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An Inversion Technique for the Laplace Transform with Application to Approximation

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Properties of a sequence of positive operators defined by the Widder Laplace inversion formula are studied in order to obtain practical methods for the inversion of the Laplace transform, practical error formulae, and useful approximations to given functions. The approximation procedure retains essential structural characteristics of the original function, e.g., nonnegativity, monotonicity, and convexity. Thus a distribution function is approximated by distribution functions. Enhancement techniques are provided for the improvement of accuracy for a given order of approximation. The methods are illustrated by applications to renewal theory and to the covariance and recovery functions of telephone traffic theory.

I. INTRODUCTION

The Laplace transform occurs frequently in investigations of queueing theory and telephone traffic models in which it usually represents a probability distribution function. Although the mean and variance of the distribution can be readily obtained from the transform, there are many investigations in which the distribution itself is needed; in particular, good analytic and numerical approximations for the complementary distribution when the argument is large. This is the case, for example, when studying waiting times of queues, time delays of work through a computer system, and delays of message progress through data networks.

Numerical methods which have been made thus far[16–18] concentrate on accurate numerical approximation on some interval $[0, T]$, the difficulty of accurate inversion increasing with increasing T. Methods depending on Gauss-Legendre quadrature applied to the defining Laplace integral with subsequent interpolation are discussed in Ref. 19. These methods require the solution of large order linear systems whose matrices are severely ill-conditioned; thus they can bog down in meaningless calculations. Much ingenuity has been used in specific cases to circumvent this problem. Asymptotic formulae may sometimes be used to approximate the complementary distribution for large argument; however, in many practical cases, good accuracy was obtained only when the argument was so large that the corresponding probabilities were too small to be of practical significance. One of the methods of this paper, namely, the α enhancement procedure, specifically attacks this problem by imitating the exponential decay of the original in $[T, \infty]$ while simultaneously providing accurate approximation in $[0, T]$. The transition region is sufficiently well approximated for most practical uses.

The well-known Laplace inversion formula of Widder[1,3] has not been actively used in practical work. It has been the experience of the author that an investigation of the Widder formula qua functional transformation can provide useful practical techniques for inversion and also inequalities and limit relations between the approximations and the original function. Accordingly, it is the object of this paper to study the properties of a sequence of positive operators defined by the Widder formula in order to obtain practical methods for Laplace inversion, practical error formulae, and useful approximations to given functions.

In II the Widder inversion formula is obtained and a sequence of positive operators, L_n, which form the subject of the paper, are introduced. The L_n map a function $f(t)(t \geq 0)$ to a sequence of functions $f_n(t) = L_n f$ which converge uniformly on $[0, \infty]$ to $f(t)$. This viewpoint enables one to study the approximation characteristics of the sequence $f_n(t)$, thus providing a means of approximating a given $f(t)$ besides effecting the inversion of its Laplace transform, $\bar{f}(s)$. Several representations are given for $f_n(t)$ in terms of $f(t)$.

In III properties of the sequence $\{f_n(t)\}$ are developed which show that it possesses many desirable characteristics. In many applications it is preferable that the approximating functions globally imitate the original function in qualitative structural features rather than to the attainment of very high numerical accuracy. Thus if the original function lies between zero and one, is monotone decreasing, and is convex, then these same properties would be desired in the approximation. It is shown that the approximating sequence, $\{f_n(t)\}$, does retain those properties. A recursion relation for $f_{n+1}(t)$ in terms of $f_n(t)$, and a generating function for the sequence are also given, thus making the computation of higher approximations possibly more convenient than the direct application of the representation formulae themselves. A useful feature of the $f_n(t)$ is that, when $f(t)$ is convex, they satisfy $f_n(t) \geq f(t)$.

Part IV develops error bounds and pointwise error estimates. The results in terms of $f(t)$ reflect the use of the technique for approximation; on the other hand, the pointwise estimate of error in terms of $f_n(t)$ is especially useful for the inversion problem since then $f(t)$ is not available.

It is also shown that the successive approximations $f_0(t), f_1(t), f_2(t), \ldots$ are uniformly better for each t if $\ddot{f}(t) \geq 0$.

In practical use the initial member of the sequence, $f_0(t)$, is not an accurate approximation to $f(t)$ for t not in the neighborhoods of zero and infinity. Additionally the sequence $\{f_n(t)\}$ does not converge rapidly in n. Consequently one must go far out in the sequence to obtain adequate accuracy. Part V treats this problem. A modification, $f_{n,\alpha}(t)$, of $f_n(t)$ is introduced depending on a parameter α for which, by appropriate choice of α, $f_{0,\alpha}(t)$ is a much improved approximation to $f(t)$ than $f_0(t)$; the rapidity of convergence of the sequence $\{f_{n,\alpha}(t)\}$ is not improved over that of the unmodified sequence. However, it has been found that good accuracy is obtained by use of $f_{0,\alpha}(t)$ or $f_{1,\alpha}(t)$ as is demonstrated in the examples on covariance and recovery functions given in this paper.

In many applications, especially to complementary distribution functions, the behavior of $f(t)$ for large t must be accurately reproduced. The approximations $f_{n,\alpha}(t)$ accomplish this especially when α is related to the decrement of an exponential majorant. For functions which are exponentially small at infinity, the $f_n(t)$ do not adequately reproduce the decay of $f(t)$.

Many of the desirable features of the original method are still retained by this modification. The concept of convexity with exponent α is introduced which allows the transference of the inequality $f_n(t) \geq f(t)$ to $f_{n,\alpha}(t) \geq f(t)$. A criterion is given for deciding convexity with exponent α in terms of the transform, $\tilde{f}(s)$.

The degree of precision concept is applied to the approximation sequence in order to obtain a modified sequence, $s_n(t)$, which converges more rapidly. For sufficiently smooth functions this method is successful. The approximation $s_n(t)$ consists of a linear combination of $f_0(t), \ldots, f_n(t)$ or of $f_{0,\alpha}(t), \ldots, f_{n,\alpha}(t)$ and hence is easily applied. Its efficacy is demonstrated in the examples of this paper. Unfortunately the improvement in rapidity of convergence is so strong that the map from $f(t)$ to $s_n(t)$ is no longer positive, consequently many of the desirable structural preservation properties of the L_n are lost in favor of greater numerical accuracy.

An attempt is made to enhance the rapidity of convergence of $\{f_n(t)\}$ while simultaneously retaining the positivity of the map. This is accomplished by the construction of a new sequence, $h_n(t)$, which is also a linear combination of $f_0(t), \ldots, f_n(t)$. As is to be expected, however, the improvement is not as great as is realized with the sequence $s_n(t)$.

The pointwise error estimate developed in Part IV may be used as a correction device on $f_n(t)$ or $f_{n,\alpha}(t)$ to improve further the accuracy of computation. This, however, in the absence of an error estimate for the modification, must rely on one's understanding of the specific problem for ascertaining the reasonableness of the result.

An application of the methods of this paper to the renewal function[9] in the theory of renewal processes is made in Part VI. The remarkable accuracy of the simplest of the approximations $f_0(t)$, $f_1(t)$ is noteworthy.

Part VII presents applications of the techniques to the covariance and recovery functions of Erlang blocking models used in telephone traffic theory.[11] For the covariance function, the initial approximation, $f_{0,\alpha}(t)$, is excellent; however, in the case of the recovery function it was found that $f_{0,\alpha}(t)$, $f_{1,\alpha}(t)$ might not be considered sufficiently accurate, accordingly the linear combination, $s_1(t)$, was used. This provided sufficient enhancement of accuracy.

The generating function, $G(z,t)$, for the $f_n(t)$ can sometimes be used to obtain an explicit construction of the sequence. Some examples of this nature are treated in Part VIII.

Applications of the methods herein have been made to the complementary distributions of waiting time in $M/G/1$ queues. Also B. W. Stuck and E. Arthurs have successfully applied these techniques to the study of models of computer systems.

There are questions of an exclusively mathematical character which have not been touched upon, e.g., a semigroup interpretation and saturation phenomena. It is felt that these would be outside the essentially practical thrust of the paper. For some theorems which are applicable to the operators of this paper see Ref. 5.

A short table of operations on $f(t)$ and their corresponding maps under L_n is included to facilitate application of these methods to the construction of approximations.

II. WIDDER INVERSION—REPRESENTATIONS

Let the transform $\bar{f}(s)$,

$$\bar{f}(s) = \int_0^\infty e^{-su} f(u)du \tag{1}$$

exist for $s > 0$, then

$$\frac{(-1)^n}{n!} s^{n+1} \bar{f}^{(n)}(s) = \frac{s^{n+1}}{n!} \int_0^\infty e^{-su} u^n f(u)du \tag{2}$$

in which

$$\bar{f}^{(n)}(s) = \frac{d^n}{ds^n} \bar{f}(s). \tag{3}$$

The function $(s^{n+1}/n!)e^{-su}u^n$ is a probability density function on $(0, \infty)$ for $s > 0, n \geq 0$ whose mean is $(n + 1)/s$ and variance $(n + 1)/s^2$. When

$s = (n+1)/t$, the mean and variance are t and $t^2/(n+1)$ respectively. One has:[1]

Theorem 1 (*Widder*). Let the transform, $\bar{f}(s)$, of $f(t)$ exist for $s > 0$, let $f(t)$ be continuous at t and bounded on $[0, \infty]$, then

$$\lim_{n\to\infty} \frac{(-1)^n}{n!} s^{n+1}\bar{f}^{(n)}(s)\bigg|_{s=(n+1)/t} = f(t).$$

The convergence is uniform in every finite closed interval throughout which $f(t)$ is continuous.

Proof. Korovkin's theorem on sequences of positive functionals.[2]

The inversion theorem, in the above form, had already been stated by Feller[3] who used the law of large numbers to effect the proof. It is the purpose of this paper to study the transformation

$$L_nf = f_n = \frac{(-1)^n}{n!} s^{n+1}\bar{f}^{(n)}(s)\bigg|_{s=(n+1)/t} \tag{4}$$

so that f_n may be effectively used as an approximation to f. The representation of f_n directly in terms of f is obtainable from (2); thus,

$$f_n(t) = \int_0^\infty g_n(t, u)f(u)du \tag{5}$$

$$g_n(t, u) = \frac{(n+1)^{n+1}}{n!t^{n+1}} e^{-[(n+1)/t]u}u^n \qquad n \geq 0. \tag{6}$$

Alternative forms which will be found useful are:

$$f_n(t) = \frac{n+1}{t} \int_0^\infty \psi\left(n, (n+1)\frac{u}{t}\right) f(u)du \tag{7}$$

$$f_n(t) = (n+1) \int_0^\infty \psi(n, (n+1)u)f(tu)du \tag{8}$$

$$\psi(x, a) = e^{-a}\frac{a^x}{\Gamma(x+1)} \tag{9}$$

in which $\psi(x, a)$ is the Poisson probability distribution,

$$f_n(t) = \int_0^\infty M_n\left(\frac{t}{u}\right) f(u)\frac{du}{u} \tag{10}$$

$$M_n(x) = \frac{(n+1)^{n+1}}{n!} e^{-(n+1)/x}x^{-n-1} \tag{11}$$

in which the representation is by means of convolution on the half line

(Mellin convolution), and

$$g_n(\eta) = \int_{-\infty}^{\infty} K_n(\eta) g(\eta - \xi) d\xi \tag{12}$$

$$K_n(\eta) = \frac{(n+1)^{n+1}}{n!} e^{-(n+1)\eta-(n+1)e^{-\eta}} \tag{13}$$

$$t = e^{\eta},\, u = e^{\xi},\, f(t) = g(\eta),\, f_n(t) = g_n(\eta) \tag{14}$$

in which the representation is by means of convolution on the whole real line (Fourier convolution).

The conditions of Theorem 1 are relaxed below.

