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Theory of Analytic Modulation Systems

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A general theory of analytic modulation systems is developed where the transmitted signal is of the form $\sigma(t) = Re\,\{e^{i\omega_c t} f(z(t))\}$. Here $f(z)$ is an analytic function (modulation law), and $z(t) = x(t) + iy(t)$ is the analytic baseband signal whose real part $x(t)$ is a bounded bandlimited signal of spectral support $[-\Omega, \Omega]$ which is assumed to have a bounded Hilbert transform $y(t)$. It is shown for a large class of $\{z(t)\}$ and modulation laws that $z(t)$ may be recovered using a receiver incorporating the inverse function of f as a detector with appropriate pre- and post-detection filtering. The theory also shows that in the procedure for factoring certain positive bandlimited signals, an approximate Hilbert transform operator (bandlimited) may be used. A related result is that signals subjected to logarithmic companding (one-sided) and filtering may be recovered by a non-iterative method.

I. INTRODUCTION

In 1962, Bedrosian proposed a modulation system called single sideband phase (or frequency) modulation.* (See Ref. 5.) The modulated signal is of the form

$$Re\,\{e^{i(\omega_c t + x(t) + iy(t))}\} = e^{-y(t)} \cos\,(\omega_c t + x(t))$$

where $x(t)$ is the "baseband" signal, $y(t)$ is the Hilbert transform of $x(t)$, and ω_c is the carrier frequency. The special relation between the amplitude modulation $e^{-y(t)}$ and the phase modulation $x(t)$ results in the modulated signal having no spectrum in the interval $(-\omega_c, \omega_c)$; i.e., the amplitude modulation removes the lower sideband. However, the spectrum is still infinite in extent. We adopt the terminology "single sideband exponential modulation" (SSBEM) for this system.

Bedrosian pointed out that SSBEM was compatible with conventional FM receivers, and suggested that the single-sideband system might offer

* K. H. Powers received a U. S. Patent (No. 3,054,073) on such a system shortly before the appearance of Bedrosian's paper. See Voelcker.[21]

some savings in transmission bandwidth over the conventional system. However, since filtering operations could radically alter the zero crossings of the modulated signal, it was not clear to what degree one could maintain compatibility and at the same time realize some saving in bandwidth.

Others[2,10,17] have compared the spectral distribution of single sideband and conventional frequency modulated signals for the cases of sinusoidal and Gaussian noise modulation. They have shown that the "effective" bandwidths, as measured by central second moments, of the single-sideband signals may be greater or less than that of the conventional FM signal depending on the nature of the modulation. At any rate, it is not clear how one would translate these results into relative bandwidth requirements of the two systems, each employing a conventional FM receiver.

Aside from the compatibility question, Barnard[4] has shown that the transmission bandwidth requirements of single-sideband exponential modulation are, in a strict sense, minimal. He showed that if the modulation $x(t)$ belonged to a certain subclass of bandlimited signals with spectral support $[-\Omega,\Omega]$, that the modulation could be recovered, within an additive constant, from a knowledge of the spectral distribution of the single-sideband signal in the interval $[\omega_c,\omega_c + \Omega + \epsilon]$, provided $\epsilon > 0$. This was proved by demonstrating the convergence of an iterative recovery scheme.

Here we consider a class of single-sideband systems wherein the modulated signal is of the form

$$s(t) = Re\,\{f(z(t))e^{i\omega_c t}\}$$

where $f(z)$, the "modulation law", is an analytic function and $z(t) = x(t) + iy(t)$ is the "analytic signal" of which, say, the real part $x(t)$ is the information to be transmitted. We suppose that $x(t)$ is bounded and bandlimited with spectral support $[-\Omega,\Omega]$ and that $s(t)$ is transmitted over a channel whose transmission function is the Fourier transform of an absolutely integrable function (impulse response) and is equal to unity over $(\omega_c,\omega_c + \alpha)$. It is shown under fairly weak conditions on $f(z)$ and $z(t)$ that $x(t)$ may be recovered (by a relatively simple non-iterative method) from the received signal, provided $\alpha > \Omega$.

