and Fig. 3).

Three curves are shown. The first is the percent savings for the growth situation assumed so far: initial additional line penetration 5 percent, and saturating growth. The second curve shows the effect of decreasing initial growth rate so that the growth becomes linear throughout the 20-year period, while the third curve shows the effect of reducing the initial additional line penetration to zero, with saturating growth. In both these latter cases, savings are higher, as there are fewer additional lines at any given time, and so lower future costs.

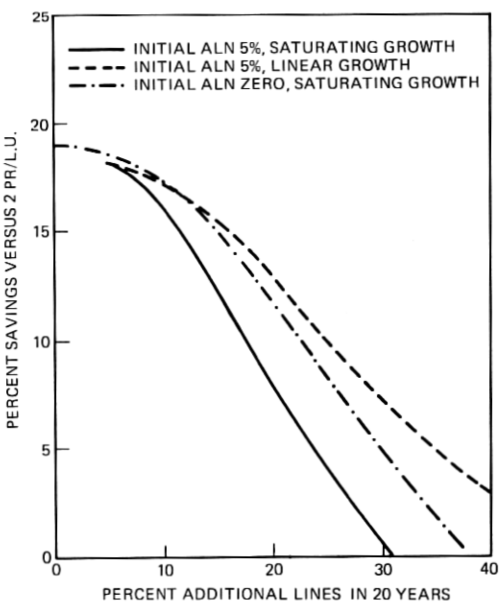

Fig. 5—Variation in cost savings with additional line (ALN) growth rate.

3.4 Buried plant

From Figs. 3 and 4, we can get an idea of the situation for buried plant. High trenching costs would probably increase the cost of cable relief severalfold. Other future costs would not be expected to increase if the terminals were pedestal-type and accessible; and it is unlikely that completely out-of-sight plant, with buried terminals, would be used with fewer than two pairs per living unit. However, since Fig. 4 shows that most future costs are for cable relief, the savings for buried plant would be expected to drop precipitously as soon as these relief costs became appreciable.

Figure 3 is appropriate since buried plant is most likely to be in stable one-family environments. This figure indicates that 1.3 pr/l.u. would probably not be satisfactory at all, but that 1.5 pr/l.u. might still provide useful savings for lower additional line penetrations. Actual results for buried plant can be computed, given the appropriate costs.

IV. CONCLUSIONS

Computer models have been developed to study the sizing of the distribution plant network. These include both initial construction costs and future costs which might be incurred for cable relief, terminal blockage, break connect-throughs, and assignment. The models are applicable to a wide variety of serving area parameters and additional line growth rates.

These models can be used as a flexible evaluation tool, allowing new or rehabilitated distribution plant to be sized appropriately, given the local conditions.

A set of examples of the application of the models to various aerial plant networks is presented. These show that, on the average, total cost savings in the 10–15 percent range can be obtained by using fewer than two distribution pairs per living unit, if future additional line penetration is less than about 20 percent. Examples are given of the effect of various parameter variations on these savings.

APPENDIX A

Cable Relief and Terminal Blockage Models

In both this appendix and the next, attention is focused on additional lines. For SAC distribution plant, it is here assumed that the ultimate required number of primary lines is provided (and dedicated, if desired). Hence all the future costs considered—cable relief, terminal blockage, break connect-through, and assignment costs—are due to inadequate provision for *additional* lines. Because of this, the emphasis and costs are somewhat different than those in Refs. 4 and 5, which consider primary lines as well, and which are also concerned with the feeder network.

In this appendix the cost of cable relief and terminal blockage is derived. These may be treated together because of the assumption made that terminal blockage is relieved in a lumped fashion by terminal throw or terminal addition, rather than by piecemeal rearrangements. This assumption is justified by the low level of terminal blockage costs. It allows us to treat terminal blockage as a single event, like cable relief. In what follows, we shall discuss cable relief. Exactly the same formulation and model apply to terminal blockage except that, instead of considering the whole cable, we consider a group of terminals in multiple.

A.1 The present worth integral

An assumption which allows simplification of the mathematics is that the additional line demand process may be treated as a pure growth process, and the effect of churning (turnover) may be ignored. This is supported by Ref. 6. Thus, it is the mean growth rather than the variability of the line demand process that is important.

Leg cable relief occurs when the line demand rises above a certain value* (say, primary plus additional line demand above 85 percent of the installed cable size). The above assumption means that the relief process can be depicted by the probability of the demand being above this value, rather than by using a rigorous stochastic process approach. This probability can be called the *instantaneous probability of cable relief*, $p_c(t)$, which is the probability that the cable first needs relief at time t.

