THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 57, No. 4, April 1978
Printed in U.S.A.

Loop Plant Modeling:

Economic Evaluation of Subscriber Pair Gain System Applications

By W. L. G. KOONTZ

(Manuscript received August 20, 1977)

In recent years, the cost of subscriber pair gain systems, i.e., systems which enable more than one subscriber to be served by a single cable pair, has decreased significantly in comparison with cable. Moreover, the operating expenses associated with an all-cable loop network have increased along with the cost of labor, particularly in areas of high customer mobility and uncertain growth. For these reasons, the application of pair gain systems has become an important consideration in loop plant design. This paper presents and discusses a series of mathematical models which can be used in the economic analysis of subscriber pair gain system applications. Given the forecast requirement for loop facilities, one may use these models to evaluate and compare alternatives for meeting this requirement on a present worth basis. The alternatives may include cable only, pair gain systems only, or a combined cable/pair gain alternative (deferred cable). These models have been applied in Bell Laboratories studies of the market for pair gain systems. They are now being incorporated into guidelines which will enable the operating companies to apply pair gain systems in an economic manner.

I. INTRODUCTION

Subscriber pair gain systems, which use carrier and concentrator techniques to reduce requirements for loop cable, have been available since the 1950s. Until recently, however, their high cost relative to cable has limited their application to very long rural routes requiring expensive coarse-gauge cable. Now, as a result of improved technology, the cost of pair gain is competitive with cable in the suburban, as well as the rural, environment. In addition, the reliability of pair gain systems has been greatly improved and this has also contributed to their increased attractiveness. Thus, subscriber pair gain systems have become an important consideration in economically expanding the capacity of the loop network.

In this paper, a series of capacity expansion models which consider both pair gain and cable will be developed. These models are extensions of the models developed in Ref. 1. The capacity expansion models are used to develop the optimal strategy for adding capacity to the loop network with a combination of pair gain and cable. The optimal strategy for the basic model is developed in Section II and some specific cases are studied in Sections III and IV. The problem of network complexities and a simple method for dealing with this problem is discussed in Section V. Mathematical programming approaches, which have been implemented as computer programs, will be discussed in Section VI. Finally, some advanced models, which reflect the stochastic nature of subscriber demand and loop network activities, will be introduced in Section VII.

The operating companies have felt an increased need for guidance in the proper application of pair gain systems. The pair-gain/cable capacity expansion theory developed in this paper forms a basis for application guidelines and computer programs now used by the operating companies in planning pair gain system application. The theory has also been applied within Bell Labs to suggest new applications for pair gain systems and to develop improved designs for the loop network.

In order to follow the theoretical development, it will be useful to have additional background information regarding pair gain systems. This section will include, therefore, an overview of subscriber pair gain systems and their applications.

1.1 Subscriber pair gain systems

The basic structure of a subscriber pair gain system is illustrated schematically in Fig. 1. The system consists of a central office (CO) unit, located in the central office building, and a remote unit, located in the field. A given number, say L, of 1-party subscribers* are connected to the remote unit by individual wire pairs which will be called subscriber lines. The remote unit is connected to the CO unit by K wire pairs ($K < L$) which will be called CO links. The CO unit effectively converts the K links into L line appearances at the CO. The CO unit may be integrated into the switching equipment such that physical expansion of the K links into L lines is not required. The *pair gain,* which is defined as the dif-

* For the purposes of this paper, multiparty subscribers may be grouped into equivalent 1-party subscribers.

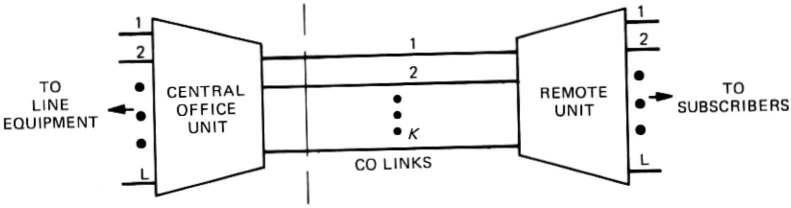

Fig. 1—Structure of a pair gain system.

ference $L - K$, is the net reduction in cable pair requirements achieved by the pair gain system.

There are two basic approaches to achieving pair gain. In a *carrier* system, time or frequency division multiplexing is used to derive additional voice and signaling channels over each CO link. For example, the *SLC*™-1* (Subscriber Loop Carrier: $L = 2, K = 1$) derives a second line from a single wire pair by means of amplitude modulation. Another example is the *SLC*-40 ($L = 40$, $K = 2$), which uses a delta modulation scheme to derive 40 channels over 2 wire pairs (using digital repeaters).

The other basic approach is concentration. In a concentrator, each of the L subscribers has access through a switching network to either all, or a subset of, the K links. When a subscriber goes off-hook, an idle link is connected to his line. The LSS (Loop Switching System: $L = 192$, $K = 66$) is a concentrator in which each subscriber has access to 7 links. The LSS switching network employs miniature relays under microprocessor control. If no idle link can be connected to the off-hook subscriber, the call is blocked. A concentrator must be designed and operated to maintain a low probability of blocking, consistent with grade of service objectives.

A system does not have to be pure carrier or concentrator. The *SLM*™ ($L = 80$, $K = 2$) concentrates 80 lines down to 24 channels which are derived from 2 links via delta modulation.

Additional discussion of pair gain techniques is beyond the scope of this paper. The interested reader may find more information in Ref. 2. In the balance of this paper, pair gain systems will be entirely characterized by L, K, and cost parameters.

* The actual pair gain systems referred to in this paper are Bell System products. Similar systems are available through the general trade.

1.2 *Pair gain system applications*

Clearly, pair gain systems reduce the need for subscriber cable pairs and therefore the obvious application of pair gain is as an alternative to additional cable. However, the determination of an economic policy for pair gain application is not simply a matter of deciding whether to use pair gain or cable. Loop network capacity expansion is a dynamic process involving the questions of when to add capacity and how much new capacity to add. The pair gain alternative adds the question of by what means to add new capacity.

Consider a route which is experiencing growth and whose existing capacity is exhausted. Any of the following alternatives may be appropriate:

(*i*) Place a new cable.

(*ii*) Place one or more pair gain systems, using existing cable pairs as links.

(*iii*) Place one or more pair gain systems initially, using existing cable pairs as links. When these systems exhaust, remove them and place a new cable.

