THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 57, No. 4, April 1978
Printed in U.S.A.

Loop Plant Modeling:

Optimal Operating Policies for Serving Areas Using Connect-Through Administration

By H. T. FREEDMAN

(Manuscript received August 20, 1977)

Connect-Through administration is the policy of leaving the pair from a customer's premises to the central office intact when the customer disconnects from the network. A pair in this idle state is called a connect-through (CT) pair. In a serving area (the geographical entity in which all customers are served through an interface connecting the feeder and distribution parts of the network), growth may lead to the condition where no spares remain in the interface. It then becomes necessary to consider breaking CT pairs or providing additional spare pairs (relief). In this paper, two related operating decisions are examined. First, in order to determine under what conditions relief is more economical than breaking CTs, models are developed to compare the expected operating cost due to breaking CTs with relief costs. Second, when breaking a CT is the preferred procedure, it is shown that the optimal policy is to break the CT with the smallest instantaneous reuse probability, given by the hazard function of the premise vacancy time.

I. INTRODUCTION

Connect-Through administration is the policy of leaving the loop from a customer's premises to the central office intact when the customer disconnects from the network. This idle, but reserved, pair is called a connect-through (CT) pair. The savings from avoiding the disconnection operation and from having the pair available for reuse when (and if) a new customer moves into the same location may be counterbalanced by the fact that with fewer spare pairs available for new customers, costly loop network reconfigurations will be required more often.

Recent emphasis has been placed on understanding the effects of CT administration because of the decision to establish PhoneCenters in the Bell System. In a PhoneCenter environment, a customer may obtain the telephone in a PhoneCenter store. In many cases the customer can also connect the telephone to the network through jacks previously installed in his residence. This eliminates the need for any work inside the residence by the installer. If, in addition, a CT pair to this residence is available, no installer work of any kind is required. The savings from having a CT pair available for reuse are then much greater since the installer trip can be eliminated. Still, the trade-off between reuse savings and loop network reconfiguration costs must be evaluated to determine an optimal policy.

A serving area is a geographical entity (200 to 600 living units) served by feeder pairs terminated in a single interface (see Long,[1] this issue). When all of the feeder pairs in the interface are either working (in service) or CT, a new customer who cannot reuse a CT can then only be served by breaking a CT reserved for another location or by making more feeder pairs available at the interface (relief). In this paper, models are developed to determine under what conditions CT pairs should be broken in preference to providing relief, and to provide an optimal policy for deciding which CT to break when one is to be broken.

The question of whether to break a CT or to relieve is attacked by determining an optimal time for relief; this time is found by trading off operating costs and relief costs. Models for the operating costs of loop plant being administered under a CT plan are developed based on a linear growth, birth-and-death Markov model for customer demand. Expressions are developed for the expected number of CT and working pairs over time, and the average operating cost over time. Assuming that the system follows these expected trajectories exactly, the times of spare exhaust (the first time there are no spares remaining in the interface) and working exhaust (when all of the feeder pairs into the interface are working) can be calculated. The operating cost function is found to be a piecewise linear function of time which is discontinuous at the time of spare exhaust, when it becomes necessary to start breaking CT connections.

Relief timing is determined by the time at which the operating costs first exceed the levelized equivalent annual charges of relief plus post-relief operating costs (see Koontz,[2] this issue). Two types of relief are considered. The first affects only a single serving area (e.g., making additional pairs available at the interface). Solutions for this optimal relief time as a function of the system parameters are developed. The second type of relief provides additional feeder pairs to an entire allocation area (a group of two to five serving areas). This relief timing is optimized by considering the sum of the operating costs in each serving area.

The problem of which CT to break, when breaking is indicated, is solved by taking the one with the smallest instantaneous reuse rate. This rate is given by the hazard function of the premise vacancy time distribution. In addition to the exponential vacancy time distribution (equivalent to the Poisson demand model used in the relief timing models), modifications are incorporated to model three empirical observations about vacancy time.[3] First, different categories of premises are allowed since, for example, first lines and additional lines would have different vacancy time distributions. Second, the fact that some CT pairs are unreusable is modelled by permitting abandonment of premises. Third, the observation that the vacancy times have a decreasing hazard rate[3] is modelled by allowing the parameter of the exponential distribution to be a random variable. The optimal strategy in this model is shown to be breaking a CT which is the oldest in its category, with the choice of category depending on the ages of the oldest CT in each category.