Theorem 2. Let the transform, $\bar{f}(s)$, of $f(t)$ exist for $s > c$, let $f(t)$ be continuous at t and let $f(t) = 0(e^{ct})(t \to \infty)$; then,

$$\lim_{n\to\infty} f_n(t) = f(t).$$

The convergence is uniform in every finite closed interval throughout which $f(t)$ is continuous.

Proof. The representation (5) may be written as follows:

$$f_n(t) = \frac{(n+1)^{n+1}}{n!t^{n+1}} \int_0^{\infty} e^{-[(n-m+1)/t]u} u^n e^{-m/tu} f(u) du. \tag{15}$$

For all t in some finite closed interval, m may be chosen so that

$$e^{-(m/t)u} f(u) = 0(1)(u \to \infty)$$

hence Korovkin's theorem is again applicable and the conclusions follow.

III. PROPERTIES OF $f_n(t)$

Jensen's theorem applied to (5) proves

Theorem 3. $f(t)$ is convex on $(0, \infty)$

$$\Rightarrow f(t) \le f_n(t),\, t \ge 0,\, n \ge 0.$$

The value of an approximation method is greatly enhanced when the approximating function preserves the shape of the original and coincides closely with its behavior in the neighborhoods of zero and infinity. Theorems 4, 5, 6 establish the desired properties.

Theorem 4. $a \le f(t) \le b \Rightarrow a \le f_n(t) \le b$; a, b arbitrary real.

Proof. Direct evaluation shows that

$$L_n f = f,\, f = \alpha + \beta t. \tag{16}$$

The positivity of L_n implies

$$f \leq b \Rightarrow L_n f \leq L_n b = bL_n 1 = b. \tag{17}$$

Similarly for the lower bound.

The derivatives of $f_n(t)$ may be related to those of $f(t)$ through use of (8); thus

Theorem 5. Let $f^{(r)}(t)$ be continuous and $0(e^{ct})(t \to \infty)$, then there is an m so that $f_n^{(r)}(t)$ exists and is continuous for $n \geq m$ and

$$f_n^{(r)}(t) = (n + 1) \int_0^\infty \psi(n, (n + 1)u)u^r f^{(r)}(tu)du.$$

One may set $m = 0$ if $f^{(r)}(t) = 0(1)(t \to \infty)$.

Proof. For m sufficiently large, the integral of the theorem converges uniformly in t; hence the representation (8) may be differentiated under the integral sign r times. If $f^{(r)}(t)$ is bounded then $m = 0$ is a permissible choice since one still has uniform convergence.

Corollary 1. $f^{(r)} \geq 0 \Rightarrow f_n^{(r)} \geq 0, n \geq m$.

Proof. This follows from the positivity of the kernel.

The above corollary implies that if f has a continuous derivative and is monotone then f_n is monotone, and if f has a continuous second derivative and is convex then f_n is convex. A stronger structural result will be obtained in Theorem 6. One also has that if f is completely or absolutely monotonic then f_n is completely or absolutely monotonic respectively.

Corollary 2. $f_n^{(r)}(0+) = \lambda_{n,r} f^{(r)}(0+), n \geq m,$

$$\lambda_{n,r} = \frac{\Gamma(n + r + 1)}{n!(n + 1)^r}.$$

In particular

$$f_n(0+) = f(0+) \qquad n \geq m$$
$$\dot{f}_n(0+) = \dot{f}(0+) \qquad n \geq m.$$

Proof. Define $\lambda_{n,r}$ by

$$\lambda_{n,r} = (n + 1) \int_0^\infty \psi(n, (n + 1)u)u^r du$$

then evaluation of the integral yields the formula stated. Since the operator is bounded, the limit statements follow. Also one has $\lambda_{n,0} = \lambda_{n,1} = 1$.

Corollary 3. Let $f^{(r)}(\infty) < \infty$; then

$$f_n^{(r)}(\infty) = \lambda_{n,r} f^{(r)}(\infty) \qquad n \geq 0$$

In particular

$$f_n(\infty) = f(\infty) \qquad n \geq 0,$$

$$\dot{f}_n(\infty) = \dot{f}(\infty) \qquad n \geq 0.$$

Proof. Dominated convergence allows the interchange of limit and integral.

The following concepts will be needed to establish further structural properties.

For an arbitrary sequence in $(-\infty, \infty)$, $-\infty < t_1, t_2, \ldots, t_\ell < \infty$, the number of changes of sign is called the variation of the sequence and will be indicated by $v(t_1, t_2, \ldots, t_\ell)$; thus

$$v(3, -1, 0, 2, -2) = 3 \tag{18}$$

$$v(1, 2, 4, 6) = 0. \tag{19}$$

One sets $v(0, 0, \ldots, 0) = -1$. Let $f(t)$ be defined on $(0, \infty)$, and let $0 < t_1 < t_2 < \ldots < t_\ell < \infty$ be an arbitrary, ordered sequence in $(0, \infty)$, the quantity $\sup v(f(t_1), f(t_2), \ldots, f(t_\ell))$, in which the supremum is taken over all sequences, i.e., for all choices of $(t_1, t_2, \ldots, t_\ell)$ and for all $\ell \geq 1$, is called the variation of f and will be indicated by $v(f)$. A transformation L on a given class of f will be called variation diminishing if and only if $v(Lf) \leq v(f)$ for every f in its domain. The definition used here is adopted from Hirschman and Widder.[4]

Let $\phi(\eta)$ be a frequency function on $(-\infty, \infty)$, that is,

$$\phi(\eta) \geq 0, \int_{-\infty}^{\infty} \phi(\eta) d\eta = 1 \tag{20}$$

and let

$$\tilde{\phi}(s) = \int_{-\infty}^{\infty} e^{-s\eta} \phi(\eta) d\eta. \tag{21}$$

Define $E(s)$ by

$$E(s) = \tilde{\phi}(s)^{-1}. \tag{22}$$

Then a theorem of Schoenberg[4] states that the transformation

$$Tg = \int_{-\infty}^{\infty} \phi(\xi) g(\eta - \xi) d\xi \qquad g \epsilon BC(-\infty, \infty) \tag{23}$$

is variation diminishing if and only if

$$E(s) = e^{-Cs^2+bs} \prod_k \left(1 - \frac{s}{a_k}\right) e^{s/a_k} \tag{24}$$

$$C \geq 0,\ b,\ a_k \text{ real}, \sum_k 1/a_k^2 < \infty.$$

The designation $g \epsilon BC(-\infty, \infty)$ means that $g(\eta)$ is bounded and continuous on $(-\infty, \infty)$. It may be observed that the mean of ϕ is b and the variance

$$2C + \sum_k 1/a_k^2$$

The Laplace transform of $K_n(\eta)$ (13),

$$\tilde{K}_n(\mathrm{s}) = \int_{-\infty}^{\infty} e^{-s\eta} K_n(\eta) d\eta \tag{25}$$

is

$$\tilde{K}_n(s) = \frac{\Gamma(n+s+1)}{n!(n+1)^s}. \tag{26}$$

This may be written in the following forms

$$\tilde{K}_n(s) = \frac{\Gamma(1+s)}{(n+1)^s} \prod_{k=1}^{n} \left(1 + \frac{s}{k}\right) \tag{27}$$

$$\tilde{K}_n(s)^{-1} = e^{s\nu_n} \prod_{k=n+1}^{\infty} \left(1 + \frac{s}{k}\right) e^{-s/k} \tag{28}$$

$$\nu_n = ln\ (n+1) + \gamma - \sum_{j=1}^{n} \frac{1}{j} \tag{29}$$

in which $\gamma = 0.5772157$ is Euler's constant. Thus by Schoenberg's theorem, the transformation $T_n g = g_n$, $g \epsilon BC(-\infty, \infty)$, defined in (12) is variation diminishing. The mean and variance of K_n are respectively ν_n as given above, and σ_n^2 given by

$$\sigma_n^2 = \frac{\pi^2}{6} - \sum_{j=1}^{n} \frac{1}{j^2}. \tag{30}$$

Since the map $t = e^\eta$ is monotone, the following theorem has been established.

Theorem 6. The transformation L_n defined on $f \epsilon BC(0, \infty)$ is variation diminishing, i.e.,

$$v(f_n) \leq v(f).$$

Corollary. f_n does not cross any straight line more often than f. In particular, if f is monotone then f_n is monotone, and if f or $-f$ is convex then f_n or $-f_n$ is convex.

Proof. From (16) and Theorem 6,

$$v(L_n(f - \alpha - \beta t)) = v(L_n f - \alpha - \beta t) \leq v(f - \alpha - \beta t). \qquad (31)$$

Clearly f is monotone $\Leftrightarrow v(f - \alpha) \leq 1$ for arbitrary α, and f or $-f$ is convex $\Leftrightarrow v(f - \alpha - \beta t) \leq 2$ for arbitrary α, β.

It is clear from (8) that the approximating sequence to $f(at)(a > 0)$ is $f_n(at)$; this may be expressed in a more illuminating way as follows. Define the operator A by

$$Af(t) = f(at) \qquad a > 0 \qquad (32)$$

then A and L_n commute; thus

$$L_n Af = AL_n f \qquad (33)$$

Hence the eigenfunctions of A, which are t^r, are also the eigenfunctions of L_n. In fact one easily obtains

$$L_n t^r = \lambda_{n,r} t^r, \; r \geq 0, \; n \geq 0. \qquad (34)$$

It may be observed that if L_n is defined by (8) instead of by (4) then (34) remains valid even for $r < 0$ provided n is large enough.

Other operations with the same eigenfunctions will also commute with L_n. Of importance in discussing the convergence of $f_n^{(r)}(t)$ is the operator

$$\theta \equiv t \frac{d}{dt} \qquad (35)$$

One has

Theorem 7. The operators $L_n (n \geq 0)$, θ commute; thus

$$L_n(\theta^r f) = \theta^r f_n.$$

It is assumed that the rth derivative of f exists and is continuous on $(0, \infty)$.

Proof. It is observed that the eigenfunctions of θ are t^r; alternatively the result follows directly from (8).

Corollary. Proceeding inductively, one can now establish that if $f^{(r)}$ is continuous and $0(e^{ct})(t \to \infty)$ then

$$\lim_{n \to \infty} f_n^{(r)} = f^{(r)}$$

In addition to shape preserving properties, another way of assessing the adequacy of an approximation process is by comparison of moments; the rth moment of $f(t)$ is here taken to be $\int_0^\infty t^r f(t)dt$. The Mellin transform, $\bar{f}(s)$, given by

$$\bar{f}(s) = \int_0^\infty t^{s-1} f(t)dt \tag{36}$$

is the appropriate tool. Since the transform of $M_n(s)$, eq. (11), is

$$\overline{M}_n(s) = \frac{(n+1)^s \Gamma(n-s+1)}{n!} \tag{37}$$

one has from (10)

$$\bar{f}_n(s) = \frac{(n+1)^s \Gamma(n-s+1)}{n!} \bar{f}(s). \tag{38}$$

For the following, it is convenient to use the factorial symbol

$$n^{(0)} = 1, n^{(r)} = n(n-1)\ldots(n-r+1), r > 0 \text{ (integral)}. \tag{39}$$

The following theorem may now be stated.

Theorem 8. Let the rth moment of $f(t)$ exist ($r \geq 0$, integral); then the rth moment of $f_n(t)$ exists for $n > r$ and one has

$$\int_0^\infty t^r f_n(t)dt = \frac{(n+1)^{r+1}}{n^{(r+1)}} \int_0^\infty t^r f(t)dt.$$

Special cases are

$$\int_0^\infty f_n(t)dt = \frac{n+1}{n} \int_0^\infty f(t)dt \qquad n \geq 1 \tag{40}$$

$$\int_0^\infty t f_n(t)dt = \frac{(n+1)^2}{n(n-1)} \int_0^\infty t f(t)dt \qquad n \geq 2. \tag{41}$$

When $\int_0^\infty t^{-1} f(t)dt$ exists, an interesting special case of (38) occurs for $s = 0$; thus

$$\int_0^\infty t^{-1} f_n(t)dt = \int_0^\infty t^{-1} f(t)dt. \tag{42}$$

Formulae (40), (41) may be used to ensure equality of moments. Thus if it is required that the zeroth order moments agree, then, according to (40), one may use as the approximating sequence $nf_n(t)/(n+1)$. If the zeroth and first moments are to agree simultaneously, one may use a linear combination of $f_n(t)$ and $f_m(t)$; for example, $\frac{3}{2}f_3(t) - \frac{2}{3}f_2(t)$.

