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Loop Plant Modeling:

A Simple Model for Studying Feeder Capacity Expansion

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Using a very simple model of feeder cable sizing facilitates the discussion of many interesting questions: How sensitive are sizing decisions to various items of data? How does the need for expensive conduit affect cable size? What kind of economy-of-scale can be expected from consolidating routes into larger backbone configurations? What effect might randomness or uncertainty in the demand forecast have on sizing? How might sizing be affected by limits on available capital? The simple sizing model discussed assumes linear growth of demand over an infinite horizon in an isolated feeder section. The cost of cable or conduit is assumed to be composed of a fixed charge plus a cost per unit of capacity added.

I. INTRODUCTION

As described by N. G. Long[1] (this issue), the *feeder* portion of the overall loop plant consists of cables, conduit, and various other hardware. It provides communication paths, usually consisting of a pair of copper wires, between the central office and the distribution plant. Additional cables, and perhaps conduit to house those cables, are added to the feeder over time as existing spare is depleted by growth in demand. Optimally sizing such additional cables and conduit is an investment decision problem known as a *capacity expansion* problem.[2]

A sophisticated computer program, called EFRAP,[3] has been developed for solving a more general version of the feeder capacity expansion problem than we shall consider here. Our aim in this paper is to develop a manageable "analytic" model of feeder sizing. While we thus ignore some aspects of the problem, such as demand in more than one gauge, which are included in the more sophisticated approach, we can more easily include others, such as the use of temporary pair gain systems (see

W. L. G. Koontz,[4] this issue). Furthermore, a simpler model is easier to understand.

For our basic model we assume that demand for additional feeder pairs through some section of plant is increasing linearly at the rate g over an infinite horizon. The cost to install and maintain forever a cable of x pairs is assumed to be expressible as $a + bx$ dollars per year per foot. In Section II we make some observations on sensitivity and on economies of scale based on this model.

In the following section we study the problem from a *cost of the future,* or backward dynamic programming viewpoint. This makes it possible to analyze some complications such as conduit, partial conduit, removal of existing sheaths, and the use of temporary pair-gain systems.

In Section IV we show that when we allow nonlinear demand in the near term, the dynamic programming formulation becomes more "computational" in nature. Also, we briefly consider a generalization from the linear deterministic demand to a stochastic demand process with stationary independent increments. We show that, except when the expected growth is very low compared to its standard deviation, we essentially get the same results as with the deterministic model.

In Section V we show that when the current cable relief budget is limited, we might still be able to calculate cable sizes on a case-by-case basis provided we can estimate an appropriate Lagrange multiplier value.

Finally, in the last section we mention some other applications of the simple feeder sizing model.

II. THE BASIC MODEL

We focus our attention on a single link of the feeder network, called a *feeder section* (see Long,[1] this issue). We assume that the demand for additional feeder pairs at time t in the future is given by $D(t) = gt$. In general, the demand may not always be homogeneous—customers far from the central office being routed through this section may need a coarser gauge of wire. This more general case is treated in Ref. 3, but not here. We also assume that $D(t)$ includes a *fill-at-relief* margin to account for the fact that additional cable is placed—i.e., relief is provided—well before all pairs are actually in use (e.g., see Koontz,[5] this issue).

We model the cost of a cable of size x, that is, one having x pairs, as $a + bx$ dollars per year per foot. This cost is an annual equivalent to the total present worth cost of supplying x pairs, taking into account the costs of material, maintenance, return on capital, and taxes over the life of the cable. In most studies, the details of these costs can be relegated to a side calculation in which an *annual charge factor* is developed relating equivalent annual costs to installed first cost for various classes of plant (see the Appendix). Once a cable is added, we assume, for purposes of calculating costs, that it will be "maintained forever"; i.e., that it will be replaced at the end of its life by equipment of the same cost. This is done mostly as a matter of mathematical convenience. Cables tend to have very long lives (e.g., 45 years) and it makes little difference on a present worth basis precisely what is assumed.

Under these circumstances, we will clearly use equal-sized cables, x^*, which minimize the present worth cost

$$\mathrm{PW} = \sum_{j=0}^{\infty} \left(\frac{a + bx}{r}\right) e^{-rjx/g} = \frac{(a + bx)/r}{1 - e^{-rx/g}} \tag{1}$$

where r is the discounting rate, and we have assumed that $a + bx$ is a continuous annuity, compounded continuously. Figure 1 plots a sample PW versus x. If we wish to consider only those discrete sizes which are actually available, the minimum can be found by trying several of them. We will show shortly, however, that only small errors result from small deviations in the size. For the rest of this paper it will be convenient to assume a continuum of sizes. We can easily show that PW is a convex function of x, and so, setting its derivative to zero yields an expression for the minimizing value, x^*,