II. RELIEF TIMING MODELS

2.1 Customer demand model

Demand for pairs is assumed to be the net result of individual customers moving into and out of "premises" according to independent Poisson processes. Section 2.1.1 examines the case where the number of premises is time-invariant, an appropriate model for non-growth areas. In Section 2.1.2 the number of premises is allowed to grow linearly over time.

2.1.1 Saturating exponential growth model

The system under consideration consists of a single serving area, served (by definition) by a single interface. It is assumed that there are a fixed number, p, of potential points of demand ("premises") in the serving area. The actual number of premises is assumed known, although in most situations it will be estimated from other data. Each premise without service ("vacant") generates inward moves according to a Poisson distribution of parameter λ, and each in-service ("working") premise generates outward moves by a Poisson distribution of parameter μ. The values of these parameters are not directly obtainable, but can be estimated from other data as will be described in Section 2.2.1. The reciprocals of these parameters are, respectively, the mean vacancy time, τ_v, and the mean occupancy time, τ_o. At a time when there are w working premises in the system, the expected inward and outward movement rates for the serving area as a whole are

$$\lambda_{\text{TOT}}(w) = \lambda(p - w)$$
$$\mu_{\text{TOT}}(w) = \mu w \tag{1}$$

The state of the demand model at any time is represented by the number of working pairs in the interface. Since each working premise requires a single feeder pair, the number of working pairs is equal to the number of working premises. The number of working pairs increases by one with every inward move, and decreases by one with every outward move. Thus the rate of change in the expected number of working pairs equals the difference between the inward and outward movement rates:

$$\frac{dw}{dt} = \lambda(p - w) - \mu w \tag{2}$$

The solution to this differential equation is

$$w(t) = w(\infty) + (w(0) - w(\infty))e^{-(\lambda+\mu)t} \tag{3}$$

where

$$w(\infty) = p\lambda/(\lambda + \mu)$$

is the steady-state number of working pairs. The exact probability of there being w working pairs at time t is derived in Feller[4] but is not necessary here since the present approach will deal only with expected values.

2.1.2 Linear growth model

Consider the saturating exponential growth model, with the number of premises allowed to vary with time instead of being fixed. In particular, assume that the number of premises grows linearly with time, so that

$$p(t) = G_p t + p(0) \tag{4}$$

The values of the constants G_p and $p(0)$ are not directly measurable; in Section 2.2.1 their estimation from other available data is described. With the number of premises time-varying, the differential equation for w [eq. (2)] still holds, but its solution is now

$$w(t) = C_1 e^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda+\mu}\left(G_p t + p(0) - \frac{G_p}{\lambda+\mu}\right) \tag{5}$$

where

$$C_1 = w(0) - \frac{\lambda}{\lambda+\mu}\left(p(0) - \frac{G_p}{\lambda+\mu}\right)$$

For large t, the first term goes to zero so that the effect of the initial number of working pairs becomes negligible. Then the number of working pairs also increases linearly with time at a rate smaller than the premise growth rate. This can be represented as

$$w(t) = G_w t + w_o \tag{6}$$

where

$$G_w = \frac{\lambda G_p}{\lambda + \mu} \tag{7}$$

and

$$w_o = \frac{\lambda}{\lambda + \mu}\left(p(0) - \frac{G_p}{\lambda + \mu}\right) \tag{8}$$

The parameters G_w and w_o for a given area will generally be obtainable from telephone company data. The parameter G_w is an estimate of the working pair growth and w_o represents the number of working pairs at the beginning of the study period. It will be assumed in the rest of this paper that the system has been operating for a sufficient time so that the exponential term of eq. (5) is negligible and the growth is linear.

2.2 CT levels over time

To determine the expected number of CT pairs, z, as a function of time, two phases have to be considered. The spare assignment phase (while the number of spare pairs in the interface is positive) lasts until the time of spare exhaust, T_s. The CT breaking phase lasts until the time of relief, T_R. As will be seen later, T_R must be between T_s and the time of working exhaust (when all of the feeder pairs are in service), T_w.