The total present worth cost of relief for any cable may then be written as an integral over time of the product of $p_c(t)$ and the present worth adjusted cost of relief at time t. This latter quantity may be broken up into the product of the actual cost of relief at t and a present worth factor. Hence the overall *present worth integral* becomes

$$\mathrm{PW}_{\mathrm{REL}} = \int_0^T p_c(t) \cdot C_c(t) \cdot F_{\mathrm{pwc}}(t) \cdot dt \qquad (1)$$

where F_{pwc} is the present worth factor, C_c is the cost of relief, and T is the study period (here taken as 20 years).

C_c is assumed constant with time (inflation can be taken into account by adjusting the present worth factor). The present worth factor is

$$F_{\mathrm{pwc}}(t) = \frac{A_c}{r}\,(e^{-rt} - e^{-rT}) \qquad (2)$$

where A_c is the annual charge factor for the cable (30 year) account, r is the force of interest, and T is the 20-year study period.

Once the instantaneous relief probability $p_c(t)$ is known, eq. (1) may

* The backbone is assumed to be relieved when first leg relief occurs.

be integrated numerically to give the expected cost of relief of any specific cable. Appropriate addition gives the expected relief cost for the serving area. In what follows, we show how $p_c(t)$ is calculated.

A.2 Instantaneous relief probability

The mathematical simplification used to get $p_c(t)$ is the modeling of the line demand on the cable as a normal distribution. To do this, the Central Limit Theorem is invoked, as well as our previous assumption that we can consider line demand a smoothly growing function. The line demand for the cable is assumed to be the sum of independent demands from the houses supplied by the cable. Suppose there are N_T terminals on the cable, and H houses per terminal. If each house has a line demand distribution with mean μ_h and standard deviation σ_h, the corresponding mean and standard deviation for the cable are $\mu_c = HN_T\mu_h$ and $\sigma_c = \sqrt{HN_T}\,\sigma_h$. We shall return to the calculation of μ_h and σ_h in the next section.

The line demand for the cable is then assumed normal, with mean $\mu_c(t)$ and standard deviation $\sigma_c(t)$. Thus the probability that the cable *has required relief* by time t is the probability that this normal variate is greater than some value X (85 percent of the cable size); that is

$$\frac{1}{\sqrt{2\pi}}\int_{\beta(t)}^{\infty} e^{-x^2/2}dx \tag{3}$$

where

$$\beta(t) = \frac{X - \mu_c(t)}{\sigma_c(t)}$$

Equation (3) represents the probability that the cable was relieved at time t or before. The instantaneous relief probability $p_c(t)$ is obtained by differentiating eq. (3):

$$p_c(t) = -\frac{1}{\sqrt{2\pi}}\frac{d\beta(t)}{dt}e^{-\beta(t)^2/2} \tag{4}$$

A.3 House distribution parameters

The house distribution parameters μ_h and σ_h are obtained by considering the number of lines required by a house as the sum of the lines required by each living unit in the house. The number of lines required by a living unit and the number of living units are both random variables in the most general case. Assume they have means and standard deviations μ_L, σ_L (lines) and μ_U, σ_U (living units). Then for a house, standard probability theory gives

$$\left.\begin{aligned} \mu_h &= \mu_L\mu_U \\ \sigma_h^2 &= \mu_U\sigma_L^2 + \mu_L^2\sigma_U^2 \end{aligned}\right\} \tag{5}$$

Assume that there are a maximum of three lines per living unit (with penetrations $\alpha_1, \alpha_2, \alpha_3$ for first, second, and third), and three living units per house (with probabilities f_1, f_2, f_3 for one, two, and three). Then manipulation of eq. (5) produces

$$\left.\begin{aligned} \mu_h &= LF \\ \sigma_h^2 &= (\alpha_1 + 3\alpha_2 + 5\alpha_3)F - L^2\{(F-1)^2 + 1 - 2f_3\} \end{aligned}\right\} \tag{6}$$

where

$$\begin{aligned} L &= \alpha_1 + \alpha_2 + \alpha_3 \\ &= \text{expected lines per living unit} \\ F &= f_1 + 2f_2 + 3f_3 \\ &= \text{expected living units per house} \end{aligned}$$

As an example, in the case of all one-family houses ($F = 1, f_3 = 0$) with no third lines ($\alpha_3 = 0$), and 100 percent first line penetration ($\alpha_1 = 1$), eq. (6) reduces to

$$\begin{aligned} \mu_h &= 1 + \alpha_2 \\ \sigma_h^2 &= \alpha_2(1 - \alpha_2) \end{aligned}$$

APPENDIX B

Break Connect-Through and Assignment Models

In both the break connect-through (BCT) and assignment models, the present worth cost is obtained from an integral similar to eq. (1). The principal difference is that both these processes occur continuously with time, rather than once only as in the case of cable relief. Also, churn (turnover) now becomes an important factor, whereas for relief we considered growth only.