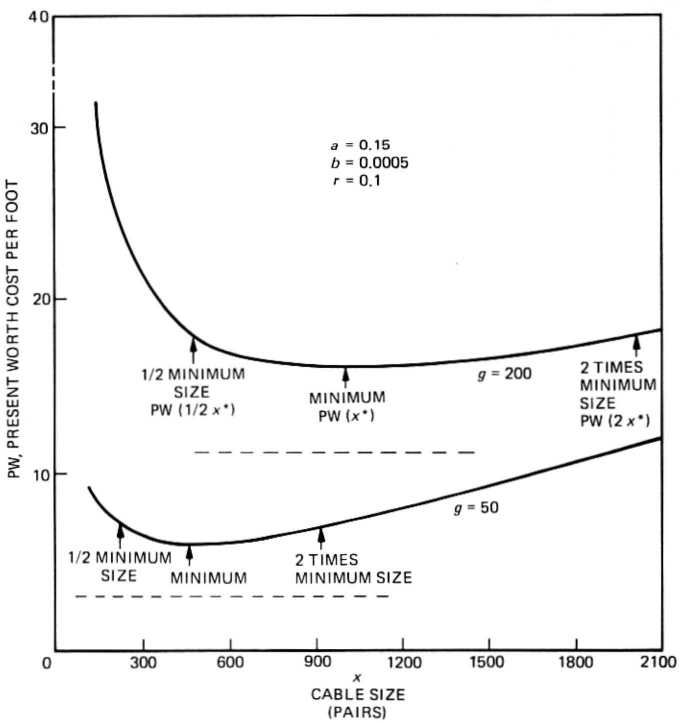

Fig. 1—Present worth cost versus size.

$$e^{rx^*/g} - \frac{rx^*}{g} - 1 = \frac{ar}{bg}$$

A quick approximation which is good for situations with short relief intervals $t^* = x^*/g$ is found by using a Taylor approximation for the exponential:

$$x^* \approx \sqrt{2\left(\frac{a}{b}\right)\left(\frac{g}{r}\right)} \tag{2}$$

Figure 2 shows x^* versus g/r for several values of the a/b ratio, with the approximation displayed for $a/b = 300$.

2.1 Sensitivity to parameters

The sizing curves tend to be shallow. Figure 1 shows that even with size varying from ½ to 2 times the optimum, the present worth varies by about 10 percent for the case of 200 pairs per year growth and 15 percent for the 50 pairs per year case. This point is even stronger if we consider that according to approximation (2), our estimate of growth rate would have to be in error by about a *factor of four* in order to make that much error in size!

Having made such a sweeping statement, we caution the reader that *percent* of present worth may not always be an appropriate measure of

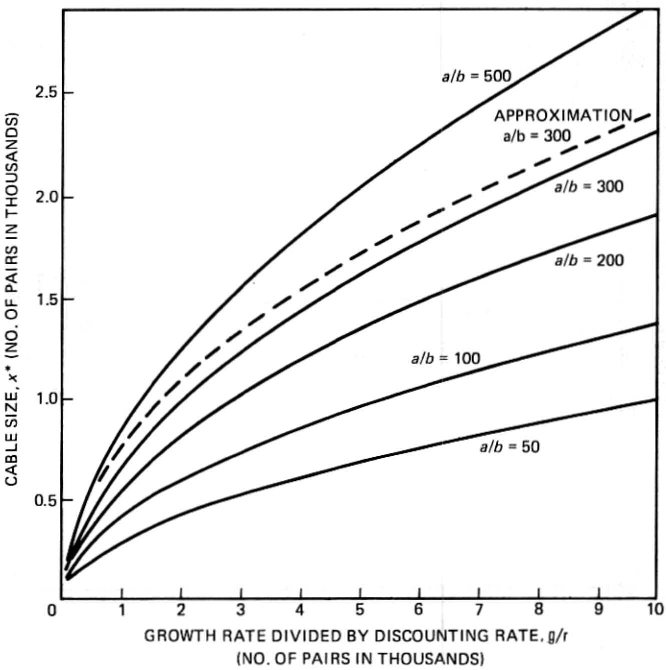

Fig. 2—Economic cable size.

the penalty for incorrect decisions. It may be more appropriate, for example, to first subtract obviously "uncontrollable" components from the total. One such component is the "b-cost" of facilities in service. That is, even if there were no "a" component in the $a + bx$ cost, we would install capacity continuously and *still* incur a present worth cost of

$$\mathrm{PW}_b = \int_0^{\infty} bgte^{-rt} = \frac{bg}{r^2}$$

Also, we are assuming that there is an initial shortage which implies that we *must* incur at least one "a-cost," $\mathrm{PW}_a = a/r$, at time zero. For the examples of Fig. 1, we have

$$g = 200: \qquad \mathrm{PW}_a + \mathrm{PW}_b = 11.5$$

$$g = 50: \qquad \mathrm{PW}_a + \mathrm{PW}_b = 3.5$$

Dashed lines are shown in Fig. 1 at these levels. If these amounts are first deducted from present worth, the percentage present worth penalty for doubling or halving the optimal size jumps to about 33 percent.