Another set of moment relations may be obtained from (7) involving sums of $f_n((n+1)t)$. These are given in

Theorem 9. Let the rth absolute moment of $f(t)$ exist; then

$$\sum_{n=0}^{\infty} n^{(r)} f_n((n+1)t) = t^{-r-1} \int_0^{\infty} u^r f(u) du.$$

Proof. One has

$$\sum_{n=0}^{\infty} n^{(r)} \psi(n, a) = a^r \tag{43}$$

and, from (7),

$$f_n((n+1)t) = t^{-1} \int_0^{\infty} \psi\left(n, \frac{u}{t}\right) f(u) du. \tag{44}$$

Multiplication of both sides of (44) by $n^{(r)}$ and summing, the result follows on interchanging summation and integration. Dominated convergence justifies the interchange.

Another property of $f_n(t)$ as a function of n is given in:

Theorem 10. Let $f(t) \geq 0$ on $(0, \infty)$, $0(e^{ct})(t \to \infty)$, then there is an m so that

$$f_{n+1}(t) \leq e^{-1} \left(\frac{n+2}{n+1}\right)^{n+2} f_n(t),\ t \geq 0, n \geq m.$$

Proof. Since

$$(n+2)\psi(n+1, (n+2)u) = ue^{-u} \left(\frac{n+2}{n+1}\right)^{n+2} (n+1)\psi(n, (n+1)u) \tag{45}$$

one may write

$$f_{n+1}(t) = (n+1) \int_0^{\infty} \psi(n, (n+1)u) ue^{-u} \left(\frac{n+2}{n+1}\right)^{n+2} f(tu) du. \tag{46}$$

Observing that $ue^{-u} \leq e^{-1}$, the inequality follows.

Corollary. $f(t) \geq 0 \Rightarrow \dfrac{f_n(t)}{n+1}$ is monotone decreasing in n for all $t \geq 0$, $n \geq m$.

Proof. One has

$$\frac{f_{n+1}(t)}{n+2} \leq \frac{f_n(t)}{n+1} e^{-1} \left(\frac{n+2}{n+1}\right)^{n+1}. \tag{47}$$

The result follows since

$$e^{-1}\left(\frac{n+2}{n+1}\right)^{n+1} \le 1. \tag{48}$$

A stronger monotonicity property of $f_n(t)$ is stated in Theorem 21.

A useful recurrence relation allowing one to compute the members of the sequence $f_n(t)$ successively starting with $f_0(t)$ is given in the following theorem.

Theorem 11.

$$f_{n+1}(t) = f_n\left(\frac{n+1}{n+2}t\right) + \frac{t}{n+2}\dot{f}_n\left(\frac{n+1}{n+2}t\right), n \ge 0, t \ge 0.$$

Proof. Define $\hat{f}_n(s)$ by

$$\hat{f}_n(s) = \frac{(-1)^n}{n!} s^{n+1}\tilde{f}^{(n)}(s) \tag{49}$$

then, by (4),

$$f_n(t) = \hat{f}_n(s)\Big|_{s=(n+1)/t} \tag{50}$$

One has

$$s\frac{d}{ds}\hat{f}_n(s) = (n+1)\hat{f}_n(s) - (n+1)\hat{f}_{n+1}(s) \tag{51}$$

$$\hat{f}_{n+1}(s) = \hat{f}_n(s) - \frac{1}{n+1}s\frac{d}{ds}\hat{f}_n(s). \tag{52}$$

Thus

$$f_{n+1}(t) = \left[\hat{f}_n(s) - \frac{1}{n+1}s\frac{d}{ds}\hat{f}_n(s)\right]_{s=(n+2)/t} \tag{53}$$

The recurrence relation is now obtained on performing the substitution for s.

A useful alternative method of presenting the structure of the entire sequence $\{f_n\}_0^\infty$ in terms of f_0 is by means of a generating function. This is given in the theorem below.

Theorem 12. $f(t)$ is bounded on $(0, \infty) \Rightarrow$

$$\sum_{n=0}^{\infty} z^n f_n((n+1)t) = \frac{1}{1-z} f_0\left(\frac{t}{1-z}\right).$$

The series is convergent for $|z| < 1$ and analytically continuable in the half plane Re $z < 1$.

Proof. The formula follows from (7) after interchange of summation and integration. The series is clearly convergent for $|z| < 1$ while dominated convergence justifies the interchange for Re $z < 1$.

The case $r = 0$ of Theorem 9 provides the following corollary.

Corollary.

$$f(t) \epsilon L(0, \infty) \Rightarrow \lim_{z \to 1-} \frac{1}{1-z} f_0 \left(\frac{t}{1-z} \right) = t^{-1} \int_0^\infty f(u) du.$$

The Mellin transform, $\bar{f}(s)$, of $f(t)$ may be directly obtained from its Laplace transform, $\bar{f}(s)$, by use, for example, of (38) for $n = 0$; thus

$$\bar{f}(s) = \frac{\bar{f}_0(s)}{\Gamma(1-s)}. \tag{54}$$

Accordingly one may now write (38) in the form

$$\bar{f}_n(s) = \frac{(n+1)^s \Gamma(n+1-s)}{n! \Gamma(1-s)} \bar{f}_0(s) \tag{55}$$

or, equivalently,

$$\bar{f}_n(s) = (n+1)^s \binom{n-s}{n} \bar{f}_0(s). \tag{56}$$

At times (55) or (56) provides a convenient alternative to Theorems 11 and 12 when $f_n(t)$ is required as a function of n.

The range of applicability of the Jensen inequality of Theorem 3 may be extended by use of Mellin or Fourier convolution. A sequence $\{f_n(t)\}_{n=0}^\infty$ will be called an approximation sequence if there is an $f(t)$ so that $f_n(t) = L_n f(t)$. Let $*$ designate Mellin convolution; then

Theorem 13. $f_n * g$ is the approximation sequence for $f * g$.

Proof. One has

$$\overline{L_n(f*g)} = \bar{M}_n \bar{f} \bar{g} = \overline{L_n f} \bar{g} \tag{57}$$

thus,

$$L_n(f*g) = (L_n f) * g = f_n * g. \tag{58}$$

The converse of Theorem 3 is also true.

Theorem 14. $f_n(t) \geq f(t)$ for all $n \geq 0, t \geq 0, f(t)$ is bounded on $(0, \infty) \Rightarrow f(t)$ is convex on $(0, \infty)$.

Proof. The result follows from Theorem 8 of Karlin and Ziegler.[5]

Corollary. f convex on $(0, \infty)$, $g \geq 0$ on $(0, \infty) \Rightarrow f*g$ convex on $(0, \infty)$.

Proof. One has from Theorem 3

$$f_n \geq f \tag{59}$$

and, since $g \geq 0$,

$$f_n * g \geq f * g. \tag{60}$$

Since, by Theorem 13, $f_n * g$ is the approximation sequence to $f * g$, application of Theorem 14 proves the corollary.

It may be observed that the inequality of (60) remains valid when $*$ is interpreted as Fourier convolution although, in this case, $f_n * g$ is not the approximation sequence for $f * g$.

Another set of convexity results may be obtained from (8) by considering logarithmic convexity.

Theorem 15. If $f(t)$ is log-convex on $t \geq 0$, then $f_n(t)$ is log-convex for $n \geq 0, t \geq 0$.

Proof. Equation (8) and the additivity of log-convex functions.[6]

Further one may state the following inequalities.

Theorem 16. If $f(t)$ is log-convex on $t \geq 0$ then

$$f(t) \leq e^{L_n \ell n f(t)} \leq f_n(t).$$

Proof. The inequality on the left follows from Theorem 3 applied to $\ell n f(t)$; the one on the right is a consequence of the geometric mean-arithmetic mean inequality.

IV. ERROR ESTIMATION

Error estimates take different forms depending on the class of functions for which they are intended and whether or not they are bounds or pointwise estimates. From a practical point of view the pointwise estimate is the most useful provided it may be easily evaluated in terms of the approximation itself. The next three theorems provide error bounds for different function classes; the fourth theorem provides an approximate formula for the pointwise evaluation of error, while (112) does the same but in terms of $f_n(t)$. The error of approximation, $\epsilon_n(t;f)$, is defined by

$$\epsilon_n(t;f) = f_n(t) - f(t) \tag{61}$$

Theorem 17. Let $\dot{f}(t)$ be continuous on $(0, \infty)$; then

$$|\epsilon_n(t;f)| \leq \frac{t}{\sqrt{n+1}} \sup_{t>0} |\dot{f}(t)|.$$

Proof. One has

$$\epsilon_n(t;f) = \int_0^\infty g_n(t, u)\{f(u) - f(t)\}du \tag{62}$$

$$|\epsilon_n(t;f)| \le \int_0^\infty g_n(t, u)|f(u) - f(t)|du \tag{63}$$

$$|\epsilon_n(t;f)| \le \sup_{t>0} |\dot{f}(t)| \cdot \int_0^\infty g_n(t, u)|u - t|du \tag{64}$$

$$|\epsilon_n(t;f)| \le \sup_{t>0} |\dot{f}(t)| \cdot \left\{\int_0^\infty g_n(t, u)(u - t)^2 du\right\}^{1/2} \tag{65}$$

$$|\epsilon_n(t;f)| \le \frac{t}{\sqrt{n+1}} \sup_{t>0} |\dot{f}(t)|. \tag{66}$$

The last inequality follows because the mean and variance of $g_n(t, u)$ are t and $t^2/(n + 1)$ respectively.

Theorem 18. Let $\ddot{f}(t)$ be continuous on $(0, \infty)$; then

$$|\epsilon_n(t;f)| \le \frac{t^2}{2n+2} \sup_{t>0} |\ddot{f}(t)|.$$

Proof. The Taylor expansion of $f(u)$ about t has the form

$$f(u) = f(t) + (t - u)\dot{f}(t) + \frac{1}{2}(t - u)^2\ddot{f}(\xi) \tag{67}$$

in which ξ lies between t and u. Thus

$$\epsilon_n(t;f) = \frac{1}{2}\frac{t^2}{n+1}\ddot{f}(\xi),\ \xi\epsilon(0, \infty). \tag{68}$$

The inequality of the theorem now follows.

The next theorem provides an error bound which is uniform for $t\epsilon[0, \infty]$. For this purpose the absolute first moment of $K_n(\eta)$ (13), α_n, is needed; thus

$$\alpha_n = \int_{-\infty}^\infty K_n(\eta)|\eta|d\eta. \tag{69}$$

Theorem 19. Let $\dot{f}(t)$ be continuous on $(0, \infty)$; then

$$|\epsilon_n(t;f)| \le \alpha_n \sup_{t>0} |t\dot{f}(t)|.$$

Proof. One has from (12)

$$\epsilon_n(e^\eta;f) = \int_{-\infty}^\infty K_n(\eta - \xi)\{g(\xi) - g(\eta)\}d\xi \tag{70}$$

$$|\epsilon_n(e^\eta;f)| \le \sup_{-\infty<\eta<\infty} |g'(\eta)| \cdot \int_{-\infty}^{\infty} K_n(\eta-\xi)|\eta-\xi|d\xi \qquad (71)$$

$$|\epsilon_n(t;f)| \le \alpha_n \sup_{t>0} |t\dot{f}(t)|.$$

Corollary. The convergence of $f_n(t)$ to $f(t)$ $(n \to \infty)$ is uniform for $t \epsilon [0, \infty]$.