The gist of the method can be grasped by considering periodic signals; e.g.,

$$x(t) = \frac{1}{2}\sum_{-n}^{n} X_k e^{ikt} \qquad (X_0 = 0,\ \text{say})$$

$$z(t) = \sum_{1}^{n} X_k e^{ikt}.$$

We suppose that sup $|z(t)| = m$, and $f(z)$, the modulation law, is analytic for $|z| \leq m, (f(0) = 0$, say)

$$f(z) = \sum_1^\infty a_k z^k, \ |z| \leq m.$$

Then setting $w(t) = f\{z(t)\}$ we have

$$w(t) = \sum_1^\infty W_k e^{ikt}$$

where W_k depends only on those X_j and a_j for which $1 \leq j \leq k$. This sort of dependence allows us to determine $x(t)$ from a bandlimited version of $w(t)$, say,

$$w_n(t) = \sum_1^n W_k e^{ikt},$$

by what amounts to reversion of power series, provided $f'(0) \neq 0$. We have

$$z = \phi(w) = \sum_1^\infty b_k w^k, \quad \text{for } |w| \text{ sufficiently small.}$$

$$(b_1 = (a_1)^{-1} = \{f'(0)\}^{-1})$$

Assuming that the series converges when w is replaced by $w_n(t)$ we set

$$z_n(t) = \phi\{w_n(t)\} = \sum_1^\infty b_k \{w_n(t)\}^k$$

and then by formal composition of the power series find that

$$z_n(t) = \sum_1^n X_k e^{ikt} + \sum_{n+1}^\infty c_k e^{ikt}.$$

So the first n Fourier coefficients of $z(t)$ and $z_n(t)$ agree; i.e., under the stated assumptions, $x(t)$ can be recovered by bandlimiting $\phi\{w_n(t)\}$, where ϕ is the inverse function, $z = \phi(w)$, and $w_n(t)$ is the partial sum of the Fourier series of $w(t)$.

The simplicity of this procedure owes to the fact that $z(t)$ has a one-sided spectrum and the *analytic* modulation law $f(z)$ then gives a function $w(t) = f\{z(t)\}$ which also has a one-sided spectrum. Since $z(t)$ contains no negative-frequency components, the usual difference terms do not appear; i.e., the spectrum of $w(t)$ in the frequency interval $[0,\alpha]$ depends only on the spectrum of $z(t)$ in the same interval.

The recovery procedure is not so transparent for more general bandlimited signals $x(t)$. First of all, filtering $w(t)$ with a filter whose transmission function is unity over $[0,\alpha]$ and zero for frequencies greater than

β ($\beta > \alpha$) will give a function $w_{\alpha,\beta}(t)$, analogous to $w_n(t)$, which may differ considerably from $w(t)$. It may be that $\phi\{w_{\alpha,\beta}(t)\}$ does not have a one-sided spectrum; i.e., $\phi\{w_{\alpha,\beta}(\tau)\}$, $\tau = t + iu$, is not analytic in the upper half-plane $u > 0$. Indeed $w_{\alpha,\beta}(t)$, $(-\infty < t < \infty)$, may not even be in the domain of definition of ϕ; i.e., ϕ may have a natural boundary beyond which it cannot be extended. Even if $\phi\{w_{\alpha,\beta}(t)\}$ does have a one-sided spectrum, the function ϕ need not have a power series representation over the range of $w_{\alpha,\beta}(t)$ so that one cannot use convolution arguments to show that the Fourier transforms of $\phi\{w(t)\}$ and $\phi\{w_{\alpha,\beta}(t)\}$ agree over $[0,\alpha]$. This particular problem is met by using generalizations of the Paley-Wiener theorem.