2.2.1 Spare assignment phase

When there are spares remaining in the interface, no CT pairs will have to be broken in order to provide service. An inward move will reuse a CT if there is one associated with its premises; otherwise, it will be assigned to a spare pair. Since an outward move always leaves a CT pair, the expected rate of increase in the number of CT pairs will equal the difference between the outward order rate, μ_{TOT} [eq. (1)], and the reuse rate. The reuse rate equals the inward order rate, λ_{TOT}, [eq. (1)] times the probability that an inward order will result in a reuse. Since the Poisson model implies that each vacant premise is equally likely to generate the next inward order, this probability is equal to the fraction of vacant premises which have CT pairs. Thus,

$$\frac{dz}{dt} = \mu w - \lambda(p - w)\left(\frac{z}{p - w}\right) \quad \text{for} \quad t \le T_s \tag{9}$$

Since this equation is only valid while there are spares remaining, the time of spare exhaust must be determined. Assuming that the system follows (2) and (9) exactly, T_s is found from

$$w(T_s) + z(T_s) = n \tag{10}$$

where n is the number of feeder pairs in the interface.

It should be noted that the derivations of eqs. (9) and (10) contain implicit approximations. First, T_s is not the expected time of spare exhaust (this requires first passage time calculations) but is the time when the expected number of spares becomes zero. This is a good approximation to the expected time of spare exhaust when growth is considered. Second, $z(t)$ is not exactly equal to the expected number of CT pairs at time t since a rigorous derivation from state probabilities would have to include the distribution of spare exhaust times. Again the approximation is sufficiently close for the models in this paper.

For the linear growth demand model, eqs. (4) and (6) are substituted into (9) and (10) and the large t approximation applied to get

$$z(t) = \frac{\mu}{\lambda}\left[G_w t + w_o - \frac{G_w}{\lambda}\right]$$
$$= G_z t + z_o \quad \text{for} \quad t \le \frac{n - w_o - z_o}{G_w + G_z} \tag{11}$$

If the past history of the system has progressed according to the model, then

$$z_o = \frac{\mu}{\lambda}\left(w_o - \frac{G_w}{\lambda}\right) \tag{12}$$

and

$$G_z = \frac{\mu}{\lambda} G_w \tag{13}$$

Since z_o and G_z can generally be obtained from telephone company data, they can be used along with w_o and G_w to estimate the parameters G_p, $p(0)$, λ and μ. Equations (7), (8), (12), and (13) are solved simultaneously, yielding

$$G_p = G_w + G_z$$
$$p(0) = w_o + z_o + \frac{G_w + G_z}{G_w G_z}(G_z w_o - G_w z_o) \tag{14}$$
$$\lambda = \frac{G_w G_z}{G_z w_o - G_w z_o}$$
$$\mu = \frac{{G_z}^2}{G_z w_o - G_w z_o}$$

These estimates will be used in the remainder of this paper.

2.2.2 CT breaking phase

After the spares are exhausted, every inward order results in either a reuse or the breaking of a CT. At this point, every nonworking pair in the interface will be a CT, so that

$$z(t) = n - w(t) \qquad T_s \le t \le T_R \tag{15}$$

For the linear model, this becomes

$$z(t) = n - G_w t - w_o \qquad T_s \le t \le T_R \tag{16}$$

Fig. 1 illustrates the equations describing the linear model.

2.3 Operating cost models

This section uses the results of the previous section to estimate the expected operating costs over time. The *expected operating cost per inward move*, C_{IM}, is defined as the sum, over all possible operations to provide service, of the product of the cost per operation and the probability that an inward order requires that operation. Let C'_R, C'_S, and C'_B be the absolute costs per reuse, spare assignment and breaking a CT, respectively. Then

$$C_{IM}(t) = \begin{cases} C'_R\left[\dfrac{z(t)}{p(t)-w(t)}\right] + C'_S\left[1-\dfrac{z(t)}{p(t)-w(t)}\right] & t < T_s \\ C'_R\left[\dfrac{z(t)}{p(t)-w(t)}\right] + C'_B\left[1-\dfrac{z(t)}{p(t)-w(t)}\right] & T_s \le t \le T_R \end{cases} \tag{17}$$

The *expected operating costs over time*, $b(t)$, are defined as the product of C_{IM} and the inward order rate [from eq. (1)]. To simplify the resulting equations, costs measured relative to the cost of a reuse (denoted C_R, C_S, and C_B) can be used in eq. (17) in place of absolute costs. It can be