2.2 Economies of scale

The reason we *have* a cable sizing problem is because of economies of scale in the cost of each cable. Here we have expressed that cost as $a + bx$. In general, any cost function which exhibits decreasing average cost per unit as the number of units increases is said to exhibit scale economies. We would like to buy more at once to take advantage of the lower unit cost but must balance that advantage against the penalty for having to tie up more capital sooner.

In a broader sense, we also speak of economies of scale as referring to the advantages of bigness. In the feeder relief problem, we might consider the potential advantages of using one large route in place of two parallel small ones. Our basic model can provide some insight. Figure 3 plots present worth cost versus growth rate (for the same cost parameters as in Fig. 1). The upper curve assumes that a 1000-pair cable will be used regardless of growth rate, while the lower curve assumes that an optimally sized cable will be used at each growth rate. It is straightforward to verify that for either curve the present worth cost per unit of growth decreases as the amount of growth served increases. For example, if we combine two parallel routes with a growth rate of 200 pairs per year into one with 400 pairs per year, we would save

$$\mathrm{PW}_1 = 2(16.52) - 28.78 = \$4.26 \text{ per foot}$$

using optimally sized cables, or

$$\mathrm{PW}_2 = 2(16.52) - 29.39 = \$3.65 \text{ per foot}$$

even if we had to use the 1000-pair cables in the combined route. That

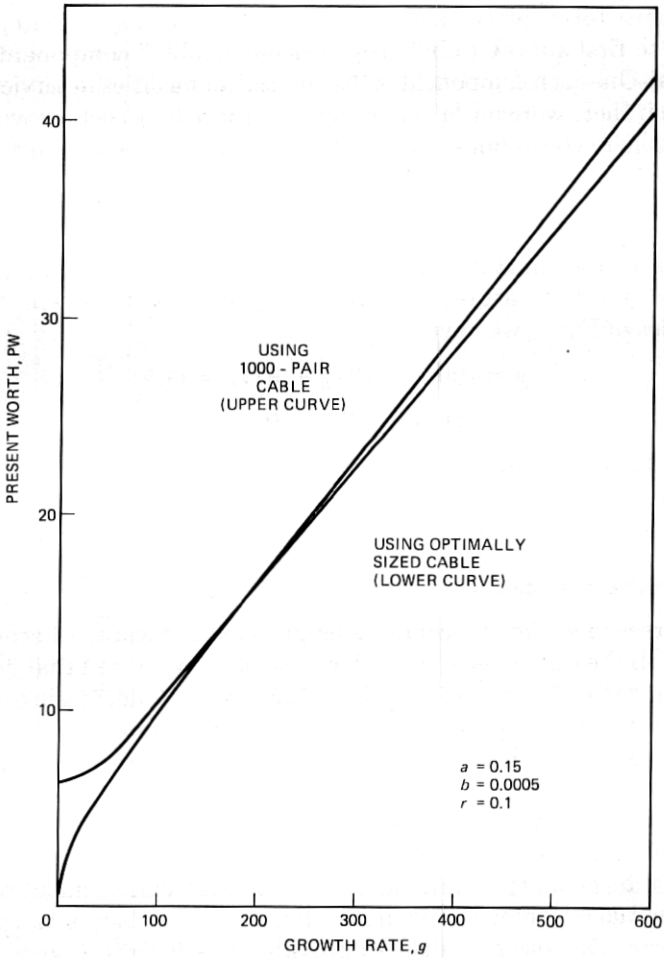

Fig. 3.—Economy of scale.

is, even though the 1000-pair cable is optimal for $g = 200$, and is non-optimal for $g = 400$, we still save by combining the routes. Intuitively, we can think of the savings in combining the routes as attributable in part to eliminating one a-cost at time zero (present worth of $0.15/0.1 =$ \$1.5 per foot), in part to utilizing excess capacity faster, and in part to being able to take advantage of a larger, lower unit cost cable.

III. CONDUIT AND OTHER COSTS

Here we expand the basic model to consider the effects of various complications, such as impending conduit shortage, and extra buried cable costs. We still retain the assumption of linear growth in a single gauge. These extensions are based on the *cost-of-the-future* formulation.