The problem arising when $\phi\{w_{\alpha,\beta}(t)\}$ does not have a one-sided spectrum ($w_{\alpha,\beta}(t)$ not in the range of the inverse function) is met by imposing restrictions on $z(t)$, namely that for sufficiently large u, the range of $z(t + iu)$ is sufficiently small that $w(t + iu)$ will be in the range of the inverse function. This implies that $w_{\alpha,\beta}(t)$ may be filtered (with a Poisson filter) to obtain $w_{\alpha,\beta}(t + ib)$, which for sufficiently large b will be in the range of the inverse function. One can then obtain $z(t + ib)$ and then use inverse Poisson filtering to recover $z(t)$.

Although one could conceivably recover $z(t)$ from $w_{\alpha,\beta}(t)$ by other procedures when $\phi\{w_{\alpha,\beta}(\tau)\}$ is not analytic for $u \geq b$, the method here avoids any decision process and gives a simple receiver model incorporating the inverse function and (possibly) a Poisson filter with its (bandlimited) inverse and appropriate low-pass filter.

Generally speaking, given $f(z)$ and the channel transmission function, one can design a receiver which will work for a certain subclass $\{z(t)\}$ of signals. Or given $f(z)$ and a fixed receiver design, one may ask for the minimum bandwidth channel required for transmitting a given subclass of signals. In this connection, some estimates are given for the bandwidth requirements of "compatible" single-sideband exponential modulation with $\{z(t)\}$ all functions of spectral support [0,1] satisfying sup $|z(t)| \leq m$. In a recent work, Werner[23] has considered the same problem for $z(t)$ in L_2 and gives upperbounds in terms of the L_2-norm of z.

There are rather dramatic mathematical simplifications in the detection theory when the signals $\{x(t)\}$ are restricted to be of the band-pass type, allowing radical changes in the system design.

The theory also shows that the factorization of certain positive bandlimited signals can be effected with an *approximate* Hilbert transform operator acting on the logarithm of the signal. A related result pertains to the signal recovery problem considered by Landau and Miranker[11,12]; viz., there is one companding function ("log") for which the signal can be recovered by a non-iterative method.

Another interesting consequence of the general theory is the fact that for n arbitrary numbers a_k, $k = 1,2,\ldots,n$, there exists an integer $\nu \geq$

n and corresponding numbers a_k, $k = n + 1, \ldots, \nu$, such that the polynomial

$$P_\nu(z) = 1 + \sum_1^\nu a_k z^k$$

is zero-free for $|z| < 1$.

Although the general theory is interesting from a mathematical viewpoint, it would appear that the practical interest in analytic modulation systems, other than the linear system, is limited to SSBEM, i.e., to the case $f(z) = e^z$ (or e^{iz}). In this case one can trade bandwidth for simplicity of detection. However, the trade-off is attractive only for moderate amplitudes of $z(t)$ where SSBEM offers an interesting alternative to other systems employing envelope detection.

It should be noted that the method here is naturally confined to bounded bandlimited signals $z(t)$, since otherwise we would require both $f(z)$ and its inverse $\phi(w)$ to be entire functions, i.e., $w = f(z) = a + bz$. The exception would be the band-pass case where f could be an entire function and ϕ replaced by an equivalent polynomial (see Section 5).

Of course, the theory here has to be extrapolated to practice with the appropriate "epsilons"; i.e., an analytic modulation law $f(z)$ can only be approximated within ϵ_1 over the disk $|z| \leq m$, and an analytic signal $z(t)$ having one-sided spectrum can be realized in practice within ϵ_2, and the impulse response of an ideal filter can be approximated (in L_1) within ϵ_3, etc. Then the continuity of the overall transformation may be used to bound the errors.