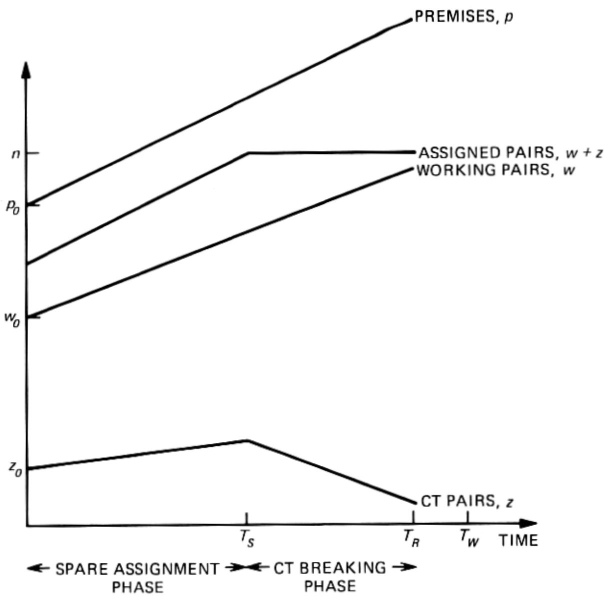

Fig. 1—Linear growth model.

shown[5] that this substitution does not affect the relief timing or strategy decisions based on the models. Since $C_R = 0$, the operating costs over time become:

$$b(t) = \begin{cases} C_S\lambda(p(t) - w(t) - z(t)) & t < T_s \\ C_B\lambda(p(t) - n) & T_s \le t \le T_R \end{cases} \tag{18}$$

For the linear model, these operating costs are

$$b(t) = \begin{cases} C_S G_p & t < T_s \\ C_B[\lambda(p(0) - n) + \lambda G_p t] & T_s \le t \le T_R \end{cases} \tag{19}$$

where $p(0)$, G_p, and λ can be estimated through eq. (14). That is, until the time of spare exhaust, the operating cost is due to the constant rate at which spares are assigned, which is equal to the premise growth rate. Once the spares exhaust, the operating costs increase linearly over time as more and more CTs must be broken to provide service. Although the operating costs given by eq. (19) were derived as expected values of the costs, they will be subsequently used as if they were deterministic, an acceptable approximation for the models in this paper.

2.4 Relief timing calculations

The optimal timing for relief projects is determined by trading off relief and operating costs. It can be shown[2] that the economically optimal time for relief occurs when the difference in operating costs of the system immediately before and after relief becomes as large as the levelized equivalent charge for relief (LEAC).[6]

Consider first, relief of a single serving area (typically this is accomplished by transferring unneeded spare pairs from a nearby interface). From eq. (19), it can be seen that the operating costs during the spare assignment phase are independent of the number of spares, so that relief should not be performed before spare exhaust. Also, at the time of working exhaust, some sort of relief must be done if service is to be provided at all. After relief, the system will again be in the spare assignment phase. Let $\overline{b}(t)$ denote the operating costs during the CT breaking phase, and $\underline{b}$ denote the initial post-relief (spare assignment) costs. Then the optimal relief time is the smallest t such that

$$\begin{gathered} \overline{b}(t) - \underline{b} \ge \text{LEAC} \\ T_s \le t \le T_w \end{gathered} \tag{20}$$

where LEAC is the levelized equivalent charge of the relief project.

There are three possible solutions to the minimization of t subject to (20). They are

$$\begin{aligned} &1: \quad t^* = T_s \text{ if } \overline{b}(T_s) - \underline{b} \ge \text{LEAC} \\ &2: \quad t^* = T_w \text{ if } \overline{b}(T_w) - \underline{b} \le \text{LEAC} \\ &3: \quad t^* \text{ is found from } \overline{b}(t) - \underline{b} = \text{LEAC otherwise} \end{aligned} \tag{21}$$

For the linear model, this becomes

$$1:\quad t^* = T_s \text{ if } (C_B - C_S)G_p \geq \text{LEAC}$$

$$2:\quad t^* = T_w \text{ if } (C_B - C_S)G_p + C_B\lambda\left(z_o + \frac{G_z}{G_w}(n - w_o)\right) \leq \text{LEAC}.$$

$$3:\quad t^* = [\text{LEAC} + C_S G_p - C_B(G_p + \lambda(w_o + z_o - n))][C_B \lambda G_p]^{-1} \text{ otherwise} \tag{22}$$

These cases are illustrated in Fig. 2.