Alternative (*i*) is the classic "all cable" solution which is emphasized in Ref. 1. Alternative (*ii*) is often called a permanent pair gain solution, since the pair gain systems are not removed. Alternative (*iii*) is called a temporary pair gain solution in which the relief cable is deferred, but once it is placed, the pair gain systems are removed. Generally speaking, the cost of the pair gain system relative to cable must be lower to justify (*ii*) rather than (*iii*). Thus (*ii*) is prevalent primarily on long rural routes while (*iii*) is more characteristic of suburban applications.

In the theoretical development which follows, neither of the above alternatives will be assumed *a priori*. Rather, a general formulation will be developed and it will be shown that each of these alternatives may be optimal under different circumstances.

Some "special" applications will be touched upon in Section VII. First of all, the application of single channel pair gain to the provision of second line service will be analyzed. Secondly, the application of pair gain systems as an alternative to network rearrangements will be studied. These latter results are quite preliminary and are included to stimulate further work.

II. BASIC CAPACITY EXPANSION MODEL

A basic model for loop network capacity expansion using pair gain and cable is derived in this section. Specifically, the model expresses the total PWAC (present worth of annual charges) associated with a generalized pair gain application policy. The minimum PWAC policy will be computed and its properties will be examined.

2.1 PWAC model

The PWAC model will be derived for the simplified route illustrated schematically in Fig. 2. The route consists of a single feeder section between the CO and the remote terminal site for the pair gain systems. It is assumed that at time $t = 0$, no pair gain systems are in place and the existing cable has just exhausted. Subscriber demand is assumed to be growing linearly with growth rate g.

The generalized application policy is as follows: From time $t = 0$ to $t = T$ ($T \geq 0$), additional capacity will be provided by means of pair gain. At time $t = T$, all pair gain systems are removed and a relief cable of size S is placed. Both T and S are design parameters to be optimized. The optimal values of T can be related to the three alternatives discussed in Section I as follows:

(*i*) $T = 0$ (all cable)
(*ii*) $T = \infty$ (permanent pair gain)
(*iii*) $0 < T < \infty$ (temporary pair gain)

It is implicitly assumed that the existing cable can supply the pairs necessary for links.

During the time interval $[0,T]$, a pair gain system cost is incurred. This cost generally includes the cost of the pair gain equipment. It is assumed here that pair gain systems are "rented" from a central "supplier" for a given annual charge. This annual charge is incurred for each pair gain system from the time it is installed until the time it is removed. The amount of the annual charge depends upon the cost of the pair gain equipment, its service life, characteristics of the supplier, and other factors. The details of computing the annual charge will not be discussed here.

The installation and removal costs are incurred whenever a pair gain system is installed or removed. Depending on the tax status of these costs, they may be treated as one time charges or levelized over the period during which the pair gain system remains at a particular location. The details of computing installation and removal charges will also be omitted from this discussion.

In the basic derivation which follows in this section, the total pair gain system cost will be expressed as an annual charge rate $\gamma(t,T)$. The annual charge rate is time varying since additional pair gain systems may be installed during $[0,T]$. The annual charge rate may also depend on T if installation or removal charges are levelized over the period during which the pair gain systems are applied. Note that one time charges will result in impulses in $\gamma(t,T)$.

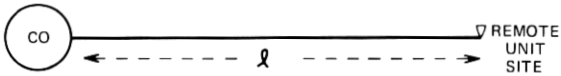

Fig. 2—Basic pair gain application.

The cable cost is expressed as an annual charge which begins at time T and continues forever. The annual charge is assumed to be of the form $(A + BS)\ell$ where S is the number of cable pairs provided and ℓ is the length of the cable. This cable will exhaust at time $t = S/g$ (assume $S/g > T$). All cash flows beyond this time are represented here by an equivalent present worth cost of the future C_F which is incurred at time $t = S/g$.

Figure 3 illustrates the cash flow assumed for the pair gain/cable capacity expansion model. The total PWAC for pair gain, cable, and all future relief is given by

$$\text{PWAC} = \int_0^T \gamma(t,T)e^{-rt}dt + \frac{1}{r}e^{-rT}(A + BS)\ell + C_F e^{-rS/g} \quad (1)$$

where r is the convenience discounting rate. In the linear growth case, with no conduit or other complications, the future capacity expansion starting at time $t = S/g$ is identical to the one starting at time $t = 0$. If the same T and S are used *ad infinitum*, $C_F = \text{PWAC}$ so that

$$\text{PWAC} = (1 - e^{-rS/g})^{-1} \times \left[\int_0^T \gamma(t,T)e^{-rT}dt + \frac{1}{r}e^{-rT}(A + BS)\ell\right] \quad (2)$$

For $T = 0$, eq. (2) reduces to the PWAC equation for the corresponding cable sizing problem.

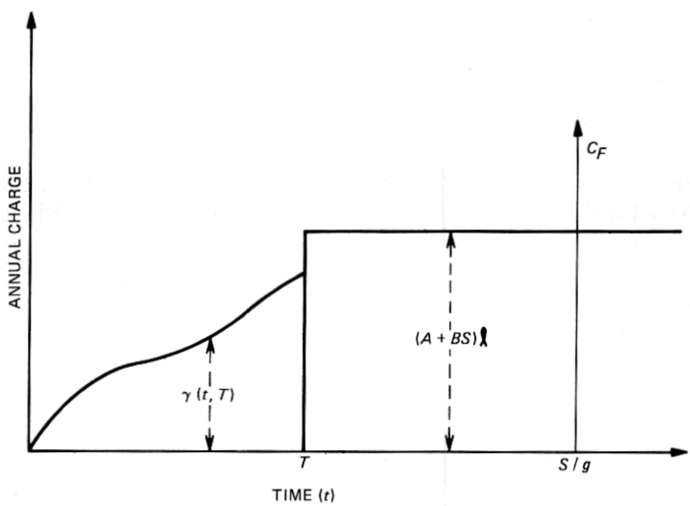

Fig. 3—Cash flow for general pair gain/cable policy.

2.2 Minimum PWAC policy

A set of equations for the optimal solution, (T^*,S^*) follows from the necessary conditions

$$\left.\frac{\partial \mathrm{PWAC}}{\partial T}\right|_{\substack{T=T^*\\S=S^*}} = 0 \tag{3}$$

and

$$\left.\frac{\partial \mathrm{PWAC}}{\partial S}\right|_{\substack{T=T^*\\S=S^*}} = 0 \tag{4}$$

The basic equations which follow from carrying out the differentiation and rearranging are

$$\gamma(T^*,T^*) + \int_0^{T^*} \left.\frac{\partial\gamma(t,T)}{\gamma T}\right|_{\substack{T=T^*\\S=S^*}} e^{-rt}dt = (A + BS^*)\ell \tag{5}$$

and

$$e^{rS^*/g} - rS^*/g - 1 = \frac{r}{gB}\left[A + re^{rT^*}\int_0^{T^*} \ell^{-1}\gamma(t,T^*)e^{-rt}dt\right] \tag{6}$$

These equations, although somewhat complex, can be readily interpreted. Equation (5) requires that the pair gain systems be removed when the effective annual charge for pair gain equals the annual charge for the relief cable. Equation (6) is the standard cable sizing equation except that a positive term has been added to the cable A cost. This means that when temporary pair gain systems are used, the relief cable is oversized in comparison to the all cable solution.