In order to deal rigorously with "communication type" signals which do not have ordinary Fourier transforms a considerable amount of preliminary mathematics is required. However, one can follow the theory assuming that the signals are either periodic or have ordinary Fourier transforms, with one cautionary note in mind. Abrupt bandlimiting operations (spectral projections) such as convolution with $\sin t/\pi t$ are not permissible (not defined) for the general signals of interest.

II. PRELIMINARIES

A measurable function $g(t)$ is said to belong to $L_p(-\infty,\infty)$ abbreviated hereafter as L_p, $(1 \leq p < \infty)$ if

$$\int_{-\infty}^{\infty} |g(t)|^p dt < \infty.$$

The L_p-norm of g is defined by

$$\|g\|_p = \left\{ \int_{-\infty}^{\infty} |g(t)|^p dt \right\}^{1/p}$$

and if α is a scalar

$$\|\alpha g\|_p = |\alpha| \cdot \|g\|_p.$$

If $|g(t)|$ is uniformly bounded, with the possible exclusion of a set of measure zero, g is said to belong to L_∞ and the norm of g is

$$\|g\|_\infty = \operatorname*{essup}_t |g(t)|$$

where "essup" over t is the essential supremum of $|g(t)|$, which is the infimum of numbers M such that

$$|g(t)| \leq M \text{ for almost all } t.$$

We will be mainly concerned with continuous bounded functions $g(t)$ in which case

$$\|g\|_\infty = \sup_t |g(t)|.$$

For $1 \leq p \leq \infty$, the L_p norm satisfies the *triangle inequality*

$$\|g_1 + g_2\|_p \leq \|g_1\|_p + \|g_2\|_p \tag{1}$$

which for any sequence of numbers a_k satisfying $\Sigma|a_k| < \infty$ and sequences of functions g_k such that $\|g_k\| \leq M$, leads to

$$\|\sum a_k g_k\|_p \leq \sum |a_k| \, \|g_k\|_p. \tag{2}$$

There are functions which belong to L_p for only one value of p. However, it is easy to see by considering the set where $|g(t)| \leq 1$ and the set where $|g(t)| > 1$, that if g belongs to L_r and L_s where $1 \leq r < s$, then g belongs to L_p for every p satisfying $r \leq p \leq s$. For example, the function $\sin t/t$ belongs to L_p for every $p > 1$.

Associated with the space L_p is the *conjugate* or *complementary* space L_q where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

For functions in complementary spaces we have *Hölder's inequality*

$$\left| \int_{-\infty}^{\infty} g(t)h(t)dt \right| \leq \|g\|_p \|h\|_q, \quad 1 \leq p \leq \infty, \tag{3}$$

which for the case $p = q = 2$ is the familiar *Schwarz's inequality.*

In connection with Hölder's inequality we note that the norm of a function may be equivalently defined as

$$\|g\|_p = \sup_h \left| \int_{-\infty}^{\infty} h(t)g(t)dt \right|, \quad \|h\|_q = 1, q = p/(p-1).$$

A convolution kernel K in L_1 carries L_p into L_p. We have

$$K \otimes g(t) = \int_{-\infty}^{\infty} g(s)K(t-s)ds \quad \begin{matrix} g \text{ in } L_p \\ K \text{ in } L_1 \end{matrix}$$

and

$$\|K \otimes g\|_p \leq \|K\|_1 \|g\|_p \tag{4}$$

which amounts to a generalization of (2) to weighted sums of translates of g. The convolution integral is not in general defined for each t unless K belongs to the conjugate space of g. In general the convolution is defined as the limit in L_p of

$$g_m(t) = \int_{-\infty}^{\infty} g(s)K_m(t-s)ds$$

where K_m is a sequence of bounded functions of L_1 (hence K_m belongs to L_q) satisfying

$$\lim_{m\to\infty} \int_{-\infty}^{\infty} |K(t) - K_m(t)|dt = 0.$$

2.1 The Fourier transform on L_p

A function g in L_p has an ordinary Fourier transform, provided $1 \leq p \leq 2$, (a theorem of M. Riesz, cf. Ref. 20) in the sense that