In general, serving areas are administratively grouped into allocation areas, consisting of from two to five serving areas, and often the entire allocation area will be relieved at once (see Marsh,[7] this issue). Let the parameters for serving area i be denoted by the subscript i and let N_s be the number of serving areas in the allocation area. Assuming that relief of individual serving areas is not feasible, the optimal relief time for the allocation area is the smallest t such that

$$\sum_{i=1}^{N_s} b_i(t) \geq \text{LEAC} + \sum_{i=1}^{N_s} \underline{b}_i \tag{23}$$

$$\hat{T}_s \leq t \leq \hat{T}_w$$

where $\hat{T} = \min_i T_i$

Since some of the serving areas may not have reached spare exhaust at

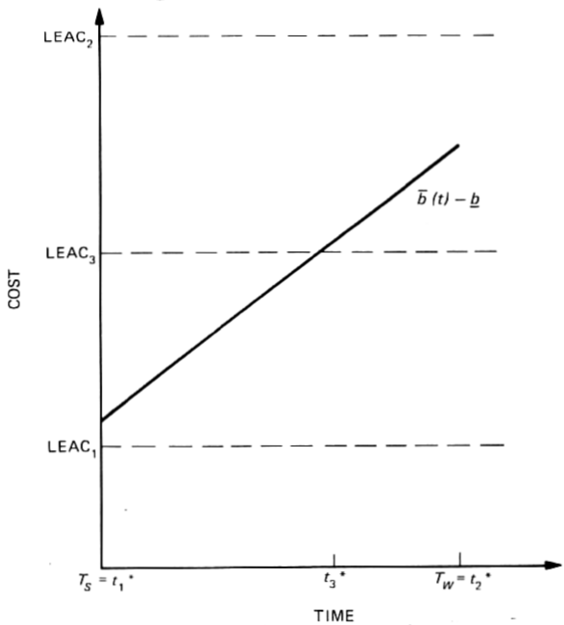

Fig. 2—Relief timing for one serving area.

the time of relief, the left side of (23) cannot specify whether pre- or post-spare exhaust costs should be used. This would be determined for each serving area from the limit in eq. (11).

III. CT BREAKING STRATEGY

Given that no spares are available at an interface, so that some CT must be broken in order to provide service to a new customer, the question of interest is which CT to break. The policy chosen should be the one which minimizes the present worth of the operating costs. It is shown in the appendix that a policy of breaking the one with the smallest instantaneous reuse probability is an excellent approximation to a minimum present worth strategy. Estimates of the reuse probabilities for each CT pair will depend on the model used for the demand for service. In particular, the exponential premise vacancy time distributions are allowed to be more general than before.

Let $f(t)$ be the probability distribution function of vacancy time at a premises, and $F(t)$ be the cumulative distribution function. Then the instantaneous reuse rate for a CT pair which has been idle for time t is given by the hazard function

$$h(t) = \frac{f(t)}{1 - F(t)} \tag{24}$$

and the probability of reuse in a small amount of time, dt, is given by $h(t)dt$. The hazard function is used in reliability theory as the measure of instantaneous failure rate, where $f(t)$ is the lifetime distribution of a system component.[8] In the above model, premises vacancy time is analogous to the component lifetime and a reconnection at a vacant premises corresponds to the component failure. Following this analogy, the time that a pair has been idle as a CT will be referred to as its age. If a CT is to be broken, the one with the smallest reuse probability, and thus the smallest $h(t)$ should be chosen.

Four different vacancy time distributions are considered here. In addition to the commonly used exponential function, modifications to allow categorization, abandonment, and variability in the rate parameter are considered.

3.1 Exponential model

The exponential distribution is commonly used for modeling phenomena such as vacancy times due to its analytic simplicity. The Poisson demand model of Section II is equivalent to exponential vacancy and occupancy times. For a premises with an exponential vacancy time distribution of parameter λ,

$$f(t) = \lambda e^{-\lambda t}$$

$$h(t) = \lambda \tag{25}$$

Thus, the instantaneous reuse rate for any premises is constant over time, independent of when the premises became vacant. In addition, since all premises in a serving area are assumed to have the same demand parameter, λ, the reuse probabilities are the same for every CT in the interface. Thus, randomly selecting which CT to break is as good a strategy as any. This unappealing result leads to several modifications of the basic model.