3.1 Cost of the future formulation[6]

Instead of starting with equal cable sizes for all future relief, suppose we assume that relief cables will be sized $x_0, x_1, x_2, \ldots$ for a total present worth cost of

$$\mathrm{PW} = \sum_{i=0}^{\infty} \frac{a + bx_i}{r} e^{-rt_i} \tag{3}$$

where $t_0 = 0$ and $t_i = \Sigma_{j=0}^{i-1} x_j/g$ for $i > 0$. We can also write (3) as

$$\mathrm{PW} = \frac{a + bx_0}{r} + \mathrm{PW}_F e^{-rx_0/g} \tag{4}$$

with

$$\mathrm{PW}_F \equiv \sum_{i=1}^{\infty} \frac{a + bx_i}{r} e^{-rt'_i} \tag{5}$$

where $t'_1 = 0$ and $t'_i = \Sigma_{j=0}^{i-1} x_j/g$ for $i > 1$. In this form, we note that PW_F, the *cost of the future,* is independent of x_0, the size of the initial cable. Given a sequence of cables for all but the initial shortage, or just its cost, PW_F, we can use (4) to find the optimal size of the initial cable. With a continuum of sizes, x_0, available, the minimizing size is the one for which the derivative of (4) is zero, treating PW_F as a constant:

$$x_0^* = \frac{g}{r} \ln \frac{r^2 \mathrm{PW}_F}{bg} \tag{6}$$

To actually minimize (4) over the entire sequence of relief cables, we must clearly use the minimal PW_F; but that implies minimizing (5) which is mathematically identical to (3). Thus we have a recursive, or backward dynamic programming formulation. It can be shown that, starting with any positive value for PW_F, if we successively use (6) (truncating any negative sizes to zero) to get improved estimates of size and (4) to get improved estimates of PW_F, we converge to the optimal solution. A sample computation in the next section (Table Ia, first three columns) illustrates.

3.2 Including conduit

Suppose that placement of each cable, regardless of its size, uses up a conduit duct and that when all ducts are used up a new conduit system must be built at a cost of $\alpha + \beta N$ dollars, annual charge, per foot for N ducts.

A slight generalization of the formulation of Section 3.1 gives us a handy algorithm. Let PW_i be the total present worth cost of placing all cable and conduit starting from a time when there are i spare conduit ducts and no spare cable available; and let x_i be the corresponding optimal cable size. Note that these cables are numbered backward in time unlike those of Section 3.1. Assuming that N ducts of conduit will be installed at a time, we can write

$$\mathrm{PW}_i = \min_{x_i} \left\{ \frac{a + bx_i}{r} + \mathrm{PW}_{i-1} e^{-rx_i/g} \right\} \tag{7}$$

for $i = 1, \ldots, N$, and

$$\mathrm{PW}_0 = \frac{\alpha + \beta N}{r} + \mathrm{PW}_N \tag{8}$$

Of course, each minimizing x_i can be very quickly found by using the appropriate PW_F in (6). It can be easily shown that the PW_i of (7), and hence the optimal x_i, form a monotone sequence with x_i approaching the size minimizing (1) and PW_i the corresponding PW. In view of (8), the sizes must decrease as more spare cable spaces are available. That is, as a conduit system is filled, it becomes optimal to install larger cables to defer the impending cost of building another conduit system.

We can also find the optimal *conduit size* if we replace (8) with

$$\mathrm{PW}_0 = \min_{N} \left\{ \frac{\alpha + \beta N}{r} + \mathrm{PW}_N \right\}$$

It turns out that the term in brackets is unimodal in N, so that we can stop at the first local minimum. Table I shows a sample calculation using discounting rate, $r = 0.1$; cable cost, $a + bx = 0.15 + 0.0005x$; conduit cost, $\alpha + \beta N = 1.0 + 0.1N$ dollars per year per foot; and $g = 200$ pairs/year. Note the convergence of x_i^* and PW_i^* to the solutions of Section II in the first major iteration. Of course, we could have stopped at $i = 7$ since we had found the minimum with that calculation. Note also the rather rapid convergence even with an initial guess of $\mathrm{PW}_0 = 1000$ compared to the optimal $\mathrm{PW}_0 = 31.3$.

3.3 Buried cable, aerial cable, partial conduit

We can extend the above analysis to various situations such as the availability of spaces for direct burial of cable or pole-line spaces for additional aerial cable. We start with an estimate of PW_F, the present worth cost of all future relief after the initial spaces are used up. Since size depends on the logarithm of PW_F in (6), our decisions are not usually very sensitive to this value; and so, we might use PW_0 as calculated in Section 3.2, for example. We then size cables for the initially available spaces, starting with the last space, using (7).