$$\hat{g}_T(\omega) = \int_{-T}^{T} g(t)e^{-i\omega t}dt$$

converges in norm as $T \to \infty$ to a function $\hat{g}(\omega)$ belonging to the complementary space L_q, i.e., there exist a function $\hat{g}$ in L_q such that

$$\lim_{T\to\infty} \|\hat{g} - \hat{g}_T\|_q = 0.$$

However the Fourier transform on L_p, $1 \leq p \leq 2$, does not carry L_p into all of L_q except in the case $p = 2$. In particular, the Fourier transform of a function g of L_1 is a continuous function. Furthermore, (the Riemann-Lebesgue Lemma)

$$\lim_{\omega\to\pm\infty} \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = 0$$

for g in L_1. Unfortunately, there is no simple description of functions $\hat{g}(\omega)$ which are the Fourier transforms of functions $g(t)$ of L_1. A useful sufficient cordition is that $\hat{g}(\omega)$ belong to L_2 and have a "derivative in L_2" (meaning only that $\hat{g}(\omega)$ is the integral of a function of L_2 denoted by $d\hat{g}/d\omega$). The sufficiency of this condition may be seen by writing

$$\int_{-\infty}^{\infty} |g(t)|dt = \int_{-\infty}^{\infty} \frac{1}{|t+ia|} \cdot |(t+ia)g(t)|dt \quad (a > 0)$$

and then applying Schwarz's inequality and Parseval's theorem. The result is (choosing the best value of a)

$$\left\{\int_{-\infty}^{\infty} |g(t)|dt\right\}^2 \le \|\hat{g}\|_2 \cdot \left\|\frac{d\hat{g}}{d\omega}\right\|_2. \tag{5}$$

In obtaining this result we use the fact that $\hat{g}$ in L_2 *and* $d\hat{g}/d\omega$ in L_2 imply that $\hat{g}(\omega)$ tends to zero at $\pm\infty$. (By Schwarz's inequality, $\hat{g}d\hat{g}/d\omega$ belongs to L_1, so $\{\hat{g}(\omega)\}^2$ is absolutely continuous and tends to limits at $\pm\infty$. The limits must be zero in order for $\hat{g}$ to belong to L_2.)

2.2 Bounded functions whose Fourier transforms vanish over certain sets

It is not necessary to attempt to define the Fourier transform of a bounded function $g(t)$ in order to give precise meaning to the statement that the Fourier transform of $g(t)$ vanishes over some open set E. This can be done in a way which is consistent with the ordinary Fourier transform, should it exist, of $g(t)$ vanishing over E. Here we restrict E to be the union of a finite number of disjoint open intervals.

Definition: The Fourier transform of a bounded function g is said to vanish over E if and only if

(*i*) $$\int_{-\infty}^{\infty} g(t)\bar{h}(t)dt = 0$$

for all h in L_1 *whose Fourier transforms satisfy*

(*ii*) $$\hat{h}(\omega) \equiv \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt = 0, \quad \omega \notin E.$$

This definition has its logical basis in Parseval's formula for functions of L_2. The bar over h in (*i*) denotes the complex conjugate of h. It is readily verified that (*i*) may be replaced by

(*iii*) $$\int_{-\infty}^{\infty} g(t)h(-t)dt = 0$$

which is more directly applicable to convolutions. That is, if the Fourier transform of g vanishes over E we have

$$(g \otimes h)(t) = \int_{-\infty}^{\infty} g(s)h(t-s)ds \equiv 0 \tag{6}$$

for all h in L_1 whose Fourier transforms vanish outside E.

We also note that in case the set E is symmetric with respect to the origin, $\bar{h}(t)$ in (*i*) may be replaced by $h(t)$. [See Ex. 4 below.]