Equations (5) and (6) define the general solution to the pair gain/cable capacity expansion model. In Sections III and IV, some specific cases will be explored.

III. SINGLE CHANNEL APPROXIMATION

The first case to be studied is an approximation to a single channel pair gain system ($L = 2, K = 1$). Let $\gamma(t,T) = \gamma g t$ where the constant γ is roughly interpreted as the annual charge per pair gained. This approximation ignores installation and removal charges and the effect of discretization. These effects will be considered in Section IV. For this special case, eqs. (5) and (6), after some manipulation, become (dropping the $*$ notation)

$$\gamma g T = (A + BS)\ell \tag{7}$$

and

$$e^{rS/g} - 1 = \frac{\gamma}{B\ell}(e^{rT} - 1) \tag{8}$$

from which it can be determined that the optimal S satisfies

$$e^{\beta(\alpha+rS/g)} - \beta e^{rS/g} = 1 - \beta \quad (9)$$

where

$$\alpha = \frac{rA}{gB} \quad (10)$$

and

$$\beta = \frac{B\ell}{\gamma} \quad (11)$$

The α parameter of eq. (10) appears in the cable sizing equations derived in Ref. 1. The β parameter of eq. (11) is the ratio of incremental cable cost to cost per pair gained.

Equation (9) can be easily solved by standard numerical techniques. Figures 4 and 5 illustrate the solution as a function of β for various values of the other parameters. The optimal deferral period T is plotted in Fig. 4 and the optimal cable size S is plotted in Fig. 5. These curves illustrate some important points about the application of pair gain. First of all, the optimal deferral period increases with β. This result reflects the fact that when the cable cost is high relative to pair gain (e.g., when the loop length ℓ is large), longer deferrals are economical. As β increases to 1, T increases without bound and $\beta \geq 1$ corresponds to a permanent application of pair gain. Secondly, the curves illustrate the impact of other

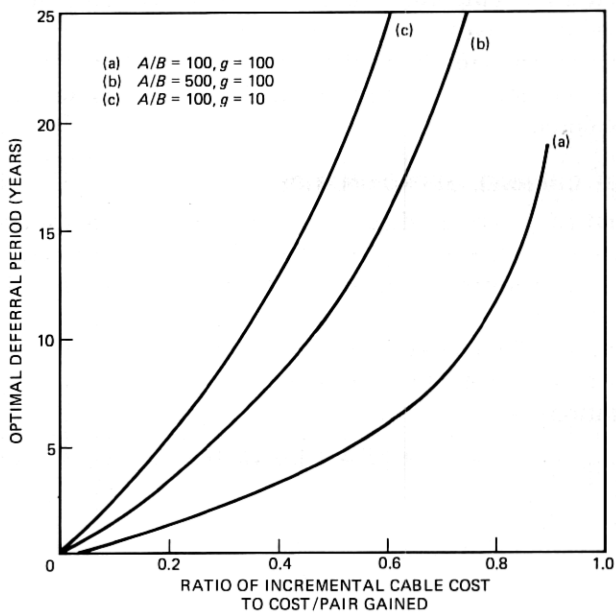

Fig. 4—Optimal deferral period curves.

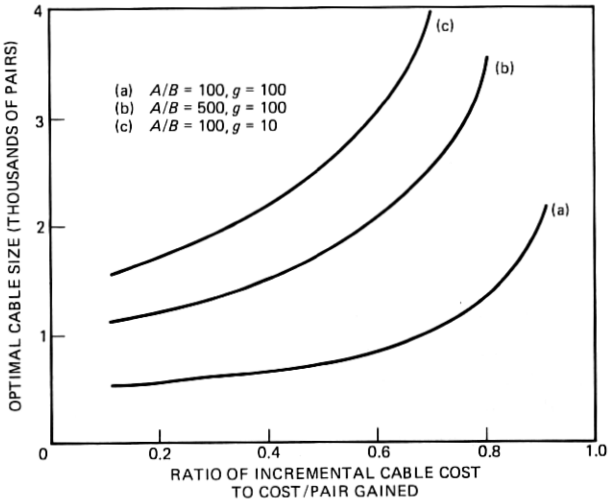

Fig. 5—Optimal cable size curves.

important parameters: A/B and the growth rate g. For larger A/B or less growth, longer deferrals are economical. Thus, the prime areas for pair gain application are those with slow growth and a high "fixed cost" for cable placement. Finally, the curves in Fig. 5 show quantitatively the increase in relief cable size that results from cable deferral with pair gain.

Note that for any $\beta > 0$, pair gain can be economically applied. Of course, it is not practical to apply pair gain for a very short time because of the cost of installation and removal. However, this result suggests that if installation/removal costs are low enough, short term deferrals will pay.

The results derived above show that, under certain assumptions, the all-cable solution is optimal only for very small β, the temporary pair gain solution is optimal for $\beta < 1$, and the permanent pair gain solution is optimal for $\beta \geq 1$. Even if $\beta < 1$, however, permanent pair gain may "prove in," i.e., compare favorably on a PWAC basis with the all-cable solution. Consider the following example:

$$A = \$0.167/\text{ft} \qquad \gamma = \$50./\text{pair gained}$$
$$B = \$0.00167/\text{ft} \qquad r = 0.07$$
$$\ell = 20\ \text{Kft} \qquad g = 50\ \text{lines/yr.}$$

The all-cable solution is obtained by solving the standard cable sizing equation,

$$e^{rS/g} - rS/g - 1 = \frac{rA}{gB} \tag{12}$$

which yields $S = 347$ pairs. The PWAC for the all-cable solution, which is obtained from eq. (2) with $T = 0$, is \$554K. The PWAC for the permanent carrier solution is given by

$$\text{PWAC} = \int_0^\infty \gamma g t e^{-rt} dt = \frac{\gamma g}{r^2} \tag{13}$$

which, for this example, is \$510K. Therefore, permanent carrier proves in by \$44K over the all-cable solution.