B.1 The present worth integral for BCTs

The present worth cost of BCTs is obtained by integrating over a product of three factors: the rate at which BCTs occur, R_b [replacing relief probability in eq. (1)], the cost of a BCT, C_b, and a present worth factor, F_{pwb}. All factors vary with time:

$$\text{PW}_{\text{BCT}} = \int_{T_s}^{T_M} R_b(t) \cdot C_b(t) \cdot F_{\text{pwb}}(t) \cdot dt \tag{7}$$

Integration extends up to a time T_M, which is the lesser of the expected time of relief and the study period T. (It is assumed that no BCTs occur after relief). Integration does not start at time zero, but rather at a *spare exhaust time* T_s, at which all additional line pairs have service wires attached to them. Prior to this time, a pair with no service wire (spare pair) could be used to provide service if no connected-through pair existed at the service location.

For this calculation, terminals on a cable are assumed multipled in groups, with the additional line pairs shared fully among terminals of one group, but no multipling between groups (the group may often be all the terminals on the cable). The integral (7) is calculated for each terminal group, and the costs added for all groups to give the serving area BCT cost.

The full access provided by this multipling ensures that any remaining spare pairs in the terminal group can always be used to fill an inward service order which is not CT. To relate this paper to the more comprehensive treatment of Ref. 4, note that no reterminated connections* can occur before spare exhaust. After spare exhaust, every reterminated connection is either a BCT, or the terminal group is blocked.

Four functions must be further specified in the integral (7): T_s, $R_b(t)$, $C_b(t)$ and $F_{\text{pwb}}(t)$. Of these, $F_{\text{pwb}}(t)$ is the simplest, though it is more complicated than the present worth factor for relief [eq. (2)] as the 10-year station account is involved. If r is the force of interest and A_s is the annual charge factor for the station account,

$$F_{\text{pwb}}(t) = \left.\begin{aligned} &\frac{A_s}{r}\{e^{-rt}(1-e^{-10r})\} \quad \text{if} \quad t \le T_M - 10 \\ &= \frac{A_s}{r}\{e^{-rt} - e^{-rT_M}\} \quad \text{if} \quad t > T_M - 10 \end{aligned}\right\} \tag{8}$$

Before going on to detail the other functions $C_b(t)$, $R_b(t)$, and T_s, let us introduce some common notation. Let N_U be the number of living units under consideration; this will be the number of living units served by a single group of multipled terminals. If the terminal group contains N_T terminals, the number of living units per terminal is $u = N_U/N_T$. To supply the N_U units, n additional line pairs are provided. At spare exhaust, all of these n pairs will have service wires attached. Finally, of the n pairs, $w(t) = \alpha_a(t)\, N_U$ are working at time t. The additional line penetration is $\alpha_a(t) = \alpha_2(t) + \alpha_3(t)$ (in the notation of Appendix A†).

B.2 Cost of breaking a connect-through

The cost of a BCT, $C_b(t)$, depends on whether one terminal must be visited (cost C_1) or two (cost C_2). Two must be visited if the terminal where service is desired does not contain a connected-through pair. Thus

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* This term is used in Ref. 4 to describe the restoration of service to a location that has had a CT broken.

† Following eq. (5).

$$C_b(t) = C_1(1 - p_{nt}) + C_2 p_{nt} \tag{9}$$

where $p_{nt} = p_{nt}(t)$ is the probability of no connected-through pair in a terminal. Of the N_U living units, the total number which do not have a connected-through pair is $N_U - n + w$ (such living units are either working or have no connected service wire). The probability p_{nt} is determined by selecting at random a subset of size u from the N_U living units, and so by probability theory is given by

$$p_{nt} = \binom{N_U - n + w}{u} \bigg/ \binom{N_U}{u} \tag{10}$$

B.3 Probability of a connect-through

The key to the determination of $R_b(t)$ and T_s is the calculation of the probability that a connected-through pair will be found at a location where additional line service is required. This probability, $p_{\mathrm{CT}} = p_{\mathrm{CT}}(t)$ can be simply modeled for one-family houses by assuming that additional line demand occurs at random—that is, all houses without a working additional line are equally likely to need one. In that case,

$$\begin{aligned} p_{\mathrm{CT}} &= \Pr(\text{CT pair} \mid \text{pair is not working}) \\ &= (\text{number of CT pairs})/(\text{number of nonworking pairs}) \\ &= \frac{n - w}{N_U - w} \\ &= \frac{n/N_U - \alpha_a}{1 - \alpha_a} \end{aligned} \tag{11}$$