The flexibility of this procedure is illustrated by considering the following *partial conduit* problem. Suppose we can install a buried cable plus a single conduit duct (costing an extra \$0.05 per foot, annual charge) at the current shortage. Then at the time of the next shortage, we must build manholes costing \$0.02 per foot annual charge and place a cable in the duct. From the following shortage onward we will build conduit and place cable as in Section 3.2, $\mathrm{PW}_F = 31.3$ less the cost of the man-

Table I — Iterations for cable and conduit size

Ia: Using initial guess of $PW_0 = 1000$

i (Number of ducts)	x_i^* (6)	PW_i^* (7)	$\frac{\alpha + \beta i}{r} + PW_i^*$ (8)	
1	3000	239.6	250.6	(Maximum size cable available is assumed to be 3000 pairs)
2	3000	70.0	82.0	
3	3000	32.1	45.1	
4	2333	23.2	37.2	
5	1680	19.9	34.9	
6	1376	18.4	34.4	minimum
7	1218	17.6	34.6	
8	1129	17.1	35.1	
⋮	⋮	⋮	⋮	
∞	1004	16.5	∞	

Ib: Using $PW_0 = 34.4$ from Table Ia

i	x_i^*	PW_i^*	$\frac{\alpha + \beta i}{r} + PW_i^*$	
1	2295	23.0	34.0	
2	1664	19.8	31.8	
3	1368	18.3	31.3	minimum
4	1213	17.6	31.6	

Ic: Using $PW_0 = 31.3$ from Table Ib

i	x_i^*	PW_i^*	$\frac{\alpha + \beta i}{r} + PW_i^*$	
1	2282	22.9	33.9	
2	1658	19.8	31.8	
3	1365	18.3	31.3	minimum
4	1212	17.6	31.6	

The optimal solution is to build 3-duct conduits and place cables of 1365, 1658, and 2282 pairs as shortages occur.

holes already built, or $PW_F = 31.3 - 0.2/r = \$29.3$ per foot. These charges are shown on the schematic of Fig. 4. Our solution proceeds backward, starting with x_1:

$$x_1^* = \frac{g}{r} \ln \frac{r^2 PW_F}{bg} = \frac{200}{0.1} \ln \frac{(0.1)^2(29.3)}{(0.0005)(200)} = 2150$$

$$PW_1^* = \frac{a + bx_1}{r} + PW_F e^{-rx_1^*/g} + \frac{0.2}{r} = 24.3$$

$$x_2^* = \frac{g}{r} \ln \frac{r^2(24.3)}{bg} = 1776$$

$$PW_2^* = \frac{a + bx_2^*}{r} + PW_1^* e^{-rx_2^*/g} + \frac{0.05}{r} = 20.9$$

Thus we should place a cable of 1776 pairs along with the conduit duct, and later fill the conduit duct with a cable of 2150 pairs for a total present worth cost of \$20.9 per foot. We note that this is considerably less than the \$31.3 for going directly to a conduit system.

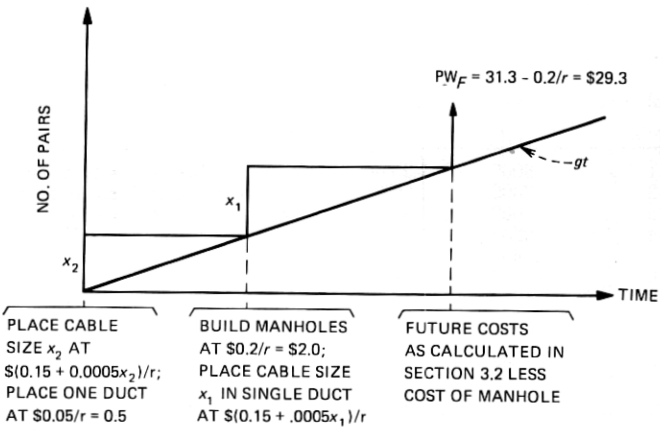

Fig. 4—A partial conduit problem.

It is interesting to ask how much of this savings is attributable to the availability of the partial conduit duct. If such a duct were not available, we might have buried a cable initially and gone directly to the conduit system when more capacity was required. The calculations for this case would be identical to some of those done earlier. In fact, the solution can be read from the $i = 1$ row in Table Ic. It is to use a 2282-pair cable for a total present worth cost of \$22.9 per foot. Thus the availability for the partial conduit duct saves about \$2.0 per foot in this example.

IV. MORE SOPHISTICATED DEMAND MODELS

Here we briefly consider two formulations with more realistic assumptions about demand. In the first, we allow demand to be some nonlinear, but still deterministic, function over the early part of a study. In the second we suppose that demand is a random process and that we wish to make decisions to minimize expected present worth cost.

4.1 Nonlinear demand in the near term

Let $D(t)$ be a nondecreasing function which represents the cumulative number of pairs required over $t = 0$ to T. Beyond T, $D(t) = D(T) + g(t - T)$. Let $\text{PW}(t)$ be the present worth cost of meeting all future demand starting from a shortage at time t. If $t > T$, we assume $\text{PW}(t) = \text{PW}_F$, independent of time. The optimal relief schedule can be found from the following dynamic program

$$\text{PW}(t) = \min_{\tau > t} \left\{ \frac{a + b[D(\tau) - D(t)]}{r} + \text{PW}(\tau) e^{-r(\tau - t)} \right\}$$

Note that we can build up the PW function by working backward from T, considering only discrete relief times (or, equivalently, discrete relief sizes).