On the other hand, the optimal policy for this example is to use pair gain for 10.72 years and then place a 702 pair cable. The optimal PWAC is \$430K which is an additional \$80K savings. Thus, temporary pair gain must always be considered, even for long routes where permanent pair gain proves in. Of course, there may be additional benefits which favor permanent rather than temporary application. In the next section, however, it will be shown that some of these benefits can and should be accounted for in the economic analysis.

IV. LUMPED PAIR GAIN MODELS

The single channel approximation in Section III does not adequately represent larger "lumped" pair gain systems. A lumped system provides pair gain in discrete steps. For example, one unit of *SLC*-40 provides a pair gain of 38 (40 lines − 2 links). Also a lumped pair gain system incurs substantial installation and removal costs. In this section, more complex forms of $\gamma(t,T)$ will be developed to represent lumped systems.

4.1 Annual charge model for lumped systems

The cost of a lumped pair gain system can be characterized by three components, an annual charge a, an installation charge I, and a removal charge R. The annual charge represents the cost of the pair gain equipment (both CO unit, remote unit, and repeaters) annualized over its effective service life. The installation charge is incurred whenever a system is installed and is assumed to be levelized over the period during which the system remains in place. The removal charge is assumed to be a one time charge which occurs when a system is removed.

For example, suppose a system is installed at time $t = 0$ and removed at time $t = T$. Then the total PWAC for the system application is

$$\begin{aligned} \text{PWAC} &= \int_0^T [a + m(T)I] e^{-rt} dt + Re^{-rT} \\ &= M(T)I + \int_0^T a e^{-rt} dt + Re^{-rT} \end{aligned} \tag{14}$$

where $M(T)$ is a factor giving the present worth of annual costs associated with each dollar of capital which is to be recovered over T years. For example, if simple straight-line depreciation is used for both book and tax purposes,

$$M(T) = 1 + \phi(1 - (1 - e^{-rT})/rT) \tag{15}$$

where ϕ is the income tax factor (Ref. 3). With modern tax laws, eq. (15) is liable to be considerably more complex, but the above form can be used here for illustrative purposes.

It may be that taxes will be calculated on the basis of an average value of T rather than the actual value. In this case, $M(T)$ is a constant and the PWAC calculations are much simpler. In the derivations which follow, however, the more general case, where the annual charge factor for I depends on T, will be assumed.

Now consider the route of Fig. 2 and assume that the demand is met by placing a sequence of N pair gain systems, each having a pair gain of η. The nth system $(1 \le n \le N)$ is installed at time $t = \tau_n$ where $\tau_n = (n - 1)\eta/g$. At time $t = \tau_{N+1} = N\eta/g$, all N systems are removed and a cable of size S is placed. When the cable exhausts, the relief cycle is repeated. The annual charge for the nth system is given by

$$\text{a.c.} = \begin{cases} a + m(\tau_{N+1} - \tau_n)I + \delta(t - \tau_{N+1})R & \tau_n \le t \le \tau_{N+1} \\ 0, \quad \text{otherwise} \end{cases} \tag{16}$$

where $\delta(t)$ is a Dirac delta function. The total annual charge function $\gamma(t,N)$ (for convenience, N is used as a control variable rather than T) is therefore given by

$$\gamma(t,N) = \begin{cases} na + \sum_{k=1}^{n} m(\tau_{N+1} - \tau_k)I, & \tau_n \le t < \tau_{N+1} \\ & n = 1,2, \ldots ,N \\ N\delta(t - \tau_{N+1})R & t \ge \tau_{n+1} \end{cases} \tag{17}$$

4.2 Optimal relief policy using lumped pair gain systems

The total PWAC of the relief policy outlined above follows from Eq. (2):

$$\text{PWAC} = (1 - e^{-rS/g})^{-1} \times \left[\int_0^{N\eta/g} \gamma(t,N)e^{-rt}dt + \frac{1}{r} e^{-rN\eta/g}(A + BS)\ell \right] \tag{18}$$

For a given value of N, the optimum cable size is obtained as the solution of

$$e^{rS/g} - rS/g - 1 = \frac{r}{gB}\left[A + re^{rN\eta/g} \int_0^{N\eta/g} \ell^{-1}\gamma(t,N)e^{-rt}dt \right] \tag{19}$$

The optimal policy is determined by solving eq. (19) for $N = 0,1,\ldots,$ and choosing the solution which minimizes the PWAC given by eq. (18). This process can be programmed quite easily, since the integral in eqs. (18) and (19) reduces to a summation, i.e.,

$$\int_0^{N\eta/g} \gamma(t,N)e^{-rt}dt = \frac{1}{r}\sum_{n=1}^{N} ae^{-r\tau_n} + m(\tau_{N+1} - \tau_n)I(e^{-r\tau_n} - e^{-rN\eta/g}) + N\left(R - \frac{a}{r}\right)e^{-rN\eta/g} \quad (20)$$

4.2.1 Example

The following data will be used to illustrate the above process

$A = \$.167/\text{ft}$	$a = \$2500$	$\eta = 50$
$B = \$.00167/\text{ft}$	$I = \$1000$	$r = 0.07$
$\ell = 15$ Kft	$R = \$500$	$g = 50$ lines/yr
$\phi = 0.7$		

It is assumed that $M(T)$ is given by eq. (15). Table I lists the solutions to eq. (19) and the PWAC from eq. (18) as a function of N. The minimum PWAC solution is obtained for $N = 5$. Thus, the optimum pair gain policy is to install one pair gain system per year until 5 systems have been placed, and then, at the end of year 5, remove the 5 systems and place a 520 pair cable. The total PWAC, including all pair gain installation, carrying, and removal charges and cable cost, is $373K.

The PWAC for the all-cable solution ($N = 0$) is $416K and the PWAC for the permanent carrier solution ($N = \infty$) is $553K. Therefore, in this example, permanent pair gain does not prove in over cable, but temporary pair gain provides significant savings. If the relative costs of cable

Table I — Lumped pair gain example

Number of systems	Cable size (pairs)	PWAC ($1000s)	Optimum solution
0	346	416	
1	365	398	
2	392	386	
3	429	378	
4	472	374	*
5	520	373	
6	571	374	
7	624	375	
8	679	378	
9	735	381	
10	792	385	
∞	—	553	

and pair gain are varied, however, the result may change. For example, if ℓ = 25 Kft, the solutions are:

(*i*) All cable (N = 0)—$693K

(*ii*) Optimum (N = 19)—$527K

(*iii*) Permanent pair gain ($N = \infty$)—$553K

Now permanent pair gain proves in over cable, but temporary pair gain is still optimal. If ℓ is further increased to 40 Kft, permanent pair gain is optimal. On the other hand, if ℓ is reduced to 5 Kft, the all-cable solution becomes optimal.