3.2 Categorized exponential model

One modification of the exponential model is to reject the assumption that all premises in a serving area have the same demand parameter. Since a premises is defined as any potential point of demand for service, the characteristics of a premises should affect at least its mean vacancy time. For example, a second line would certainly have a longer expected vacancy time than a first line. Four other categorizations appropriate for premises within a serving area which have significant differences in the vacancy time parameters are[3]:

(*i*) Type of dwelling (apartment/single family residence)
(*ii*) Reason for disconnect (moving within a city/leaving city)
(*iii*) Occupation (business/professional/military)
(*iv*) Customer estimated date for reestablishment of service (less than two weeks/more than two weeks)

By using various combinations of categorizations, up to 48 different categories could be defined. If all premises in category i have exponential vacancy time distributions with parameter λ_i, the instantaneous reuse rates become

$$h_i(t) = \lambda_i \tag{26}$$

The optimal CT strategy is therefore to break any CT in the category which has the smallest λ_i (i.e., largest mean vacancy time).

3.3 Categorized exponential with abandonment model

A phenomenon which the above models do not take into account is the unreusability of some CT pairs (this is known as abandonment). This may be due either to physical abandonment of a premises or to changes in address designations which cause plant assignment procedures to ignore reuse possibilities. Assume that the vacancy time distribution of nonabandoned premises in category i is exponential with parameter λ_i and that the probability of abandonment is q_i. Then,

$$f_i(t) = (1 - q_i)\lambda_i e^{-\lambda_i t} \qquad \text{for } t < \infty$$

$$h_i(t) = \frac{\lambda_i(1 - q_i)e^{-\lambda_i t}}{1 - (1 - q_i)(1 - e^{-\lambda_i t})} \qquad \text{for } t < \infty \tag{27}$$

The reuse probabilities thus depend on t, the age of the CT.

If there is only one category, the optimal CT strategy is to break the oldest, since h_i decreases with time. This is reasonable since the longer a premises has been vacant, the more likely it is to be an abandoned one. When several categories are present, the effect of the parameters λ_i and q_i on the instantaneous reuse rate must be taken into account. The CT to break would be the oldest in its category, but the reuse probabilities for the oldest CT in each category must be compared to determine the lowest.

Figure 3 shows the optimal CT strategy as a function of the age of oldest CT in each of two categories for a case where Category I is more likely to be abandoned, but is also more likely to be reused sooner if it is not abandoned. In this hypothetical example, if the age of the oldest CT in Category I is small (less than 4 months), it is preferable to break even a new Category II CT. This occurs because the effects of abandonment are small relative to the effect of the λ_i for these values. For older Category I's (above 5.5 months), however, it may be preferable to break a newer Category I CT over an older CT in Category II.

3.4 Categorized beta type II with abandonment model

A further modification of the exponential model is to change the rate parameter, λ, from a known constant to a random variable with known

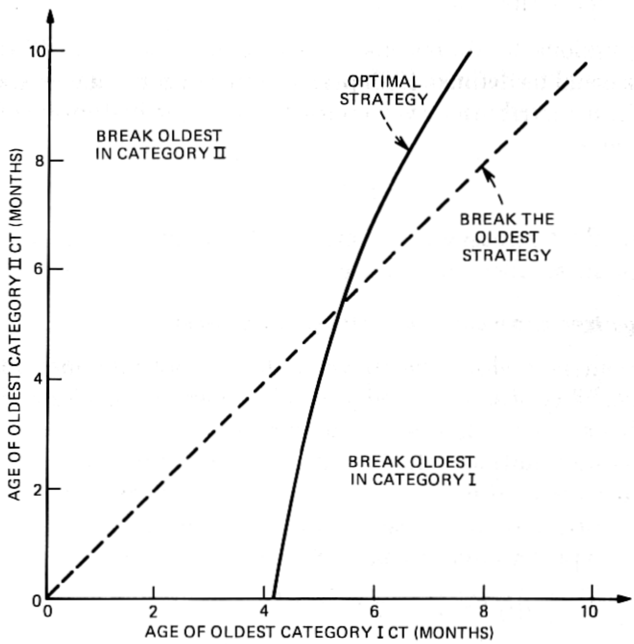

Fig. 3—CT strategy curves for hypothetical example. The parameter values used were $\lambda_1 = 0.25$, $\lambda_2 = 0.20$, $q_1 = 0.10$, $q_2 = 0.05$.

distribution. For example, any particular occupant of a premises may generate moves according to an exponential distribution with parameter which depends on the particular occupant. Then the premises would have an exponential distribution with a random parameter.