Although it is computationally easy to solve if we keep T reasonably small, this formulation actually goes beyond our analytical model framework for this paper.

4.2 Random demand

Suppose the demand is not assumed to be known deterministically but is instead assumed to be generated by a known stochastic process. Here we only consider processes with *stationary independent increments.* Intuitively, that means the additional future demand (positive or negative) is statistically the same no matter what time or current demand level we start with. An example of such a process is random inward and outward movement of customers according to independent Poisson processes;[7] another is Brownian motion.[2]

Our development will be heuristic rather than mathematically rigorous. Let τ_x be the (random) time until we first get x more customers than we currently have (i.e., the first-passage time). With no spare pairs at $t = 0$, τ_x is the time of the next shortage if we place a cable with x pairs. We would thus like to minimize the expected value of

$$\mathrm{PW} = \frac{a + bx}{r} + \mathrm{PW}_F e^{-r\tau_x}$$

where both PW_F and τ_x are random variables. Because of the statistical independence assumption, PW_F and $e^{-r\tau_x}$ are independent and the expected value of their product is the product of their expected values

$$E[\mathrm{PW}] = \frac{a + bx}{r} + E[\mathrm{PW}_F]E[e^{-r\tau_x}] \tag{9}$$

where $E[\cdot]$ denotes expected value. The reader might recognize the factor involving τ_x as the Laplace transform of the first-passage time, evaluated at r. We can think of τ_x as the sum of x independent, identically distributed first-passage times to one more unit of demand, τ_1. The Laplace transform of the sum, τ_x, is the product of the individual Laplace transforms, thus

$$E[e^{-r\tau_x}] = (E[e^{-r\tau_1}])^x$$

Since the Laplace transform is a number between zero and one, we can define an *equivalent* (positive) *growth rate,* g_{eq}, such that

$$e^{-r/g_{\mathrm{eq}}} \equiv E[e^{-r\tau_1}]$$

Then we can rewrite (9) as

$$\overline{\mathrm{PW}} = \frac{a + bx}{r} + \overline{\mathrm{PW}}_F e^{-rx/g_{\mathrm{eq}}}$$

where the bar denotes expected value; and we have precisely the form of (4) with g_{eq} replacing g. That is, we can solve this stochastic problem exactly as we would a deterministic one if we only use the equivalent growth rate in place of the deterministic one.

To get an idea of how the equivalent growth rate relates to more familiar quantities, we have plotted g_{eq} versus g_{av} for various σ^2 in Fig. 5, where g_{av} is the expected number and σ^2 is the variance of the additional number of customers per unit time. These curves are derived in Ref. 3 for the Poisson inward/outward movement model. Their most notable feature is that unless the variance is *very* large compared to the average, the equivalent growth rate is only slightly larger than the average growth rate. Thus we conclude that randomness of this type may be ignored for most cable-sizing problems.

V. SIZING UNDER A BUDGET CONSTRAINT

What if, for some reason, we had to get by with less than the ideal overall feeder relief budget for some year? How should we modify our sizing? We model the situation as a constrained optimization. Letting i index all of the relief projects subject to the constraint, and assuming there is only one such constraint,

$$PW_{\text{total}} = \underset{\text{all } x_i\text{'s}}{\text{minimum}} \left\{ \sum_{\text{all } i} \frac{a + bx_i}{r} + PW_i e^{-rx_i/g_i} \right\} \tag{10}$$

subject to the budget constraint

$$\sum_{\text{all } i} \frac{a + bx_i}{r} \leq \beta \tag{11}$$

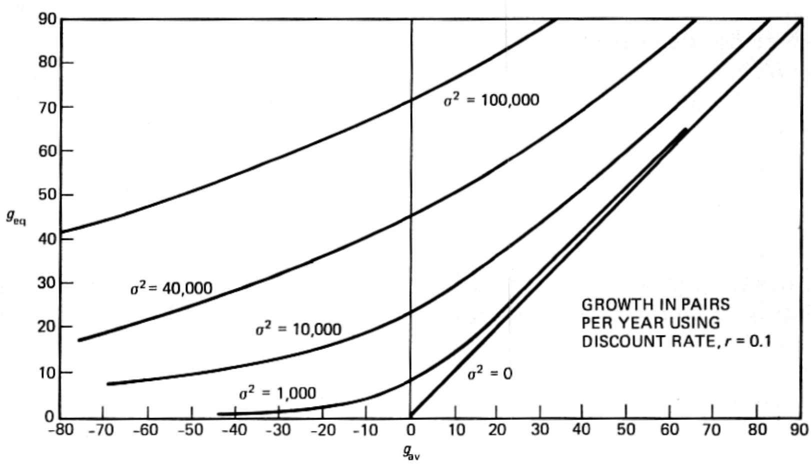

Fig. 5—Equivalent growth rate versus average growth rate.