For multifamily houses, however, a correction must be applied, because a living unit can also use a pair connected through to another living unit in the same house. Thus

$$p_{\mathrm{CT}} = \frac{n/N_U - \alpha_a}{1 - \alpha_a} + p_{\mathrm{corr}} \tag{12}$$

where

$$\begin{aligned} p_{\mathrm{corr}} &= \Pr(\text{no direct CT}) \{\Pr(\text{one neighbor l.u.}) \cdot \Pr(\text{neighbor CT}) \\ &\qquad + \Pr(\text{two neighbor l.u.}) \cdot \Pr(\text{either is CT})\} \\ &= \frac{1 - n/N_U}{1 - \alpha_a} \left\{ \frac{2f_2}{F} \cdot p_{n\mathrm{CT}} + \frac{3f_3}{F} (2p_{n\mathrm{CT}} - p_{n\mathrm{CT}}^2) \right\} \end{aligned}$$

where

$$\begin{aligned} p_{n\mathrm{CT}} &= \Pr(\text{neighbor is CT}) \\ &= \frac{n - \alpha_a N_U}{N_U - 1} \end{aligned}$$

and

$$F = f_1 + 2f_2 + 3f_3$$

as in Appendix A [following eq. (6)]. We neglect the "minus one" in the denominator of $p_{n\text{CT}}$, and obtain

$$p_{\text{corr}} = \frac{1 - n/N_U}{1 - \alpha_a} \cdot \frac{n/N_U - \alpha_a}{F} \{2f_2 + 3f_3(2 - n/N_U + \alpha_a)\}$$

From eq. (12), the probability of *no* CT may be written

$$1 - p_{\text{CT}} = \frac{1 - n/N_U}{1 - \alpha_a} \left[1 - \frac{n/N_U - \alpha_a}{F} \{2f_2 + 3f_3(2 - n/N_U + \alpha_a)\} \right] \tag{13}$$

B.4 BCT rate and spare exhaust time

The probability $1 - p_{\text{CT}}$ enables us to determine both $R_b(t)$, the rate of BCTs, and T_s, the time of spare exhaust. Once spare exhaust has occurred, each non-CT additional line order requires a BCT. Thus if $L_r(t)$ is the rate of inward additional line orders,

$$R_b(t) = L_r(t) \cdot (1 - p_{\text{CT}}(t)) \tag{14}$$

$L_r(t)$ is calculated using the quantity $\alpha_a F$, which is the expected number of additional lines per house:

$$L_r(t) = \left\{ \frac{\alpha_a F}{\tau_0} + \frac{d}{dt} (\alpha_a F) \right\} N_h \tag{15}$$

Here τ_0 is the mean occupancy time* for additional lines, and N_h is the number of houses served by the terminal group. In eq. (15), the first term represents additional line orders due to churn, and the second term those due to growth.

To determine T_s, we note that if spare exhaust has *not* occurred, each non-CT additional line order requires another pair to have a service wire connected. The analog of eq. (14) is therefore

$$\frac{da(t)}{dt} = L_r(t) \cdot (1 - p_{\text{CT}}(t)) \tag{16}$$

where $a(t)$ is the number of pairs with service wires connected (assigned pairs). T_s is obtained by numerical integration of eq. (16) from a specified initial value of a to the value $a = n$.

Having found T_s, overall cost is obtained by substituting eqs. (8), (9), (10), (14), (13), (15) in eq. (7) and integrating.

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* The mean time that such a line is working.

B.5 Assignment costs

Assignment costs follow from the foregoing derivation of BCT costs. The basic integral (7) is replaced by

$$\mathrm{PW}_{\mathrm{AST}} = \int_0^T L_r(t) \cdot C_a \cdot F_{\mathrm{pwb}}(t) \cdot dt \qquad (17)$$

This integral extends from zero to the end of the study period T. In the integrand, $R_b(t)$ of (7) is replaced by $L_r(t)$, the inward rate of additional line service orders, since every order must be assigned, whether connected through or not. C_a is the assignment cost, a constant, representing the difference in cost of performing an assignment with the records appropriate to two pairs per living unit, and with those appropriate to less than two pairs per living unit. The present worth factor F_{pwb} is the same as in the BCT case [eq. (8)].

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