If a gamma distribution of scale parameter $1/d_i$ and shape parameter c_i is chosen to represent the known distribution of λ_i, the vacancy time distribution for a premises in category i becomes

$$f_i(t) = (1 - q_i)c_i d_i^{-1}(1 + t/d_i)^{-c_i-1}$$

$$h_i(t) = (1 - q_i)c_i d_i^{-1}[q_i(1 + t/d_i)^{c_i+1} + (1 - q_i)(1 + t/d_i)]^{-1} \quad (28)$$

The derivation of (28) is given in Mann et al.[8] where it is called the exponential conditional failure distribution. Note that c_i/d_i is the expected value of λ_i and c_i/d_i^2 is its variance.

This distribution was used (under the name beta type II distribution) by Hoadley[3] to model premise vacancy times based on the empirical observations of abandonment and of decreasing reuse probabilities with CT age. In particular, the empirical evidence showed that probability of reuse within the first few weeks is very high. Overall, 50 percent of the premises were reoccupied within 60 days, with some categories finding 80 percent reconnection within that time span. If such numbers are generally applicable, a high percentage of inward orders will result in reuses, so that the savings from using a good CT policy should be very high.

Although both the beta type II distribution and the categorized exponential with abandonment model give decreasing reuse probabilities over time when the abandonment probabilities are positive, only the former has this property when $q_i = 0$.

Under the assumption of beta type II distributed vacancy times, the optimal CT strategy when there is a single category is to break the oldest, since h decreases with time. For multiple categories, the reuse probabilities for the oldest in each category would have to be compared to find the lowest. Again, there will be cases where it is more advantageous to break a newer CT.

3.5 Summary

Although the exponential model is the simplest analytically, it is apparent that it does not account for empirical observations about vacancy time distributions. Both the categorized exponential with abandonment and the beta type II models are more realistic. Both have reuse probabilities decreasing with CT age, but of different functional form. The optimal strategy for breaking CTs under either model is to break one which is the oldest in its category, with the category determined by comparing the instantaneous reuse rates for the oldest in each category.

The use of an optimal strategy gives lower operating costs than the random strategy used in the relief timing derivations. This would have the effect of lowering the operating cost curve in Fig. 2, thus postponing the optimal relief time. Although analytic operating cost models to assess the exact extent of this effect have not been developed, typical serving areas were simulated under the various demand assumptions, using both a random breaking strategy and the policy of breaking the one with the minimum instantaneous reuse probability. The reduction in operating costs from using the optimal policy ranged up to twelve percent, depending on the demand parameters.

IV. CONCLUSIONS

In this paper, optimal procedures have been developed for two network operation decisions. The optimal time for relief of a serving area or allocation area is found as the time when operating costs (determined from a linear growth, birth-and-death demand model) exceed the levelized equivalent charges for relief. If the interface exhausts its spares and relief is not yet appropriate, CT pairs will have to be broken to provide service on some inward orders. The optimal CT to break is the one with the smallest instantaneous reuse probability; this will be one that is the oldest in its category, but the category will depend on the CT ages.

The question to be resolved before a model of this type can be implemented involve the data requirements and how to estimate the model parameters. Data (e.g., growth rates for working and CT pairs) may only be available at an aggregate level (e.g., by allocation area), so that a means of disaggregation may be required for these models. Although eq. (14) provided a means for estimating some of the model parameters, procedures for obtaining others (e.g., abandonment rates) remain to be developed.

The models developed here provide optimal operating policies for serving areas; however, serving areas constitute only a portion of the present loop plant. Extensions of these models to other loop network configurations is discussed by Koontz[2] elsewhere in this issue.

APPENDIX

Derivation of minimum cost CT breaking strategy

This appendix will derive the minimum present worth operating cost strategy for breaking CTs and show that it is approximately the same as minimizing the instantaneous reuse probability as given by the hazard function. The derivations are minor modifications of those originally developed by J. Freidenfelds in unpublished notes.