Proof. It is necessary to show that

$$\lim_{n\to\infty} \alpha_n = 0.$$

The expression for α_n (69) is rewritten as follows

$$\alpha_n = \rho_n \int_{-\infty}^{\infty} K(\eta)^{n+1}|\eta|d\eta \qquad (72)$$

$$\rho_n = \frac{1}{n!}\left(\frac{n+1}{e}\right)^{n+1}, K(\eta) = e^{1-\eta-e^{-\eta}}. \qquad (73)$$

Use of the power series expansion for $e^{-\eta}$ yields

$$\alpha_n \sim \rho_n \int_{-\infty}^{\infty} e^{-[(n+1)/2]\eta^2}|\eta|d\eta = \frac{2}{en!}\left(\frac{n+1}{e}\right)^n. \qquad (74)$$

Stirling's formula now shows that

$$\alpha_n \approx \sqrt{\frac{2}{\pi n}}. \qquad (75)$$

Some numerical values of α_n are $\alpha_0 = 1.0160$, $\alpha_1 = 0.6388$, $\alpha_2 = 0.5006$, $\alpha_3 = 0.4247$, $\alpha_4 = 0.3751$, $\alpha_5 = 0.3396$, $\alpha_6 = 0.3126$, $\alpha_7 = 0.2911$; the asymptotic formula (75) is sufficiently accurate for $n > 7$.

To continue the study of $\epsilon_n(t;f)$, it is useful to obtain an explicit formula of Peano type, that is an integral transform of $\ddot{f}$.
Let

$$x_+ = x \qquad x \ge 0 \qquad (76)$$

$$= 0 \qquad x \le 0 \qquad (77)$$

then the Taylor expansion of $f(t)$ with remainder is

$$f(t) = f(0) + \dot{f}(0)t + \int_0^{\infty} (t-v)_+\ddot{f}(v)dv. \qquad (78)$$

From (5) and (16), one has

$$f_n(t) = f(0) + \dot{f}(0)t + \int_0^{\infty} \ddot{f}(v)dv \int_v^{\infty} g_n(t,u)(u-v)du. \qquad (79)$$

Thus

$$\epsilon_n(t, f) = \int_0^\infty E_n(t, v)\ddot{f}(v)dv \tag{80}$$

$$E_n(t, v) = \int_v^\infty g_n(t, v)(u - v)du - (t - v)_+. \tag{81}$$

The kernel $E_n(t, v)$ (the Peano kernel for error representation) is, clearly,

$$E_n(t, v) = L_n(t - v)_+ - (t - v)_+. \tag{82}$$

The explicit evaluation of the kernel may be most simply carried out by means of (24) since the Laplace transform of $(t - v)_+$ is e^{-sv}/s^2. Let

$$S_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!} \tag{83}$$

and

$$\ell_n(a) = e^{-(n+1)a}[S_n((n + 1)a) - aS_{n-1}((n + 1)a)] \tag{84}$$

then

$$L_n(t - v)_+ = t\ell_n(a) \qquad a = v/t \tag{85}$$

$$E_n(t, v) = t[\ell_n(a) - (1 - a)_+]. \tag{86}$$

In particular, one has

$$E_0(t, v) = te^{-v/t} - (t - v)_+ \tag{87}$$

$$E_1(t, v) = (t + v)e^{-2v/t} - (t - v)_+. \tag{88}$$

Since $(t - v)_+$ is a convex function of t for each v, (82) and Theorem 3 establish

$$E_n(t, v) \geq 0 \text{ for all } t \geq 0, v \geq 0. \tag{89}$$

The moments of the kernel, $E_n(t, v)$, may be obtained by substituting the functions $f(t) = t^r$ $(r \geq 2)$ into (80), and using (34) and (61) for evaluation of $\epsilon_n(t;t^r)$; the following is obtained:

$$\int_0^\infty v^r E_n(t, v)dv = \frac{\lambda_{n,r+2} - 1}{(r + 1)(r + 2)} t^{r+2} \qquad r \geq 0. \tag{90}$$

In particular

$$\int_0^\infty E_n(t, v)dv = \frac{t^2}{2}\frac{1}{n + 1} \tag{91}$$

$$\int_0^\infty vE_n(t, v)dv = \frac{t^3}{6}\frac{3n + 5}{(n + 1)^2}. \tag{92}$$

One may now obtain an approximate evaluation of $\epsilon_n(t;f)$.

Theorem 20. Let $\ddot{f}(t)$ be continuous on $(0, \infty)$ and $0(e^{ct})(t \to \infty)$; then there is an m so that

$$\epsilon_n(t;f) \approx \frac{t^2}{2n+2}\ddot{f}\left(t\,\frac{3n+5}{3n+3}\right), n \geq m;$$

also if $\ddot{f}(t)$ is convex, then the approximation is a lower bound.

Proof. The one point Gaussian quadrature formula for $\int_0^\infty E_n(t, v)f(v)dv$ is of the form $Af(\alpha)$ in which the constants, A, α, are determined by requiring the quadrature to be exact for all linear functions. Use of (91), (92) now yields the formula of the theorem. The inequality follows from the nonnegativity of $E_n(t, v)$ (89) and Jensen's inequality.

Since by the Corollary to Theorem 7, $\ddot{f}_n(t)$ approximates $\ddot{f}(t)$, in practice the required value of $\ddot{f}(t)$ is approximated either from the analytic form of $f_n(t)$ or numerically from a table or curve already computed for $f_n(t)$.

At this point another property of the sequence $\{f_n(t)\}_0^\infty$ can be proved.

Theorem 21. Let $\ddot{f}(t) \geq 0$, continuous on $(0, \infty)$, and $0(e^{ct})(t \to \infty)$; then there is an m so that

$$f_{n+1}(t) \leq f_n(t) \text{ for all } t \geq 0, n \geq m.$$

Proof. Clearly the monotonic decreasing character of $f_n(t)$ as a function of n will hold if $\epsilon_n(t;f)$ has the same property. The nonnegativity of $\ddot{f}(t)$ and (80) shows that the result is implied if $E_n(t, v)$ is monotonically decreasing in n; in turn, by (86), this will follow if $\ell_n(a)$ is monotonically decreasing in n for each $a \geq 0$. From (84), by direct calculation,

$$\frac{d}{da}\ell_n(a) = -e^{-(n+1)a}S_n((n+1)a) \tag{93}$$

$$\frac{d^2}{da^2}\ell_n(a) = \frac{(n+1)^{n+1}}{n!}a^n e^{-(n+1)a}. \tag{94}$$

Let

$$h_n(a) = \ell_{n-1}(a) - \ell_n(a) \qquad n \geq 1 \tag{95}$$

then, from (94),

$$\frac{d^2}{da^2}h_n(a) = r_n(a)\frac{d^2}{da^2}\ell_{n-1}(a) \tag{96}$$

$$r_n(a) = 1 - \left(1 + \frac{1}{n}\right)^{n+1} ae^{-a}. \tag{97}$$

It is clear from (94) that the sign of

$$\frac{d^2}{da^2} h_n(a)$$

is the same as that of $r_n(a)$. There exist two points $0 < a_0(n) < a_1(n)$ with the following properties:

$$r_n(a) \geq 0 \qquad 0 \leq a \leq a_0(n) \tag{98}$$

$$r_n(a) < 0 \qquad a_0(n) < a < a_1(n) \tag{99}$$

$$r_n(a) \geq 0 \qquad a \geq a_1(n). \tag{100}$$

Since

$$\ell_n(0) = 1, \frac{d}{da} \ell_n(0) = -1, \quad n \geq 0 \tag{101}$$

it follows that

$$h_n(0) = 0, \frac{d}{da} h_n(0) = 0, \quad n \geq 1. \tag{102}$$

One has the following integral representations for $h_n(a)$:

$$h_n(a) = \int_0^a db \int_0^b r_n(c) \frac{d^2}{dc^2} \ell_{n-1}(c) dc, \tag{103}$$

$$h_n(a) = \int_a^\infty db \int_b^\infty r_n(c) \frac{d^2}{dc^2} \ell_{n-1}(c) dc. \tag{104}$$

Thus (98) and (103) imply

$$h_n(a) \geq 0 \qquad 0 \leq a \leq a_0(n); \tag{105}$$

similarly (100) and (104) imply

$$h_n(a) \geq 0 \qquad a \geq a_1(n). \tag{106}$$

The function $h_n(a)$ cannot be negative in $(a_0(n), a_1(n))$ since then it would have at least one local minimum; however, (99) shows that in $(a_0(n), a_1(n))$

$$\frac{d^2}{da^2} h_n(a) < 0$$

which is a contradiction. Thus

$$h_n(a) \geq 0, a \geq 0, n \geq 1 \tag{107}$$

and the theorem is proved.

It is possible to estimate conveniently $\epsilon_n(t;f)$ directly from $f_n(t)$ if $\dot{f}_n(t)$ and $\ddot{f}_n(t)$ are readily obtainable, at least possibly numerically from values of $f_n(t)$. From (38) one has

$$\bar{f}(s) = \overline{M}_n(s)^{-1}\bar{f}_n(s). \tag{108}$$

Expansion of $\Gamma(n+1-s)$ into a power series in s and substitution into (37) provides the following series

$$\overline{M}_n(s) = 1 + \nu_n s + \frac{\sigma_n^2 + \nu_n^2}{2} s^2 + \ldots \tag{109}$$

$$\overline{M}_n(s)^{-1} = 1 - \nu_n s + \frac{\nu_n^2 - \sigma_n^2}{2} s^2 + \ldots \tag{110}$$

Thus

$$\bar{\epsilon}_n(s;f) = \left[\nu_n s + \frac{\sigma_n^2 - \nu_n^2}{2} s^2 + \ldots\right]\bar{f}_n \tag{111}$$

$$\epsilon_n(t;f) \approx -\nu_n \theta f_n(t) + \frac{\sigma_n^2 - \nu_n^2}{2}\theta^2 f_n(t). \tag{112}$$

To facilitate the use of (112) some values of the coefficients are given in Table I.

The following readily obtained asymptotic formulas may be used for values of n beyond the table:

$$\nu_n \approx \frac{1}{2(n+1)} + \frac{1}{12(n+1)^2} \tag{113}$$

$$\sigma_n^2 \approx \frac{1}{n+1} + \frac{1}{2(n+1)^2}$$

$$\frac{\sigma_n^2 - \nu_n^2}{2} \approx \frac{1}{2(n+1)} + \frac{1}{8(n+1)^2}.$$

V. ENHANCEMENT OF ACCURACY

The excellent behavior of the operator L_n in constructing approximations to a given $f(t)$ which preserve its structural properties and its limiting values and which provide inequalities exacts a penalty in the form of slow convergence. A high value of n is required to attain high

Table I — Coefficients

n	ν_n	σ_n^2	$(\sigma_n^2 - \nu_n^2)/2$
0	0.5772	1.6449	0.6559
1	0.2704	0.6449	0.2859
2	0.1758	0.3949	0.1820
3	0.1302	0.2838	0.1334
4	0.1033	0.2213	0.1053
5	0.0856	0.1813	0.0870
6	0.0731	0.1535	0.0741
7	0.0638	0.1331	0.0645
8	0.0566	0.1175	0.0572

numerical accuracy. In many practical problems, fortunately, very high accuracy is not needed; notwithstanding, the value of n required may still be inconveniently high. Considering that one starts with the Laplace transform, $\bar{f}(s)$, of $f(t)$ and uses (4), or constructs $f_0(t)$ and uses the recursion of Theorem 11, a high value of n implies obtaining a correspondingly high order of derivative of $\bar{f}(s)$ or of $f_0(t)$ which can be a time-consuming operation. Thus it would be useful to modify the basic approximation, $L_n f$, while still preserving many of its original characteristics so that the accuracy for a given value of n may be increased.