Note that this is equivalent to constraining the total installed first cost if all annual charge factors are the same (see Section II); a slight modification would allow individual annual charge factors for each project.

Applying a Lagrange multiplier, λ, to the constraint (11) and adding it to the objective function (10),

$$\mathrm{PW_{total}} = \min_{x_i\text{'s}} \sum_i \left\{ \frac{a + bx_i}{r} + \mathrm{PW}_i e^{-rx_i/g_i} \right\} - \lambda \left(\beta - \sum_i \frac{a + bx_i}{r} \right)$$

or

$$\mathrm{PW_{total}} = \sum_i \min_{x_i} \left\{ (1 + \lambda) \frac{a + bx_i}{r} + \mathrm{PW}_i e^{-rx_i/g_i} \right\} - \lambda\beta$$

Carrying out each minimization,

$$x_i = \frac{g_i}{r} \ln \frac{r^2 PW_i}{bg_i} - \frac{g_i}{r} \ln (1 + \lambda) \tag{12}$$

Thus the optimal solution involves subtracting from the unconstrained optimum (first term), a number of pairs which is directly proportional to growth rate, and is increasing with λ. That is, if we can estimate an appropriate value of the Langrange multiplier, λ, we can continue to do our sizing on a case-by-case basis even in this constrained situation, by simply replacing (6) with (12).

Of course a general formulation for dealing with budgetary constraints would be considerably more complex. It would include the possibility of different budget constraints in different periods so that we may want to install some cables early, for example, to avoid a pinch in later years. We may also wish to consider deferring construction out of a tightly constrained period at the expense of temporarily increasing the operating costs. (Recall that additional cable is installed before spare is completely exhausted. Some of that spare margin could be used up at a cost.)[5]

VI. FURTHER OBSERVATIONS

There are many problems which can conveniently be studied with a feeder sizing model of this type. We have touched on some; others include the following.

(*i*) How is the optimal size affected by relieving earlier or later than the nominal relief time? The reader may wish to check that any excess spare or pent-up demand at relief should be subtracted from or added to, respectively, the optimal size.

(*ii*) Would it pay to remove a small existing cable and replace it with a larger one instead of building conduit right away? A straightforward calculation will show that it is often economical provided the existing cable is small enough and the cost of removal is not too large.

(*iii*) Would it pay to relieve with pair-gain systems instead of wire pairs? That question is explored by Koontz[4] in this issue.

(*iv*) What is the cost of losing a few feeder pairs, for example, because they are defective? The cost in feeder relief is essentially just the advancement of some appropriate PW_F.

Another use for this model has been to obtain approximate solutions within the more sophisticated EFRAP[3] sizing algorithm.

We have given many specific feeder sizing problems in which the cost-of-the-future approach works. Of course, it will not *always* be helpful. Generally, it will only be helpful when we can define an appropriate cost of the future which is *independent of time* and at least relatively *independent of prior decisions.*

VII. ACKNOWLEDGMENTS

Many of the ideas presented here had their genesis in unpublished work at Bell Laboratories. Most notably, the simplest model [eqs. (1) through (2)] was studied by D. R. Hortberg, and the cost-of-the-future approach was used by C. E. Warren.

APPENDIX

Levelized Equivalent Annual Cost Associated With a Capital Expenditure—Annual Charge Factors

In calculating the cost of some equipment or service it is necessary to distinguish between costs that are classified as *expense* and *capital.* Due primarily to income tax laws, the impact of a capital expenditure includes not only the immediate cash flow, but additional future financial consequences as well. The Internal Revenue Service (IRS) classifies certain expenditures as *expense;* e.g., most routine service, maintenance, and items which are used up in less than a year. These expenses are immediately deductible from income in calculating income tax. Other expenditures, primarily associated with durable equipment, are classified as *capital.* Tax deductions for capital items are spread out over their useful life (i.e., the items are *depreciated*). The allowed depreciation schedule (i.e., how much can be deducted from income in each year) is liable to be quite complex, with current regulations allowing more deduction in earlier than later years (called *accelerated* depreciation). Furthermore, an *investment tax credit* (a reduction of tax obligation) is generally allowed in the year following a capital outlay. In addition to these tax consequences, the Bell System also includes the effects of its *accounting* system (*book depreciation* is generally different from tax depreciation).