These results parallel the results obtained from the single channel approximation in Section III. Thus, the general nature of the optimal pair gain application policy is not affected by the considerations of installation and removal cost and the discrete sizes of pair gain systems.

V. RELIEF PROJECT DEFERRAL—A PRACTICAL APPROACH

Up to now, consideration has been limited to the simple network illustrated in Fig. 2. In practice, however, loop networks are much more complex.[4] This section discusses some of the complexities of the loop network which must be considered and provides a simplified approach to dealing with them. In Section VI, more sophisticated mathematical programming approaches are outlined.

5.1 Loop network complexities

The simple network of Fig. 2 consists of a single cable section and a single point at which pair gain systems may be placed. A real loop network is composed of many interconnected cable sections and many potential pair gain system sites. In general, the capacity of the network may be expanded by placing additional cable or deploying pair gain systems throughout the network. Even for a moderately complex network, the number of alternatives for providing additional capacity is enormous.

Specifically, whenever a facility shortage occurs anywhere in the network, one or more of the following steps may be taken.

(*i*) Place additional cable (where? how much?)

(*ii*) Place or remove pair gain systems (where? how many? what kind?)

Clearly, the basic model of Section II cannot handle this complex problem. On the other hand, a truly general formulation is not practically solvable even by sophisticated mathematical programming techniques (see Section VI). Thus, it is necessary to simplify the general problem to one which is amenable to available techniques. In Section 5.2, it is shown that the problem can be simplified to the extent that a variation of the approach developed in Sections II–IV can be applied.

5.2 Deferral of a feeder relief project

Although feeder relief cables are sized independently for each feeder section,[1] the actual provision of relief is through a sequence of relief projects. A project generally provides relief for a feeder route for a period of at least two years by relieving one or more feeder sections. The consolidation of section relief cables into route relief projects is a practical measure which strikes a balance between the PWAC penalty for advancing the relief of some sections and the costs of complex splicing between sections as well as project overhead.

It will be assumed that a relief project is indivisible and its make-up (cable sizes, etc.) and cost are fixed. The only variable is the time at which the project is placed. It is also assumed that the project clears all shortages in the network. Finally, it is assumed that there is a single site at which pair gain systems may be located so as to reduce the cable pair demand in the sections requiring relief.

Figure 6 illustrates this formulation of the problem. The remote unit site is connected to the CO through a series of feeder sections and the relief project spans one or more of these sections. Subscribers beyond the remote unit site may be served by pair gain systems resulting in a net reduction in demand in the feeder sections shown.

Under these assumptions, the optimization problem is greatly simplified. Whenever a shortage occurs, only two options are available:

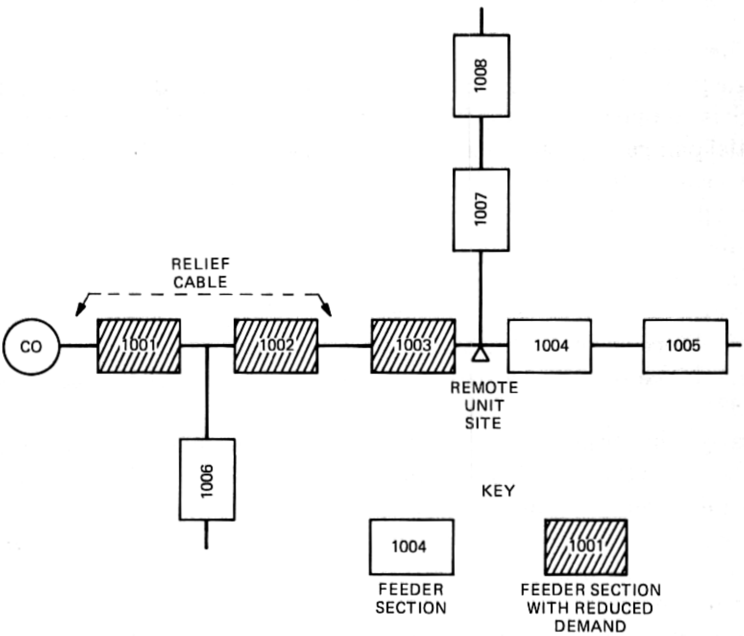

Fig. 6—Relief project deferral.

(*i*) Place an additional pair gain system
(*ii*) Place relief cable and remove all pair gain systems.

Moreover, once the project is done, there are no further decisions until the next project comes along. Thus, the basic question is how many systems should be placed, or, equivalently, how long should the project be deferred.

5.3 PWAC model

As in Section IV, it is assumed that a sequence of N pair gain systems are placed at times $\tau_1, \tau_2, \ldots, \tau_N$ and, at time τ_{N+1}, the N systems are removed and the project (cable) is placed. It is necessary, however, to generalize the definition of the τ_n to be the time at which the pair demand just exceeds the capacity of the cable network augmented by $n - 1$ pair gain systems. For τ_n defined this way, the pair gain annual charge function $\gamma(t,N)$ is still given by eq. (17). If the annual charge for the relief project is A, then the total PWAC is given by

$$\text{PWAC} = \int_0^{\tau_{N+1}} \gamma(t,N) e^{-rt} dt + \frac{1}{r} A e^{-r\tau_{N+1}} + C_F e^{-rT_F} \tag{21}$$

where T_F is the time at which the demand exceeds the capacity of the relief project and C_F is the PWAC for all future relief.

It is assumed that, for the optimal (minimum PWAC) value of N, $\tau_{N+1} < T_F$ and that both C_F and T_F do not depend on N. This assumption is reasonable for small N. The consequences of relaxing this assumption are discussed in Section 5.5.

The τ_n are determined by the demand/facility relationships in the feeder sections spanned by the relief project. Let the demand in the kth feeder section be given by:*

$$d_k(t) = d_k(0) + g_k t \tag{22}$$

If $n - 1$ pair gain systems are in place, and each realizes a pair gain of η, then the demand in each feeder section is reduced by $(n - 1)\eta$. Therefore, if the kth feeder section contains S_k pairs, it will exhaust at time τ_n^k where

$$d_k(0) + g_k \tau_n^k - (n - 1)\eta = S_k$$

or

$$\tau_n^k = \frac{S_k + (n - 1)\eta - d_k(0)}{g_k} \tag{23}$$

Since something must be done as soon as any feeder section exhausts, it follows that

* The linear demand assumption is not necessary, but it simplifies the discussion.

$$\tau_n = \min_k \tau_n^k \tag{24}$$

The form of eq. (24) may be simplified somewhat if it is assumed that one section, say section cr, always exhausts first. Then, if the time scale is chosen such that $S_{cr} = d_{cr}(0)$, τ_n is given by

$$\tau_n = (n-1)\eta/g_{cr} \tag{25}$$

where g_{cr} is the growth in the section which is exhausting.