In many cases the transform, $\bar{f}(s)$, has the property that for some $\alpha > 0, \bar{f}(s - \alpha)$ converges for $s > 0$. This property is used to construct a new approximation, $f_{n,\alpha}(t)$, defined by

$$f_{n,\alpha}(t) = e^{-\alpha t} L_n(e^{\alpha t} f(t)) \tag{114}$$

and, correspondingly, a new operator

$$L_{n,\alpha} f = f_{n,\alpha}. \tag{115}$$

The following theorem permits the construction of $f_{n,\alpha}(t)$ directly from $f_n(t)$.

Theorem 22.

$$f_{n,\alpha}(t) = \frac{e^{-\alpha t}}{\left(1 - \dfrac{\alpha t}{n+1}\right)^{n+1}} f_n\left(\frac{t}{1 - \dfrac{\alpha t}{n+1}}\right).$$

Proof. From (5) and (6), one has

$$L_n f = \frac{(n+1)^{n+1}}{n! t^{n+1}} \int_0^\infty e^{-[(n+1)/t]u} u^n f(u) du \tag{116}$$

$$L_n(e^{\alpha t} f) = \frac{(n+1)^{n+1}}{n! t^{n+1}} \int_0^\infty e^{-[(n+1)/t]u + \alpha u} u^n f(u) du. \tag{117}$$

Thus

$$L_n f \Big|_{\frac{t}{1-\alpha t/(n+1)}}$$

$$= \left(1 - \frac{\alpha t}{n+1}\right)^{n+1} \frac{(n+1)^{n+1}}{n! t^{n+1}} \int_0^\infty e^{-[(n+1)/t]u + \alpha u} u^n f(u) du. \tag{118}$$

Comparison of (118) with (117) shows that

$$L_n(e^{\alpha t} f) = \frac{1}{\left(1 - \dfrac{\alpha t}{n+1}\right)^{n+1}} f_n\left(\frac{t}{1 - \dfrac{\alpha t}{n+1}}\right) \tag{119}$$

hence, the result follows from (114).

The approximations $f_{n,\alpha}(t)$ satisfy theorems similar to those proved for $f_n(t)$; however, modifications are required. Only Theorem 3 will be discussed. A function $f(t)$ will be said to be convex over an interval with exponent α if and only if $e^{\alpha t}f(t)$ is convex over the same interval.

Theorem 23. If $f(t)$ is convex on $(0, \infty)$ with exponent α then

$$f(t) \leq f_{n,\alpha}(t).$$

Proof. One has from Theorem 3,

$$e^{\alpha t}f(t) \leq L_n(e^{\alpha t}f(t)). \tag{120}$$

The result now follows from the nonvanishing of $e^{\alpha t}$ and (114).

The error of approximation by $f_{n,\alpha}(t)$ will be designated $\epsilon_{n,\alpha}(t;f)$ and defined by

$$\epsilon_{n,\alpha}(t;f) = f_{n,\alpha}(t) - f(t). \tag{121}$$

Clearly

$$\epsilon_{n,\alpha}(t;f) = e^{-\alpha t}\epsilon_n(t;e^{\alpha t}f). \tag{122}$$

also, if the condition of Theorem 21 is satisfied, one has

$$\epsilon_{n,\alpha}(t;f) \geq 0. \tag{123}$$

One of the useful aspects of the approximation, $f_{n,\alpha}(t)$, is that it more accurately reflects the asymptotic behavior of $f(t)(t \to \infty)$ than $f_n(t)$ does for a given value of n. In the later applications this will be an important characteristic.

Clearly, ordinary convexity corresponds to convexity with exponent zero; however, the following theorem relates convexity with exponent α to log-convexity.

Theorem 24. Let $\ddot{f}(t)$ be continuous on some interval I; then $f(t)$ is log-convex on I if and only if it is convex with exponent α on I for all α.

Proof. One has

$$\ell n\ (e^{\alpha t}f(t)) = \alpha t + \ell n f(t) \tag{124}$$

hence $e^{\alpha t}f(t)$ is convex with exponent α on I for all α if $f(t)$ is log-convex on I. The derivative condition for convexity with exponent α on I is

$$\ddot{f}(t) + 2\alpha\dot{f}(t) + \alpha^2 f(t) \geq 0 \text{ on } I \tag{125}$$

and the derivative condition for log-convexity on I is

$$f(t)\ddot{f}(t) - \dot{f}(t)^2 \geq 0 \text{ on } I. \tag{126}$$

The choice

$$\alpha = -\dot{f}(t)/f(t) \tag{127}$$

which is always possible since $f(t) > 0$ on I, in (125) verifies (126).

Convexity with exponent α and, hence, by Theorem 24, log-convexity may be decided by means of the Laplace transform and the use of the Hausdorff-Bernstein theorem.[7]

Theorem 25. Let $\ddot{f}(t)$ be continuous on $(0, \infty)$, then $f(t)$ is convex with exponent α on $(0, \infty)$ if and only if

$$(s + \alpha)^2 \bar{f}(s) - (s + 2\alpha) f(0+) - \dot{f}(0+)$$

is completely monotonic in s on $(0, \infty)$ and is absolutely convergent on $s > 0$.

Proof. The expression cited is the Laplace transform of

$$e^{-\alpha t} \frac{d^2}{dt^2} (e^{\alpha t} f(t))$$

whose nonnegativity is the necessary and sufficient condition for convexity of $f(t)$ with exponent α. The Hausdorff-Bernstein theorem now completes the proof.

It may be observed that the quantities $f(0+)$, $\dot{f}(0+)$ are obtainable from

$$\lim_{s \to \infty} s\bar{f}(s) = f(0+) \tag{128}$$

$$\lim_{s \to \infty} \{s^2 \bar{f}(s) - s f(0+)\} = \dot{f}(0+). \tag{129}$$

Another method of enhancement is related to the concept of "degree of precision." An approximation operator T, i.e., $Tf \approx f$, in which the functions t^r, suitably restricted to an appropriate interval ($r \geq 0$, integral), are in its domain, is said to have degree of precision k if $Tt^r = t^r$ for $0 \leq r \leq k$ and $Tt^r \neq t^r$ for $r = k + 1$. Thus the singular operators L_n studied here have degree of precision one.

The enhancement method consists of the following: coefficients $\delta_j (0 \leq j \leq k - 1)$ are determined by the moment conditions

$$\sum_{j=0}^{k-1} \delta_j L_j(t^r) = t^r \qquad 0 \leq r \leq k \tag{130}$$

and, accordingly, the linear combination

$$s_{k-1}(t) = \sum_{j=0}^{k-1} \delta_j f_j(t) \tag{131}$$

is now taken to approximate $f(t)$. Clearly the map from f to s_{k-1} has degree of precision k, however, unfortunately, it is not positive. If f is sufficiently smooth there will result a significant improvement in accuracy over the use of f_{k-1} alone. The system (130) may be expressed in terms of $\lambda_{n,r}$ as follows

$$\sum_{j=0}^{k-1} \delta_j \lambda_{j,r} = 1 \qquad 1 \leq r \leq k. \tag{132}$$

Accordingly special cases of (131) are

$$s_1 = -f_0 + 2f_1 \tag{133}$$

$$s_2 = \frac{1}{2} f_0 - 4f_1 + \frac{9}{2} f_2. \tag{134}$$

A method of enhancement consisting of a linear combination of the $f_n(t)$ similar to (131) which, however, retains the positivity of the map will now be constructed. The accuracy attained will usually not be as great as that of (131) for a given set of values $\{f_j(t)\}_0^n$. The new sequence will be designated $h_n(t)$ and is defined by

$$h_n(t) = \sum_{j=0}^{n} \rho_j f_j(t). \tag{135}$$

Define $W_n(u)$ by

$$W_n(u) = \sum_{j=0}^{n} \rho_j (j+1)\psi(j, (j+1)u) \tag{136}$$

then the coefficients, ρ_j, are constrained by

$$\sum_{j=0}^{n} \rho_j = 1, \; W_n(u) \geq 0 \text{ for all } u \geq 0. \tag{137}$$

Theorem 26.

$$|h_n(t) - f(t)| \leq t \sup_{t>0} |\dot{f}(t)| \left\{ \sum_{j=0}^{n} \frac{\rho_j}{j+1} \right\}^{1/2}.$$

Proof. From (8), one has

$$h_n(t) = \int_0^\infty W_n(u) f(tu)\,du. \tag{138}$$

Also, from (137),

$$\int_0^\infty W_n(u)\,du = 1 \tag{139}$$

hence

$$h_n(t) - f(t) = \int_0^\infty W_n(u)[f(tu) - f(t)]\,du. \tag{140}$$

The nonnegativity of $W_n(u)$ now permits the following inequality

$$|h_n(t) - f(t)| \leq \int_0^\infty W_n(u)|f(tu) - f(t)|du; \tag{141}$$

hence,

$$|h_n(t) - f(t)| \leq t \sup_{t>0} |\dot{f}(t)| \int_0^\infty W_n(u)|u - 1|du. \tag{142}$$

The Cauchy-Schwartz inequality applied to (142) yields

$$|h_n(t) - f(t)| \leq t \sup_{t>0} |\dot{f}(t)| \left\{ \int_0^\infty W_n(u)(u - 1)^2 du \right\}^{1/2}. \tag{143}$$

Evaluation of the integral in (143) provides the inequality of the theorem.

Consider the sum, S, defined by

$$S = \sum_{j=0}^{n} \frac{\rho_j}{j + 1} \tag{144}$$

then, in order to obtain the best approximation, the ρ_j must be chosen to minimize S besides satisfying the conditions of (137). One has

$$e^u W_n(u) = \sum_{j=0}^{n} \frac{(j + 1)^{j+1}}{j!} \rho_j e^{-ju} u^j. \tag{145}$$

Let

$$z = e^{1-u} u \tag{146}$$

then the constraint of (137) may be written

$$\sum_{j=0}^{n} \frac{(j + 1)^{j+1}}{j!} e^{-j} \rho_j z^j \geq 0 \qquad 0 \leq z \leq 1. \tag{147}$$

Define the polynomials $P(x)$ by (147) with $z = (x + 1)/2$; then one has

$$P(x) = \sum_{j=0}^{n} x^j \left[\sum_{k=j}^{n} \frac{(k + 1)^{k+1}}{k!} \binom{k}{j} (2e)^{-k} \rho_k \right] \tag{148}$$

and

$$P(x) \geq 0 \qquad -1 \leq x \leq 1. \tag{149}$$

The cosine polynomial $P(\cos\theta)$ is now obtained and written in the Fourier form

$$P(\cos\theta) = \frac{1}{2} a_0 + \sum_{j=1}^{n} a_j \cos j\theta \tag{150}$$

in which the a_j are obtained from

$$a_j = \frac{1}{\pi} \int_{-\pi}^{\pi} P(\cos\theta) \cos j\theta d\theta. \tag{151}$$

The nonnegativity of $P(\cos\theta)$ implies the following representation[8]

$$P(\cos\theta) = |h(\theta)|^2 \tag{152}$$

$$h(\theta) = x_0 + x_1 e^{i\theta} + \ldots + x_n e^{in\theta} \tag{153}$$

in which the coefficients $x_0, \ldots, x_n$ are real; thus

$$a_j = 2 \sum_{\nu=0}^{n-j} x_\nu x_{\nu+j} \qquad 0 \le j \le n. \tag{154}$$

When (154) is solved for the ρ_j in terms of $x_0, \ldots, x_n$, the problem of minimizing S subject to $\rho_0 + \ldots + \rho_n = 1$ becomes that of minimizing a quadratic form relative to another quadratic form.