Assume that a CT pair has to be broken at time zero, and that the choice of which to break has been narrowed down to CT_1 and CT_2 (for example, by applying the derived results iteratively). Define

a_i = age of CT_i at time zero
T_i = random variable representing the time (relative to time zero) when a customer returns to the location of CT_i
$g_i(t)$ = p.d.f. for T_i
$G_i(t)$ = c.d.f. for T_i
$f_i(t)$ = vacancy time p.d.f. for location of CT_i
$F_i(t)$ = vacancy time c.d.f. for location of CT_i

Note that

$$g_i(t) = f_i(t)/[1 - F_i(a_i)]$$

and

$$G_i(t) = F_i(t)/[1 - F_i(a_i)]$$

Also define

$$\Delta = \text{time between breaking CTs at the interface}$$

$$r = \text{discounting rate, and}$$

$$\theta_i(y) = E(e^{-rT_i}|y) = \frac{1}{1 - G_i(y)} \int_y^\infty e^{-rt} g_i(t)dt$$

Since, in addition to breaking a CT at time zero another one will have to be broken at time Δ, the options are to break CT_1 now and CT_2 at time Δ or CT_2 now and CT_1 at Δ. The cost, C_1, of the former option is the sum of the present worths of the reconnection cost when customer 1 returns, the reconnection cost when customer 2 returns if he returns after Δ, and the reuse cost if he returns before Δ. Letting the reuse cost = 0, and the cost of reconnection relative to reuse = C_{REC}, then

$$C_1 = C_{REC}[E(e^{-rT_1}|0) + E(e^{-rT_2}|\Delta)(1 - G_2(\Delta))]$$
$$= C_{REC}[\theta_1(0) + \theta_2(\Delta)(1 - G_2(\Delta))]$$

Similarly, the option of breaking CT_2 now costs

$$C_2 = C_{REC}[E(e^{-rT_2}|0) + E(e^{-rT_1}|\Delta)(1 - G_1(\Delta))]$$
$$= C_{REC}[\theta_2(0) + \theta_1(\Delta)(1 - G_1(\Delta))]$$

Then CT_1 should be broken if $C_1 < C_2$, or

$$\theta_1(0) + \theta_2(\Delta)(1 - G_2(\Delta)) - \theta_2(0) - \theta_1(\Delta)(1 - G_1(\Delta)) < 0$$

Let

$$\mu_i = \theta_i(0) - \theta_i(\Delta)(1 - G_i(\Delta))$$

Then CT_1 should be broken if $\mu_1 - \mu_2 < 0$ which means the CT with the smaller μ_i should be broken.

In a serving area interface, Δ tends to be very small. Thus a valid approximation to μ_i is

$$\lim_{\Delta \to 0} \mu_i$$

Since this limit is easily seen to be zero for any μ_i, we need to look at

$$\lim_{\Delta \to 0} \mu_i / \Delta$$

to get a good approximation for small Δ. Applying l'Hôpital's rule gives:

$$\lim_{\Delta \to 0} \frac{u_i}{\Delta} = \lim_{\Delta \to 0} \frac{d\theta_i(0)}{d\Delta} - \left[(1 - G_i(\Delta)) \frac{d\theta_i(\Delta)}{d\Delta} - \theta_i(\Delta) \frac{dG_i(\Delta)}{d\Delta} \right]$$

Performing the differentiation gives:

$$\begin{aligned} \lim_{\Delta \to 0} \frac{u_i}{\Delta} &= \lim_{\Delta \to 0} \frac{-g_i(\Delta)}{1 - G_i(\Delta)} [G_i(\Delta)\theta_i(\Delta) - (1 - G_i(\Delta))e^{-r\Delta}] \\ &= g_i(0) \\ &= f_i(a_i)/[1 - F_i(a_i)] \end{aligned}$$

which is the hazard function of the vacancy time distribution. Thus we would break CT_1 if and only if its hazard function were smaller than that of CT_2. In the general case, the CT with the smallest hazard function value should be chosen.

REFERENCES

1. N. G. Long, "Loop Plant Modeling: Overview," B.S.T.J., this issue.
2. W. L. G. Koontz, "An Approach to Modeling Operating Costs in the Loop Network," B.S.T.J., this issue.
3. B. Hoadley, "The Southern Bell Left-in Station Study," Bell Laboratories Memorandum, February 1971.
4. W. Feller, *An Introduction to Probability Theory and its Applications,* Vol. I, New York: Wiley, 1968, P. 481.
5. H. T. Freedman, private communication.
6. J. Freidenfelds, "Levelized Equivalent Annual Cost Associated with a Capital Expenditure—Annual Charge Factors," appendix to "A Simple Model for Studying Feeder Capacity Expansion," B.S.T.J., this issue.
7. B. L. Marsh, "The Feeder Allocation Process," B.S.T.J., this issue.
8. N. R. Mann, R. E. Schafer, and N. D. Singpurwalla, *Methods for Statistical Analysis of Reliability and Life Data,* New York: Wiley, 1974, pp. 116–147.