5.4 Optimal policy

The optimal number of pair gain systems, N, is chosen so as to minimize the partial PWAC, P_N, given by the first two terms of eq. (21) as

$$P_N = \int_0^{\tau_{N+1}} \gamma(t,N)e^{-rt}dt + \frac{1}{r}Ae^{-r\tau_{N+1}} \tag{26}$$

The minimization can be carried out by enumeration as in Section IV. If eq. (25) holds, however, a set of curves can be generated from which the optimal N can be determined given A and g_{cr}.

Since a more expensive project can be economically deferred for a longer period of time, it follows that the optimal N increases with A. The Nth breakpoint, A_N, which is that value of A at which the optimal number of systems changes from $N-1$ to N, is given by

$$\int_0^{\tau_{N+1}} \gamma(t,N)e^{-rt}dt + \frac{1}{r}A_N e^{-r\tau_{N+1}} = \int_0^{\tau_N} \gamma(t,N-1)e^{-rt}dt + \frac{1}{r}A_N e^{-r\tau_N}$$

or

$$A_N = r(e^{-r\tau_N} - e^{-r\tau_{N+1}})^{-1} \times \left[\int_0^{\tau_{N+1}} \gamma(t,N)e^{-rt}dt - \int_0^{\tau_N} \gamma(t,N-1)e^{-rt}dt\right] \tag{27}$$

A family of curves for A_N as a function of N and g_{cr} can be generated from eq. (27).

Equation (27) can be greatly simplified if it is further assumed that $m(T) = m$. In this case, A_N is given by

$$A_N = r[e^{-r(N-1)\eta/g_{cr}} - e^{-rN\eta/g_{cr}}] \times \left\{\sum_{k=1}^{N} \frac{a'}{r}[e^{-r(k-1)\eta/g_{cr}} - e^{-rN\eta/g_{cr}}] + NRe^{-rN\eta/g_{cr}} - \sum_{k=1}^{N-1} \frac{a'}{r}[e^{-r(k-1)\eta/g_{cr}} - e^{-r(N-1)\eta/g_{cr}}] - (N-1)Re^{-r(N-1)\eta/g_{cr}}\right\} \tag{28}$$

where $a' = a + mI$. After some manipulation, eq. (28) becomes

$$A_N = (a' - rR)N + \frac{r}{1 - e^{-r\eta/g_{cr}}}R \tag{29}$$

Figure 7 is a sample plot of the A_N as a function of g_{cr} for a pair gain system characterized by

$$a' = \$2500$$

$$R = \$500$$

$$\eta = 50$$

with $r = 0.07$. These curves can be used to determine the optimal N for a given project. For example, if the critical section growth is 150 pairs per year and the annual charge for the relief project is \$10K, then 3 systems should be placed.

Note that the curves are linear. This is because, except for small g_{cr}, the exponential in eq. (29) can be replaced by its linear approximation, i.e.,

$$A_N \approx (a' - rR)N + \frac{R}{\eta}g_{cr} \tag{30}$$

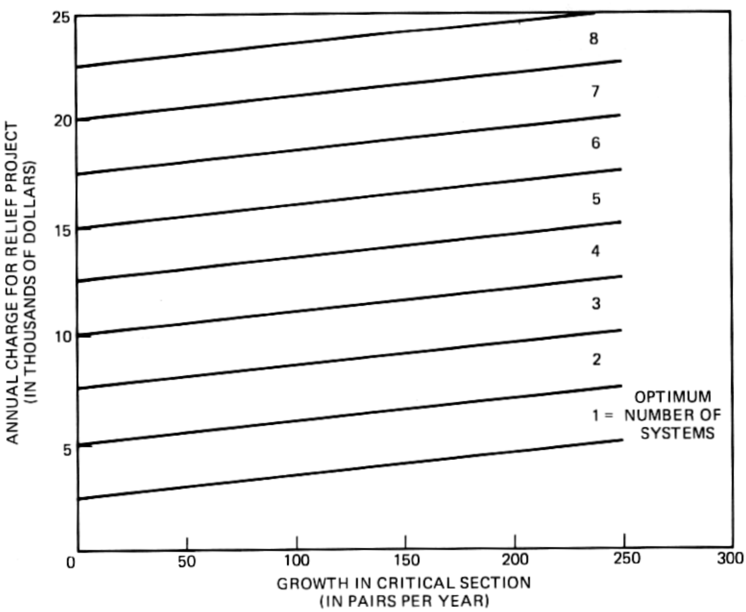

Fig. 7—Design chart for relief project deferral.

5.5 Extended deferrals

It has been assumed that the relief project is designed with no consideration toward the use of pair gain. The results of Sections II–IV, however, indicate that cable sizes are increased when pair gain is employed. Thus, when a project is deferred, that project, and all future projects, should be resized. For short (about one year or less) deferrals, the resizing is not very significant. If a project is deferred for more than one or two years, however, it should be redesigned.

When a deferred project is revised, larger cables will generally be called for [see discussion following eq. (6)] and the project annual charge will, therefore, increase. For this reason deferral of the revised project with additional pair gain systems should be considered. This repeated deferral/revision procedure is basically an iterative solution to a combined relief sizing and timing problem. It is a practical approach, however, and could be implemented by the operating telephone companies.

VI. MATHEMATICAL PROGRAMMING APPROACHES

In this section, the application of mathematical programming techniques to the pair gain/cable network capacity expansion program is discussed. These more powerful techniques can be applied to solve the important problem of *where* to place pair gain systems.

Although the methods developed in Sections II–V do not address the pair gain location problem, their importance in developing a more sophisticated approach should not be overlooked. In particular, they indicate the kinds of solutions (e.g., permanent pair gain, temporary pair gain) which may be obtained and the general conditions which favor a particular solution. These insights suggest simplifications which lead to tractable computer algorithms such as those described in this section.

The mathematical programming algorithms are described only briefly here. The reader is referred to Refs. 5 and 6 for more detail.

6.1 Permanent lumped pair gain

If temporary pair gain is precluded, the optimization problem becomes somewhat simpler. Whenever a shortage occurs in the network, either a cable or a pair gain system must be placed. The problem is further simplified if it is assumed that the cable size and the location of the pair gain system are determined on the basis of the pair gain system configuration at the time of the shortage and the projected demand. Under this assumption, each decision point (shortage) has only two alternatives: place a cable of a given size in the short section or install a pair gain system at a given location. Thus, the problem becomes a search of a binary decision tree.