The optimum ρ_j have been obtained for the case $n = 2$; the result is

$$h_2(t) = 0.146993 f_0(t) - 0.944260 f_1(t) + 1.797267 f_2(t) \tag{155}$$

with $S = 0.273952$. Thus, from Theorem 26,

$$|h_2(t) - f(t)| \le 0.523404 t \sup_{t>0} |\dot{f}(t)|. \tag{156}$$

The estimate of $\epsilon_n(t;f)$ in (112) may be effectively used to reduce error. One may take as an approximation to $f(t)$ the following

$$f(t) \approx f_n(t) + \nu_n \theta f_n(t) - \frac{\sigma_n^2 - \nu_n^2}{2} \theta^2 f_n(t). \tag{157}$$

In order to improve $f_{n,\alpha}(t)$, the approximate calculation of $\epsilon_{n,\alpha}(t;f)$ proceeds by use of (122). The practical use of (112), (157) uses difference quotients to evaluate $\theta f_n(t)$, $\theta^2 f_n(t)$ from the values already obtained for $f_n(t)$. Thus, let $h > 0$ be the distance between consecutive values of t for which $f_n(t)$ is calculated; then

$$\theta f_n(t) \approx t \frac{f_n(t+h) - f_n(t-h)}{2h} \tag{158}$$

$$\theta^2 f_n(t) \approx \theta f_n(t) + t^2 \frac{f_n(t+h) - 2f_n(t) + f_n(t-h)}{h^2}. \tag{159}$$

The following comment should prove useful in reduction of error. If a function $g(t)$ is known which approximates $f(t)$, for example, the leading term of an asymptotic expansion for $f(t)$, then one may use

$$f(t) \approx g(t) + L_n(f - g). \tag{160}$$

Evidently an appropriate $g(t)$ should always be sought before constructing practical approximations to $f(t)$.

VI. THE RENEWAL FUNCTION

In this section some of the preceding theory will be applied to obtaining approximations for the renewal function, $M(t)$, of a renewal stream.[9] Let $A(t)$, with $A(0+) = 0$, be the interarrival time distribution and $\hat{A}(s)$, given by

$$\hat{A}(s) = \int_0^\infty e^{-st} dA(t) \tag{161}$$

its Laplace-Stieltjes transform, then

$$\tilde{M}(s) = \frac{1}{s} \frac{\hat{A}(s)}{1 - \hat{A}(s)}. \tag{162}$$

The sequence of approximations, $M_n(t)$, may now be constructed from

$$M_0(t) = \frac{1}{1 - \hat{A}(1/t)} - 1. \tag{163}$$

In particular one has

$$M_1(t) = \frac{1}{1 - \hat{A}(2/t)} - 1 - \frac{2}{t} \frac{\hat{A}'(2/t)}{[1 - \hat{A}(2/t)]^2} \tag{164}$$

in which

$$\hat{A}'(s) = \frac{d}{ds} \hat{A}(s). \tag{165}$$

Let λ be the arrival rate, and σ^2 the variance of interarrival time, that is,

$$\lambda^{-1} = \int_0^\infty t dA(t) \tag{166}$$

$$\sigma^2 = \int_0^\infty t^2 dA(t) - \lambda^{-2} \tag{167}$$

then evaluation of the contribution of $\tilde{M}(s)$ at $s = 0$ provides the term

$$\lambda t + \frac{\sigma^2\lambda^2 - 1}{2}. \tag{168}$$

Thus one may introduce a new function, $f(t)$, by

$$M(t) = \lambda t + \frac{\sigma^2\lambda^2 - 1}{2} + f(t) \tag{169}$$

with

$$f_0(t) = \frac{\hat{A}(1/t)}{1 - \hat{A}(1/t)} - \lambda t - \frac{\sigma^2\lambda^2 - 1}{2}. \tag{170}$$

Since linear functions are invariants of the operators L_n, there is no reduction of error when approximating $f(t)$ by $f_n(t)$ over approximating $M(t)$ by $M_n(t)$; however, often $f(t)$ is exponentially dominated and the enhancement technique of Theorem 22 is applicable.

The following example will be considered:

$$A(t) = \operatorname{erf}\sqrt{\frac{t}{2}} \qquad \hat{A}(s) = \frac{1}{\sqrt{1+2s}} \tag{171}$$

$$\tilde{M}(s) = \frac{1}{s}\frac{1}{\sqrt{1+2s} - 1}. \tag{172}$$

Thus,

$$M_0(t) = \frac{1}{\sqrt{1+2/t} - 1} \tag{173}$$

$$M_1(t) = \frac{1}{\sqrt{1+4/t} - 1} + \frac{2}{t}\frac{1}{\sqrt{1+4/t}(\sqrt{1+4/t} - 1)^2}. \tag{174}$$

Since $\lambda = 1$, $\sigma^2 = 2$, one has

$$M(t) = t + \frac{1}{2} + f(t)$$

$$f_1(t) = \frac{1}{\sqrt{1+4/t} - 1} + \frac{2}{t}\frac{1}{\sqrt{1+4/t}(\sqrt{1+4/t} - 1)^2} - t - \frac{1}{2}. \tag{175}$$

The α transformation of Theorem 22 may be applied to $f_1(t)$. Assuming $f_1(t)$ to be ultimately of one sign, the singularity farthest to the right of $\tilde{f}(s)$, namely $-1/2$, coincides with the abscissa of convergence; hence, $\alpha = 1/2$. Table II compares the approximations for $M(t)$ given by $M_1(t)$, the enhancement procedure of (133), and $t + 1/2 + f_{1,1/2}(t)$ with more accurate values obtained from the exact solution

$$M(\mathrm{t}) = \sum_{n=0}^{\infty} \frac{1}{\Gamma\left(n + \frac{1}{2}\right)} \int_0^{t/2} e^{-u}u^{n-1/2}du. \tag{176}$$

Since $M(t) \approx t + 1/2 (t \to \infty)$, the accuracy increases with increasing t. This is characteristic of the applications to the renewal function.

An example will now be considered in which the interarrival time distribution is a mixture of exponentials since this is of frequent practical use. Accordingly let

Table II — Comparison of approximations

t	$M_1(t)$	$s_1(t)$	$t + \frac{1}{2} + f_{1,\frac{1}{2}}(t)$	$M(t)$
0	0	0	0	0
0.5	0.83333	0.85764	0.84297	0.86007
1	1.39443	1.42283	1.41014	1.42466
2	2.44338	2.47255	2.46303	2.47161
5	5.48142	5.50480	5.49568	5.49718
10	10.49350	10.50977	10.49980	10.49989

$$A(t) = 1 - \frac{7}{10} e^{-t} - \frac{3}{10} e^{-2t}. \tag{177}$$

Then,

$$\hat{A}(s) = \frac{7}{10} \frac{1}{s+1} + \frac{6}{10} \frac{1}{s+2} \tag{178}$$

$$\tilde{M}(s) = \frac{1}{s^2} \frac{20 + 13s}{17 + 10s}. \tag{179}$$

Also one has

$$M(t) = \frac{20}{17} t + \frac{21}{289} + f(t) \tag{180}$$

$$f_0(t) = -\frac{210}{289} \frac{1}{17t + 10}. \tag{181}$$

Application of Theorem 12 provides

$$f_1(t) = -\frac{8400}{289} \frac{1}{(17t + 20)^2} \tag{182}$$

$$f_2(t) = -\frac{5.67 \times 10^5}{289} \frac{1}{(17t + 30)^3}. \tag{183}$$

The exact solution for this simple example is

$$M(t) = \frac{20}{17} t + \frac{21}{289} \left(1 - e^{-(17/10)t}\right). \tag{184}$$

Table III compares calculations from $M_2(t)$ and the enhancement procedures of (134) and (155) with the exact value. The α enhancement procedure with $\alpha = 1.7$ was not used because it produces the exact result.

This example shows the operation of the enhancement procedures (134) and (155); clearly, $s_2(t)$ is very accurate since the constraint of positivity of the approximation operators is discarded in its construction.

Table III — Comparison of calculations

t	$M_2(t)$	$s_2(t)$	$h_2(t)$	$M(t)$
0	0	0	0	0
0.5	0.62652	0.62967	0.62712	0.62984
1	1.23024	1.23559	1.23127	1.23586
2	2.41812	2.42353	2.41913	2.42318
5	5.95373	5.95595	5.95407	5.95500
10	11.83713	11.83747	11.83710	11.83737

VII. THE COVARIANCE AND RECOVERY FUNCTIONS

The study of errors in switch count and continuous scan observational methods in telephone traffic engineering is facilitated by use of the covariance function of the number of busy trunks in the Erlang blocking system.[11] Specifically let x_t be the number of trunks busy at time t in an equilibrium $M/M/C$ blocking system with unit mean holding time and offered load of a erlangs, then the covariance function, $R(t)$, is

$$R(t) = E(x_0 x_t) - (Ex_0)^2. \tag{185}$$

In order to express the Laplace transform, $\tilde{R}(s)$, it is necessary to introduce the Poisson-Charlier polynomials[10,13] which may be obtained from

$$G_j(x, a) = \sum_{\nu=0}^{j} (-1)^{j-\nu} \binom{j}{\nu} \nu! a^{-\nu} \binom{x}{\nu}. \tag{186}$$

They satisfy the following recurrence

$$G_{j+1}(x, a) = \frac{x - j - a}{a} G_j(x, a) - \frac{j}{a} G_{j-1}(x, a) \tag{187}$$

$$G_0(x, a) = 1 \qquad G_1(x, a) = \frac{x}{a} - 1.$$

Also needed is the function $\alpha_j(x, a)$ given by

$$\alpha_j(x, a) = \frac{G_{j-1}(x, a)}{G_j(x, a)} \tag{188}$$

which satisfies the first order recurrence

$$\alpha_{j+1}(x, a)^{-1} = \frac{x - j - a}{a} - \frac{j}{a} \alpha_j(x, a) \tag{189}$$

$$\alpha_1(x, a) = \left(\frac{x}{a} - 1\right)^{-1}.$$

The zeros of $G_j(x, a)$ are all positive and simple; in particular, the zeros of $G_j(-s - 1, a)$ as a function of s are designated r_i and ordered by

$$r_j < r_{j-1} < \ldots < r_1 < 0. \tag{190}$$

In the approximation to be developed, r_1 will be the dominant root.

The Erlang loss function,[10] $B(c, a)$, given by

$$B(c, a) = \frac{a^c}{c!} \Big/ \sum_{j=0}^{c} \frac{a^j}{j!}$$

gives the probability that all servers are busy. In the formulae below it will be designated simply by B. The mean number of servers busy, μ, is

$$\mu = a(1 - B) \tag{191}$$

and the variance, σ^2, of the number of busy servers is

$$\sigma^2 = \mu - a(c - \mu)B. \tag{192}$$

The Laplace transform, $\tilde{R}(s)$, and the covariance, $R(t)$, are[11]

$$\tilde{R}(s) = \frac{\sigma^2 + \mu^2}{1 + s} + \frac{a\mu}{s(1 + s)} - \frac{acB}{(1 + s)^2} - \frac{\mu^2}{s} + \frac{acB}{s(1 + s)^2} \alpha_c(-s - 1, a) \tag{193}$$

$$R(t) = \sum_{j=1}^{c} A_j e^{r_j t} \qquad A_j = -\frac{a^2 B}{r_j(1 + r_j)^2} \prod_{i \neq j} \left(1 - \frac{1}{r_j - r_i}\right) \tag{194}$$

The approximation $R_0(t)$ is

$$R_0(t) = \frac{\sigma^2 + \mu^2}{1 + t} + \frac{a\mu t}{1 + t} - \mu^2 - \frac{acBt}{(1 + t)^2} + \frac{acBt^2}{(1 + t)^2} \alpha_c \left(-\frac{1}{t} - 1, a\right). \tag{195a}$$

Since the dominant root is r_1 one may choose α satisfying $0 \leq \alpha \leq -r_1$ to obtain

$$R_{0,\alpha}(t) = \sigma^2 e^{-\alpha t} g(t)$$

$$g(t) = \frac{1 + \mu^2/\sigma^2}{1 + (1 - \alpha)t} + \frac{a\mu t/\sigma^2}{(1 - \alpha t)(1 + (1 - \alpha)t)} - \frac{\mu^2/\sigma^2}{1 - \alpha t} - \frac{acBt/\sigma^2}{(1 + (1 - \alpha)t)^2} + \frac{acBt^2/\sigma^2}{(1 - \alpha t)(1 + (1 - \alpha)t)^2} \alpha_c \left(-\frac{1}{t} + \alpha - 1, a\right). \tag{195b}$$

It is known that the zeros r_j are separated by at least one so that $1 - 1/(r_j - r_i) > 0$, and hence A_j is positive for each j; thus $R(t)$ is log-convex.