Fortunately, for most outside plant studies, it is not necessary to keep track of these complex financial consequences in detail. All that is required is their present worth or, as we shall describe, their *levelized equivalent annual cost* (LEAC). For study purposes, it is generally adequate to assume some standard financial consequences of the type described above for various classes of capital expenditure. For example, any underground cable with a given installed cost may be assumed to generate an identical stream of tax depreciation allowances, investment tax credits, book depreciation, etc. Thus it is only necessary to examine the detailed financial consequences for representatives of the various classes (e.g., cable of various types, conduit, and repeaters). The same present worth will apply to each member of the class. It is often convenient to scale the results of calculations for each category per dollar of installed first cost (IFC). Furthermore, it is useful to calculate a levelized equivalent annual charge (LEAC) for each category. The LEAC is defined so that the present worth of a constant annuity of LEAC dollars per year equals the present worth of the capital expenditure and all of its associated financial consequences. The LEAC per dollar of IFC, commonly called the *annual charge factor* (ACF), is calculated, perhaps with the aid of a computer program, and tabulated for all of the common outside plant capital expenditures.

In a particular outside plant study, the total financial impact of a capital expenditure is reflected by merely assuming that a constant annual charge of LEAC = ACF × IFC is incurred starting from the time an item is placed into service. It is commonly the case in outside plant studies that a capital expenditure represents a commitment to continue providing service into the indefinite future, replacing the given equipment by similar equipment at the end of its life (*repeated plant* assumption). In that case, it is appropriate to apply the LEAC from the time an item is placed until the end of the study period (which might be infinite, for example). This allows for valid economic comparisons of plant items with different service lives. The present worth of the LEACs of all capital expenditures plus the present worth of expense* items is called the *Present Worth of Annual Charges* (PWAC). This is taken to be the fundamental economic criterion—among plans providing equal service, the smaller the PWAC, the better.

The actual calculation of the ACFs varies according to the type of plant (e.g., different tax laws apply to short-life versus long-life plant, and to low-salvage versus high-salvage items), as well as to current tax laws (e.g., the investment tax credit seems to change regularly), and to Bell System or regulatory body policy (e.g., normalization or flow-through accounting for differences between book and tax depreciation). The following equations, taken from the "new greenbook," [8] are representative of the calculations involved.

The LEAC is the constant annuity whose present value over the service life, L, is

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* In general, if there are differences in the revenues generated for the different alternatives under study, these differences should be treated in the same manner as differences in expense flows.

$$\mathrm{PW}\begin{pmatrix}\text{Capital} & + & \text{Income}\\ \text{recovery} & & \text{tax}\end{pmatrix}$$
$$= (1+\phi)[\textsc{ifc} - \mathrm{S}]$$
$$- \phi \mathrm{PW}(D_b)$$
$$- \tau(1+\phi)\mathrm{PW}(D_\tau - D_b)$$
$$- (1+\phi)[\mathrm{PW}(\textsc{tc}) - \mathrm{PW}(\textsc{atc})];$$

where

PW(·) designates present worth;

ϕ is the *income tax factor:*

$$\phi = \left(\frac{\tau}{1-\tau}\right)\left(1 - \delta\frac{i_d}{i}\right)$$

with

τ = income tax rate
δ = debt ratio
i_d = interest cost of debt
i = composite cost of debt and equity

IFC is the installed first cost at time zero

S is the net salvage obtained at the end of the service life

D_b is the book depreciation:

$$\begin{pmatrix}\text{Book}\\ \text{depreciation}\\ \text{in year } t\end{pmatrix} = \frac{\textsc{ifc} - S}{L}$$

D_τ is the tax depreciation which varies from year to year according to

$$\begin{pmatrix}\text{Tax}\\ \text{depreciation}\\ \text{in year } t\end{pmatrix} = \begin{cases} 2/L_\tau & \text{in year 1}\\ \left(1 - \dfrac{2}{L_\tau}\right)\left(\dfrac{2}{L_\tau}\right)\left(\dfrac{L_\tau + 1 - t}{L_\tau - 1}\right) & \text{in other years}\\ -S & \text{in year } L\end{cases}$$

but with the proviso that no further depreciation is allowed once the year-by-year total amount depreciated reaches the IFC

L_τ is *tax life,* generally 80% of L

TC is the investment tax credit (e.g., 10 percent of IFC in year 1)

ATC is the amortized tax credit:

$$\textsc{atc} = \textsc{tc}/L$$

This formula applies under several assumptions:

(*i*) Tax credits are flowed through rather than normalized (last two terms would be different).

(*ii*) Tax depreciation is calculated according to double-declining balance in year 1 and sum-of-years digits thereafter (or there would be a different formula for year-by-year tax depreciation).

(*iii*) The asset depreciation range (ADR) system is allowed and $L \geqq 3$ so that $L_\tau = 0.8L$.

(*iv*) Salvage is less than 10 percent of IFC so that allowable depreciation for tax purposes is the total IFC (otherwise, less year-by-year depreciation would be allowed with the difference made up in year L).

(*v*) The entire IFC is to be capitalized both for book and tax purposes (sometimes the IRS allows part of installation costs, capitalized on the books, to be treated as expense in tax calculations).

Further discussion of this formula or the assumptions behind it is beyond the scope of this paper. The interested reader is referred to the new "greenbook."[8]

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