The Long Feeder Route Analysis Program (LFRAP)[5] solves the above problem by a mathematical programming algorithm known as branch and bound. The LFRAP user provides data describing the network topology, existing cable facilities, and projected demand. The LFRAP output lists the sequence of placements of cable and pair gain systems which has the minimum PWAC over all sequences spanned by the binary decision tree.

Because of the restrictive assumptions on sizing cables and locating pair gain systems, the LFRAP solution is not truly optimal. The solutions have been shown to be quite good, however, and the restrictions are necessary in order to limit the computational requirements of the program.

6.2 Deferral of presized cables

Another way to simplify the problem is to assume that cable sizes have been predetermined and treat them as constant quantities. If, in addition, a discrete time scale is adopted, it becomes feasible to consider both installation and removal of pair gain systems.

Consider a one-year interval. If the facilities (cable and pair gain) in place at the beginning of the year do not meet the demand at the end of the year, shortages will occur. These shortages must be satisfied by some combination of cable placement and reconfiguration of the pair gain systems (installation, removal, relocation). The optimal pair gain configuration is the one which minimizes the total charge for the year including

(*i*) Annual charges for additional cable,
(*ii*) Annual charges for pair gain systems, and
(*iii*) Installation and removal charges for pair gain systems.

The optimal configuration can be determined by branch and bound. Each node in the decision tree corresponds to a candidate location for one or more pair gain systems. The decision to be made is how many systems should be in place at that node during the one year interval. The sequence of decisions determines the pair gain configuration, which, in turn, determines the total charge for pair gain and undeferred cable.

This formulation of the problem has been implemented as an experimental computer program.[6] The program obtains cable size data from the Exchange Feeder Route Analysis Program (EFRAP)[7] and computes the optimal configuration for each year of a prescribed study period. The sequence of configurations corresponds to a sequence of pair gain system installations and removals, at various locations, interleaved with cable placements. If any pair gain systems are installed, then one or more cables are deferred, resulting in a PWAC savings.

6.3 Comments

Compared to the approaches of Sections II through V, mathematical programming is very powerful. Indeed, it is the only approach which addresses the network aspect of the problem. In some cases, such as when right-of-way limitations drastically reduce the number of potential pair gain system locations, the network question is academic. Thus, a sophisticated program is not always justified. However, in complicated networks, where the planning engineer is free to choose from many alternatives, the impact of pair gain system location is very difficult to judge. It appears, therefore, that both programs such as LFRAP and charts like Fig. 7 have a place in pair gain application planning.

VII. MODELS FOR FUTURE STUDY

This paper will be concluded with a glimpse of some recently proposed models for pair gain applications which do not fit the pattern established earlier. Specifically, the application of pair gain to provide temporary second line demand and the use of pair gain to avoid facility modifications will be discussed.

7.1 Provision of second lines

In residential areas, second lines are commonly requested to provide service for teen-aged children, for burglar alarms, or simply for convenience. Because of potential second line demand, the distribution cable network[4] is sized to provide a minimum of two pairs per ultimate living unit. If second line penetration is low, many of these pairs will be unused, but they must still be provided since, at any given time, any subscriber may request a second line.

On the other hand, second line service can be provided by a single channel pair gain system. That is, the distribution network can be sized to provide one pair per ultimate living unit, and the second line can be provided by installing a single channel system at the subscriber's premises.

A rough calculation of the economics of providing second lines in this manner is fairly simple. It will be assumed that, in a given wire center serving area, the second line demand is constant over time and all second lines are provided by single channel pair gain. Under this assumption, the central office units can be treated as permanent facilities. The remote units will move from house to house in response to the second line demand.

Let p be the penetration of second line demand, i.e., in an area containing H living units, there will be pH second lines. Also, let T be the average duration of second line service at a given location. The total annual charge for pair gain, which is the total annual charge for providing second line service, consists of a pair gain equipment charge and an annual turnover charge. For each single channel system, the annual charge, a, includes the annual charge for both the central office and remote equipment and the levelized cost of installing the central office unit. The turnover charge for each system follows from the assumption that the remote unit is removed from one location and reinstalled elsewhere every T years. Thus, the turnover charge per system is approximately $(I + R)/T$ where I and R are the installation and removal costs *for the remote unit.* The total annual charge per system is, therefore,

$$a + (I + R)/T$$

and, assuming p systems per living unit, the pair gain annual charge per living unit γ is

$$\gamma = p[a + (I + R)/T] \tag{31}$$

In order to judge the economics of the pair gain second line policy, one would compare eq. (31) with the annual charge per living unit for providing the same service with cable. The annual charge for the all-cable policy includes the marginal cost of the second distribution pair (which must be provided for *every* living unit) and the cost of additional feeder cable pairs.

Equation (31) indicates that the cost of the pair gain policy is directly proportional to the second line penetration. Thus, the policy is most likely to prove in areas of low second line penetration. Also, it should be noted that the turnover cost may be neglected in some cases. If, under the all-cable policy, it is necessary to install or even just terminate a second service wire, the net installation charge for the remote unit (i.e., remote unit installation charge minus drop installation charge) may be zero or negative. If I and R can be neglected, it is not necessary to estimate T in order to evaluate the economics of the policy.

7.2 Avoidance of facility modifications

In a congested network, inward service orders are often blocked, i.e., a facility modification must be made in order to provide service. As discussed in Ref. 8, a facility modification is a minor rearrangement of the network which entails costs in addition to the normal cost of providing service. Rather than disturb the network, however, one may elect to provide the service with a single channel pair gain system. In this section, a cost model for this kind of pair gain application will be outlined. The analysis will draw upon results derived in Ref. 8.

Consider a geographic area administered as multiple outside plant[4] and served by X feeder pairs. Assume that demand in the area is growing linearly at rate g and let $S(t)$ be the number of spare feeder pairs [$S(0) = S$]. The probability that an inward order at time t is blocked is given approximately by[8]

$$\Pr\{\text{BLK}\} = e^{-kS(t)/X} \tag{32}$$

where k is the apparent access group size (i.e., serving terminal size). Assume that, whenever an inward order is blocked, it will be served by a single channel pair gain system. Then, if $n(t)$ is the number of units of pair gain in the area $[n(0) = 0]$, the rate of increase of $n(t)$ is

$$\dot{n} = ge^{-kS/X} \tag{33}$$

Since only unblocked inward orders use up spare feeder pairs, the rate of change in $S(t)$ is

$$\dot{S} = -g(1 - e^{-kS/X}) \tag{34}$$

Note that both $S(t)$ and $n(t)$ are modeled as continuous deterministic variables governed by a pair of differential equations. A more rigorous, but much more difficult approach would be to model them as discrete valued random processes driven by random arrivals and departures.