Accordingly, the following inequality is valid (Theorem 23):

$$R(t) \leq R_{0,\alpha}(t). \tag{196}$$

In order to facilitate the use of $R_{0,\alpha}(t)$ an accurate upper bound for r_1 is needed to provide a suitable choice for α. Such a bound is available in Ref. 12. Thus let

$$\ell = \sum_{\nu=1}^{c} \frac{1}{\nu} c^{(\nu)} a^{-\nu} \tag{197}$$

$$m = \ell^2 - 2 \sum_{\nu=2}^{c} \frac{1}{\nu} c^{(\nu)} a^{-\nu} \sum_{j=1}^{\nu-1} \frac{1}{j} \tag{198}$$

then

$$r_1 \leq -\frac{c}{\ell + \sqrt{(c-1)(cm - \ell^2)}} - 1. \tag{199}$$

To illustrate the practical performance of (195b) and (199), calculations were made for the cases $a = 4, 8, 12$ and $c = 8$ corresponding to medium, heavy, and very heavy loads respectively. The corresponding equilibrium blocking probabilities are $B(8, 4) = 0.030420$, $B(8, 8) = 0.235570$, $B(8, 12) = 0.422655$. Table IV compares the exact and approximate values. Figures 1(a), 1(b), and 1(c) compare the corresponding curves.

Table IV — Comparison of exact and approximate values

	$a = 4$		$a = 8$		$a = 12$	
t	$R(t)$	$R_{0,\alpha}(t)$	$R(t)$	$R_{0,\alpha}(t)$	$T(t)$	$R_{0,\alpha}(t)$
0	3.377	3.377	2.564	2.564	1.492	1.492
0.4	2.143	2.145	1.075	1.091	0.312	0.331
0.8	1.365	1.367	0.474	0.483	0.075	0.079
1.2	0.870	0.872	0.212	0.216	0.019	0.020
1.6	0.555	0.556	0.095	0.097	0.005	0.005
2.0	0.354	0.355	0.043	0.043	0.001	0.001
2.4	0.226	0.227	0.019	0.019		
2.8	0.144	0.145	0.009	0.009		
3.2	0.092	0.092	0.004	0.004		
3.6	0.059	0.059	0.002	0.002		
4.0	0.038	0.038	0.001	0.001		

The quality of approximation of (199) may be seen from the following values of α used in (195b) compared to the exact r_1 values.

a	$-r_1$	α
4	1.1218	1.1215
8	2.0000	1.9730
12	3.4778	3.3415

The transition probabilities $P_{ij}(t)$—the probability j trunks are busy at time t given i trunks are busy at time zero—may all be obtained from the transition probability $P_{cc}(t)$[11]; this probability as a function of time is called the recovery function. It may be used in a similar manner to the covariance function, $R(t)$, for the study of errors in scan measurement techniques[14]; additionally it is especially important in the analysis of telephone retrial models.

The Laplace transform, $\tilde{P}_{cc}(s)$, and the recovery function, $P_{cc}(t)$, are[11]

$$\tilde{P}_{cc}(s) = \frac{1}{s} + \frac{c}{as}\, \alpha_c(-s-1, a) \tag{200}$$

$$P_{cc}(t) = B - \sum_{j=1}^{c} B_j e^{r_j t} \tag{201}$$

$$B_j = \frac{1}{r_j} \prod_{i \neq j} \left(1 - \frac{1}{r_j - r_i}\right). \tag{202}$$

As for the covariance function, $B = B(c, a)$, and $r_j (1 \leq j \leq c)$ are the roots of $G_c(-s-1, a)$ as a function of s.

In order to apply the α enhancement procedure, the function

$$f(t) = P_{cc}(t) - B \tag{203}$$

is considered whose Laplace transform is

$$\tilde{f}(s) = \frac{1}{s}\left[1 - B + \frac{c}{a}\, \alpha_c(-s-1, a)\right]. \tag{204}$$

It may be observed from (201) and (203) that $f(t)$ is log-convex, hence the approximations obtained will constitute upper bounds. In order to demonstrate the operation of the approximations, the functions $f_{0,\alpha}(t)$, $f_{1,\alpha}(t)$, and

$$s_1(t) = 2f_{1,\alpha}(t) - f_{0,\alpha}(t) \tag{205}$$

were constructed; they are

$$f_{0,\alpha}(t) = \frac{e^{-\alpha t}}{1 - \alpha t}\left[1 - B + \frac{c}{a}\, \alpha_c\left(-\frac{1}{t} + \alpha - 1, a\right)\right], \tag{206}$$

$$f_{1,\alpha}(t) = \frac{e^{-\alpha t}}{\left(1 - \frac{\alpha t}{2}\right)^2}\left[1 - B + \frac{c}{a}\, \alpha_c\left(-\frac{2}{t} + \alpha - 1, a\right)\right] + \frac{e^{-\alpha t}}{1 - \frac{\alpha t}{2}} \frac{2c}{at}\, \alpha_c'\left(-\frac{2}{t} + \alpha - 1, a\right). \tag{207}$$

The prime on $\alpha_c(x, a)$ indicates differentiation with respect to x. The

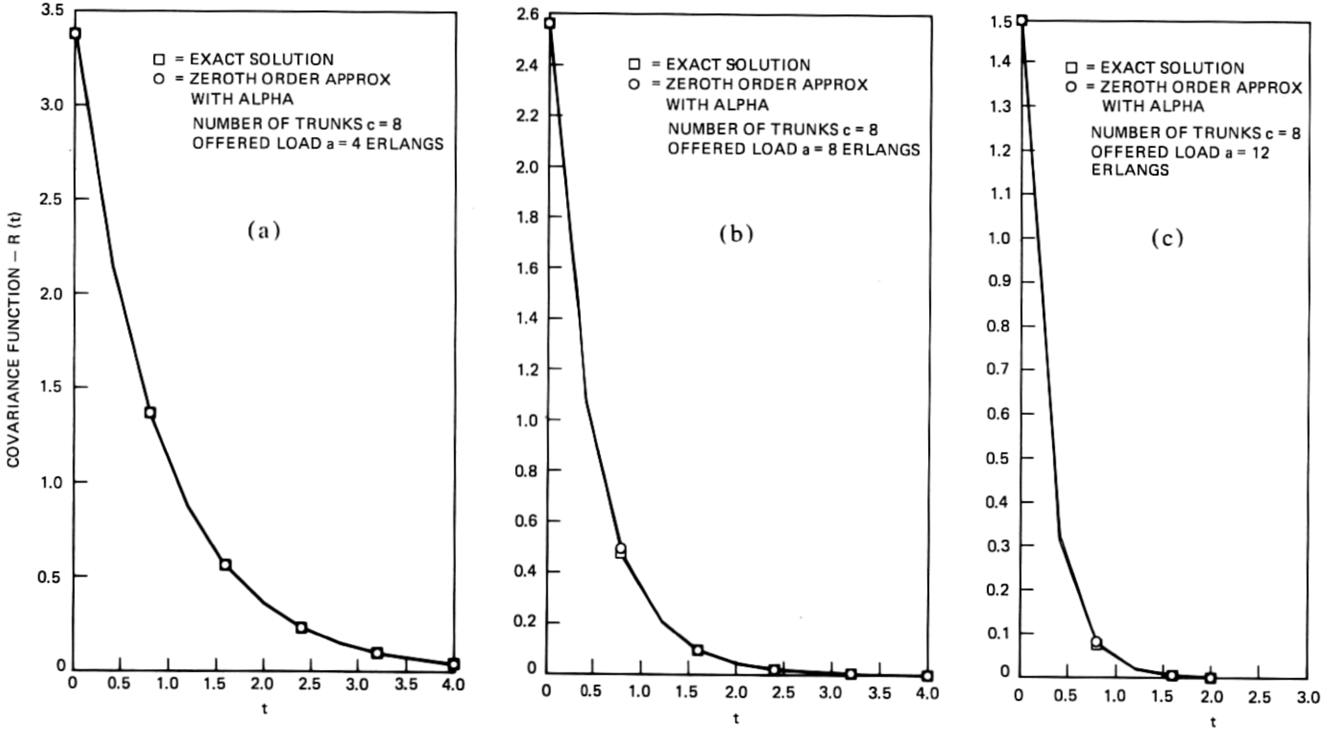

Fig. 1—Covariance function.

following recurrence relation is obtained from (189);

$$\alpha'_{j+1}(x, a) = \frac{1}{a}\,[j\alpha'_j(x, a) - 1]\alpha_{j+1}(x, a)^2 \tag{208}$$

$$\alpha'_1(x, a) = -\frac{1}{a}\,\alpha_1(x, a)^2.$$

Since $s_1(t)$ does not correspond to a positive operator, it does not provide a bound for $P_{cc}(t)$; one has, however,

$$P_{cc}(t) \leq B + f_{1,\alpha}(t) \leq B + f_{0,\alpha}(t). \tag{209}$$

The first inequality follows from Theorem 23 and the second inequality from Theorem 21.

The same cases as for the covariance function were treated. Tables V, VI, and VII compare the exact and approximate values. Figures 2(a), 2(b), and 2(c) compare the corresponding curves.

Table V — $a = 4$

t	$P_{cc}(t)$	$B + f_{0,\alpha}(t)$	$B + f_{1,\alpha}(t)$	$s_1(t)$
0	1.0000	1.0000	1.0000	1.0000
0.1	0.5178	0.5907	0.5597	0.5287
0.2	0.3304	0.4280	0.3856	0.3432
0.3	0.2380	0.3335	0.2901	0.2468
0.4	0.1844	0.2703	0.2296	0.1889
0.5	0.1497	0.2249	0.1880	0.1511
0.6	0.1256	0.1907	0.1578	0.1248
0.7	0.1080	0.1641	0.1350	0.1059
0.8	0.0947	0.1429	0.1174	0.0918
0.9	0.0844	0.1258	0.1034	0.0810
1.0	0.0762	0.1118	0.0922	0.0726

Table VI — $a = 8$

t	$P_{cc}(t)$	$B + f_{0,\alpha}(t)$	$B + f_{1,\alpha}(t)$	$s_1(t)$
0	1.0000	1.0000	1.0000	1.0000
0.1	0.5756	0.6335	0.6088	0.5842
0.2	0.4379	0.5005	0.4725	0.4445
0.3	0.3727	0.4256	0.4006	0.3756
0.4	0.3347	0.3770	0.3561	0.3352
0.5	0.3099	0.3432	0.3262	0.3092
0.6	0.2927	0.3187	0.3050	0.2913
0.7	0.2802	0.3005	0.2895	0.2786
0.8	0.2708	0.2866	0.2779	0.2692
0.9	0.2637	0.2760	0.2690	0.2621
1.0	0.2581	0.2677	0.2622	0.2567

This example will be used to show the operation of the error estimate (112). Using the increment $h = 0.1$, (158) and (159) were used to obtain $\theta[e^{\alpha t} f_{0,\alpha}(t)]$, $\theta^2[e^{\alpha t} f_{0,\alpha}(t)]$ and $\theta[e^{\alpha t} f_{1,\alpha}(t)]$, $\theta^2[e^{\alpha t} f_{1,\alpha}(t)]$ at $t = 0.5$. Equation (122) was used to estimate $\epsilon_{0,\alpha}(t)$, $\epsilon_{1,\alpha}(t)$. The error in $s_1(t)$ was