Equations (33) and (34) can be solved in closed form and the solutions are

$$S(t) = \frac{X}{k} \ln\,[1 + (e^{kS_o/X} - 1)e^{-kgt/X}] \tag{35}$$

and

$$n(t) = \frac{X}{k} \ln\,[1 + e^{-kS_o/X}(e^{kgt/X} - 1)] \tag{36}$$

If the only charge for the single channel pair gain system is the annual charge a, the total annual charge for pair gain will be $an(t)$.

If pair gain is not used, spare feeder pairs are used up at a rate of g so that $S(t) = S_o - gt$ and blockages occur at a rate of $\lambda e^{-kS/X}$ where λ is the inward order rate. Thus, the annual charge for the all-cable alternative is

$$\lambda C_{\mathrm{BLK}} e^{-k(S_o - gt)/X}$$

where C_{BLK} is the average cost of a facility modification, until the spare is used up or relief is provided.

Under the pair gain policy, the spare feeder pairs are never exhausted. Rather, as time goes on, a larger and larger proportion of subscribers are served with single channel carrier. However, under either the cable or pair gain policy, the annual charge eventually increases to a point where relief cable is justified.

7.2.1 Example

Consider a pair group characterized by the following parameters:

$$X = 2000 \text{ pairs}$$

$$k = 20 \text{ terminations}$$

$$S_o = 500 \text{ pairs}$$

$$g = 100 \text{ pairs/yr}$$

$$\lambda = 500 \text{ orders/yr}$$

Assume that C_B = \$100 and a = \$50/yr. Figure 8 is a plot of the annual charge function for both the cable and pair gain approach. In this case, the pair gain cost is uniformly lower.

7.3 Comment

The applications discussed in Section 7.1 and 7.2 have been implemented in a few areas of the Bell System. However, because of the random nature of pair gain installation and removals in these applications, conventional analysis methods like those of Sections II through VI cannot be applied. The material in this section represents an initial attempt to establish a mathematical foundation for these kinds of applications.

VIII. SUMMARY

This paper has presented a mathematical theory of pair gain appli-

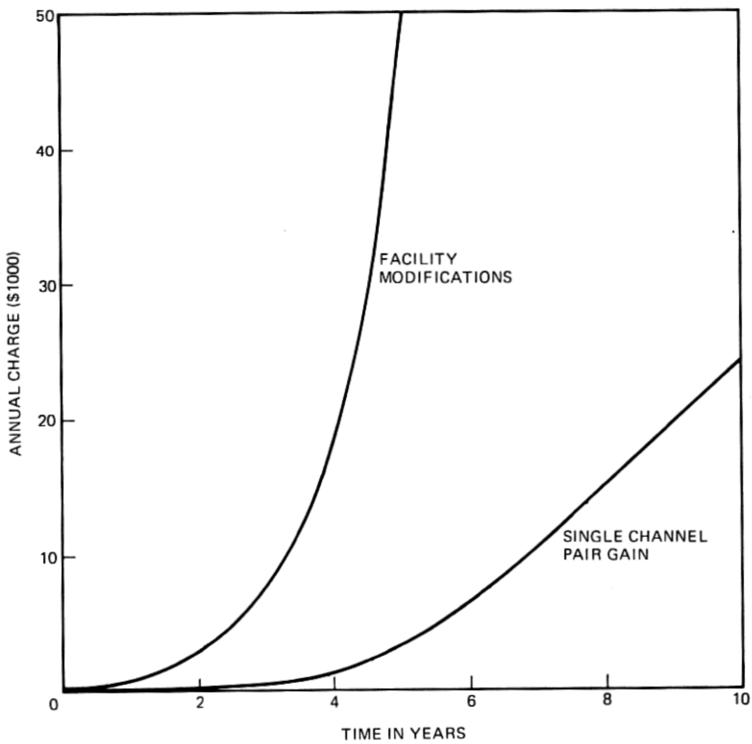

Fig. 8—Annual charge comparison of pair gain and facility modifications.

cation. The approach has been to adopt a framework based on the PWAC criterion and use it to develop both useful concepts and methods. These concepts are fairly consistent with the way pair gain has been traditionally applied. However, there are some important differences, particularly with regard to temporary versus permanent applications.

It has been shown that, under the conditions assumed by the model, the optimal strategy will be either all cable, permanent pair gain, or temporary pair gain (deferred cable). Although each of these strategies has been applied to real networks, it has not always been clear which one is best for a given situation. The operating companies now have guidelines and computer programs which will help them choose the lowest cost alternative. These pair gain planning tools have been developed as the result of mathematical modeling efforts such as those presented in this paper.

New applications, such as provision of second lines and avoidance of facility modifications, have been identified and studied. Although these applications are not very prevalent today, their importance will undoubtedly increase as improving technology continues to reduce the cost of pair gain relative to cable. Mathematical models like those developed here can serve as a guide for introducing new applications and developing future pair gain systems.

IX. ACKNOWLEDGMENTS

Many people have contributed to the results presented in this paper and their contributions are gratefully acknowledged. The author would like especially to thank R. G. Hinderliter, who suggested this area of work, N. G. Long, who encouraged the preparation of this paper, and the members of the Loop Transmission Engineering Center and the Loop Transmission System Laboratory who have participated at one time or another in the analysis of pair gain applications.

REFERENCES

1. J. Freidenfelds, "A Simple Model for Studying Feeder Capacity Expansion," B.S.T.J., this issue.
2. Technical Publication Dept., *Engineering and Operations in the Bell System,* Bell Laboratories, 1977.
3. Construction Plans Dept., AT&T, *Engineering Economy.* New York: McGraw-Hill, 1977.
4. N. G. Long, "Loop Plant Modeling: Overview," B.S.TJ., this issue.
5. B. S. Abrams and R. B. Hirsch, "Computer Aids for Rural Route Planning," Bell Laboratories Record, September 1974.
6. W. L. G. Koontz, "Optimal Temporary Deferral of Reinforcements in an Exchange Cable Network," International Symposium on Subscriber Loops and Services, Ottawa, Canada, May 1974.
7. J. Albers and C. D. McLaughlin, "Exchange Feeder Route Analysis Program—An Application of Branch and Bound. Techniques to Economic Cable Sizing," International Symposium on Subscriber Loops and Services, R-3, Ottawa, Canada, May 1974.
8. W. L. G. Koontz, "An Approach to Modeling Operating Costs in the Loop Network," B.S.T.J., this issue.