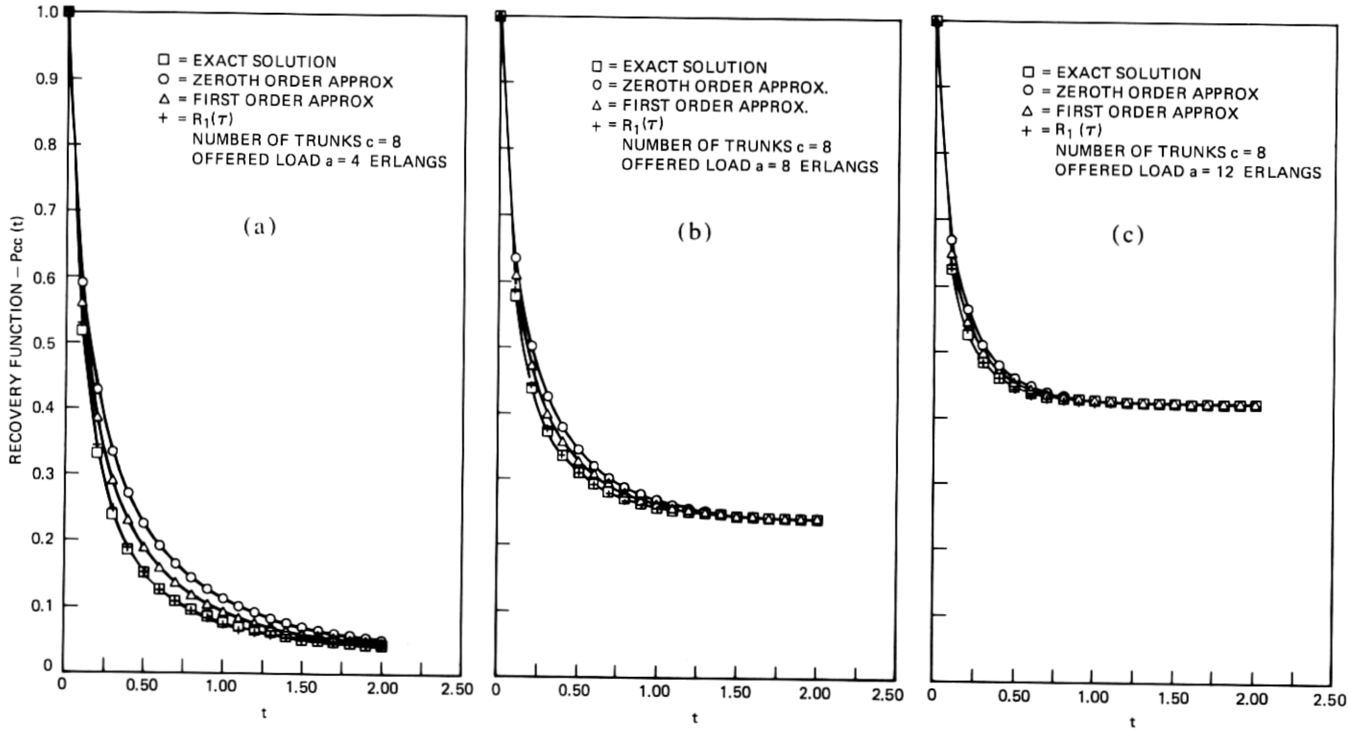

Fig. 2—Recovery function.

Table VII — $a = 12$

t	$P_c(t)$	$B + f_{0,\alpha}(t)$	$B + f_{1,\alpha}(t)$	$s_1(t)$
0	1.0000	1.0000	1.0000	1.0000
0.1	0.6245	0.6679	0.6493	0.6307
0.2	0.5255	0.5629	0.5458	0.5286
0.3	0.4834	0.5097	0.4969	0.4840
0.4	0.4611	0.4789	0.4698	0.4607
0.5	0.4479	0.4598	0.4535	0.4472
0.6	0.4396	0.4476	0.4427	0.4378
0.7	0.4342	0.4396	0.4366	0.4336
0.8	0.4306	0.4342	0.4322	0.4301
0.9	0.4282	0.4306	0.4292	0.4278
1.0	0.4265	0.4281	0.4272	0.4262

estimated by $2\epsilon_{1,\alpha}(t) - \epsilon_{0,\alpha}(t)$ in which the estimates for $\epsilon_{0,\alpha}(t)$, $\epsilon_{1,\alpha}(t)$ were used. The results obtained are given in Table VIII.

Table VIII — Error estimates at $t = 0.5$

a	$\epsilon_{0,\alpha}$	Estimate	$\epsilon_{1,\alpha}$	Estimate	$s_1 - f$	Estimate
4	0.0752	0.0714	0.0383	0.0372	0.0014	0.0030
8	0.0333	0.0317	0.0163	0.0160	−0.0008	0.0002
12	0.0120	0.0113	0.0056	0.0001	−0.0007	−0.0111

VIII. SOME APPLICATIONS OF THEOREM 12

The generating function, which will be designated $G(z, t)$, of Theorem 12, namely,

$$G(z, t) = \frac{1}{1-z} f_0\left(\frac{t}{1-z}\right) \tag{210}$$

may sometimes be used to obtain explicitly the form of $f_n(t)$. The following are some examples.

For $f(t) = \cos t$, one has $f_0(t) = 1/(1 + t^2)$, hence

$$G(z, t) = \frac{1-z}{(1-z)^2 + t^2}. \tag{211}$$

The generating function for the Chebyshev polynomials, $T_n(t)$, of first kind is[15]

$$\frac{1 - tz}{1 - 2tz + z^2} = \sum_{n=0}^{\infty} T_n(t) z^n \tag{212}$$

hence

$$G(z, t) = \sum_{n=0}^{\infty} z^n (1 + t^2)^{-(n+1)/2} T_n\left(\frac{1}{\sqrt{1+t^2}}\right). \tag{213}$$

One now obtains

$$f_n(t) = L_n \cos t = \left[1 + \left(\frac{t}{n+1}\right)^2\right]^{-(n+1)/2} T_n \left[\frac{1}{\sqrt{1 + \left(\frac{t}{n+1}\right)^2}}\right] \tag{214}$$

For $f(t) = \sin t$, one has $f_0(t) = t/(1 + t^2)$, hence

$$G(z, t) = \frac{t}{(1 - z)^2 + t^2}. \tag{215}$$

The generating function for the Chebyshev polynomials, $U_n(t)$, of second kind is

$$\frac{1}{1 - 2tz + z^2} = \sum_{n=0}^{\infty} U_n(t) z^n \tag{216}$$

hence

$$G(z, t) = \sum_{n=0}^{\infty} z^n (1 + t^2)^{-(n+1)/2} t U_n \left(\frac{1}{\sqrt{1 + t^2}}\right). \tag{217a}$$

Thus

$$f_n(t) = L_n \sin t = \frac{t}{n+1}\left[1 + \left(\frac{t}{n+1}\right)^2\right]^{-(n+1)/2} \times U_n \left[\frac{1}{\sqrt{1 + \left(\frac{t}{n+1}\right)^2}}\right]. \tag{217b}$$

The Bessel functions provide additional interesting relations with orthogonal polynomials. For $f(t) = J_0(t)$, one has $f_0(t) = 1/\sqrt{1 + t^2}$, and

$$G(z, t) = \frac{1}{\sqrt{(1 - z)^2 + t^2}}. \tag{218}$$

The Legendre polynomials, $P_n(t)$, are generated by

$$\frac{1}{\sqrt{1 - 2tz + z^2}} = \sum_{n=0}^{\infty} P_n(t) z^n, \tag{219}$$

hence

$$f_n(t) = L_n J_0(t) = \left[1 + \left(\frac{t}{n+1}\right)^2\right]^{-(n+1)/2} P_n \left[\frac{1}{\sqrt{1 + \left(\frac{t}{n+1}\right)^2}}\right]. \tag{220}$$

By the substitution of it for t, one derives immediately

$$f_n(t) = L_n I_0(t) = \left[1 - \left(\frac{t}{n+1}\right)^2\right]^{-(n+1)/2} \cdot$$

$$P_n\left[\frac{1}{\sqrt{1 - \left(\frac{t}{n+1}\right)^2}}\right]. \tag{221}$$

Since $I_0(t)$ is convex, one also has

$$I_0(t) \le L_n I_0(t) \tag{222}$$

for sufficiently large n.

As another example relating to Bessel functions, consider $f(t) = J_0(2\sqrt{t})$, then $f_0(t) = e^{-t}$ and

$$G(z, t) = \frac{1}{1-z} e^{-t/(1-z)}. \tag{223}$$

The generating function for the Laguerre polynomials, $L_n(t)$, is

$$\frac{1}{1-z} e^{-tz/(1-z)} = \sum_{n=0}^{\infty} L_n(t) z^n \tag{224}$$

hence

$$f_n(t) = L_n J_0(2\sqrt{t}) = e^{-t/(n+1)} L_n\left(\frac{t}{n+1}\right). \tag{225}$$

IX. SUMMARY

The methods of this paper have been found particularly useful in analyzing complex queueing phenomena whose Laplace transform representations are quite often implicitly defined. The error estimate of (112) has been found especially useful. Its computation is numerically effected by use of (158) and (159).

It would be desirable to have an effective method of estimating the α parameter of (114) directly from $f_n(t)$. In fact a method of this type which yields a rough evaluation has been devised and will be reported in a later paper. Of interest also would be further elaboration of the way structural properties of $f(t)$ are reflected in $f_{n,\alpha}(t)$.

The investigation of linear combinations of iterates, L_n^r, of the operators L_n may prove useful in providing additional enhancement methods. Especially, further investigation is needed concerning enhancement methods which preserve the positivity of the approximation process.

The isolated result of Theorem 16, which shows that $\exp(L_n \ell_n f(t))$ is a better approximation to $f(t)$ than $f_n(t)$ when $f(t)$ is log-convex, should be examined with the purpose of the possible construction of nonlinear approximation methods exploiting this structural characteristic.

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APPENDIX

Operations

$f(t)$	$f_0(t)$ or $f_n(t)$
$\frac{1}{t} f(t)$	$\frac{1}{t} \int_0^t f_0(x) \frac{dx}{x}$
$f(at), a > 0$	$f_n(at)$
$e^{\alpha t} f(t)$	$\left(1 - \frac{\alpha t}{n+1}\right)^{-n-1} f_n \left(\frac{t}{1 - \alpha t/(n+1)}\right)$
$\int_0^t f(x)dx$	$\frac{t}{n+1} \sum_{j=0}^{n} f_j \left(t \frac{j+1}{n+1}\right)$
$\dot{f}(t)$	$\frac{f_0(t) - f(0)}{t}, \frac{n+1}{t}\left[f_n(t) - f_{n-1}\left(\frac{n}{n+1} t\right)\right] \quad (n \geq 1)$
$\ddot{f}(t)$	$\frac{f_0(t) - f(0) - t\dot{f}(0)}{t^2}, 4\frac{f_1(t) - 2f_0(t/2) + f(0)}{t^2}$
	$(n+1)^2 \frac{f_n(t) - 2f_{n-1}\left(\frac{n}{n+1} t\right) + f_{n-2}\left(\frac{n-1}{n+1} t\right)}{t^2} (n \geq$
$tf(t)$	$tf_0(t) + t^2 \dot{f}_0(t)$
$t\dot{f}(t)$	$t\dot{f}_n(t)$
$\frac{1}{t} \int_0^t f(x)dx$	$\frac{1}{t} \int_0^t f_n(x)dx$
$f(t)*h(t)$(Mellin)	$f_n(t)*h(t)$

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