We say that *the Fourier transforms of two bounded functions g_1 and g_2 agree over E if and only if the Fourier transform of $(g_1 - g_2)$ vanishes over E*. We also say that *g has spectral support E_c* (a closed set), or *the spectrum of g is confined to E_c*, meaning that *the Fourier transform of g vanishes over the complement of E_c*.

The following are some elementary consequences of the definition. The proofs are left as simple exercises. It is understood throughout that g, g_1, and g_2 are bounded functions.

Example 1. *Suppose all the intervals composing E are finite and $K(t)$ is any function of L_1 whose Fourier transform $\hat{K}(\omega)$ satisfies*

$$\hat{K}(\omega) = 1 \quad \text{for} \quad \omega \in E. \tag{7}$$

Let

$$g_2(t) = \int_{-\infty}^{\infty} g_1(s)K(t-s)ds. \tag{8}$$

Then the Fourier transforms of g_1 and g_2 agree over E.

Example 2. *If the Fourier transform of $g(t)$ vanishes over (α,β), then the Fourier transform of $e^{i\lambda t}g(t)$ vanishes over $(\alpha + \lambda, \beta + \lambda)$.*

Example 3. *If the Fourier transform of g_1 vanishes over E_1 and the Fourier transform of g_2 vanishes over E_2, then the Fourier transform of $(g_1 + g_2)$ vanishes over $E_1 \cap E_2$.*

Example 4. *If the Fourier transform of g vanishes over E, then the Fourier transform of the complex conjugate $\bar{g}$ vanishes over $E^{(-)}$, where $E^{(-)}$ denotes the reflection of E with respect to the origin.*

Example 5. *If the Fourier transform of g vanishes over E, then the Fourier transform of Re $\{g\}$ (or I_m $\{g\}$) vanishes over $E \cap E^{(-)}$.*

Note: $E \cap E^{(-)}$ may be the null set. However, if E is symmetric with respect to the origin, $E = E^{(-)}$. Hence a class of functions whose Fourier transforms vanish over a set E which is symmetric with respect to the origin is essentially a class of real-valued functions, since the real and imaginary parts of the functions separately belong to the class.

Example 6. (Reproducing Kernels) *Suppose the spectrum of g is confined to a set E_c consisting of n finite disjoint closed intervals. Let $K(t)$ be any function of L_1 whose Fourier transform $\hat{K}(\omega)$ satisfies*

$$\hat{K}(\omega) = 1, \quad \omega \in E_c. \tag{9}$$

Then for almost all t we have

$$g(t) = \int_{-\infty}^{\infty} g(s)K(t-s)ds. \tag{10}$$

(Set $g_1 = K \otimes g$ and show that $(g_1 - g)$ is orthogonal to all of L_1.)

Note: The qualification "almost all t" arises because the definition of a function g on a set of measure zero is irrelevant to the condition for its Fourier transform to vanish outside E_c. However, $(K \otimes g)(t)$ is a continuous function of t and so we will adopt the *convention* that *a function such as g in Ex. 6 is continuous.*

We note further that the condition that K in Ex. 6 belong to L_1 can be relaxed in case g belongs to L_p for some p satisfying $1 \leq p < \infty$. In this case we can take $\hat{K}(\omega) = 0$ for $\omega \notin E_c$. It is sufficient to prove this when E_c is a single interval $[-\Omega,\Omega]$ and this has been done (Ref. 14).

2.3 The Paley-Wiener Theorems for L_∞

There is an important connection between functions whose Fourier transforms vanish over a half-line and functions analytic and of exponential type in a half-plane. The following theorems are extensions to L_∞ of the classical "one-sided" and "two-sided" Paley-Wiener Theorems[18] for L_2.

Theorem 1. The Fourier transform of a bounded function g vanishes over $(-\infty,\alpha)$ if and only if $g(t)$ is the boundary value of a function $g(\tau)$, $\tau = t + iu$, analytic in the upper half-plane $u > 0$ and satisfying

$$\sup_t |g(t + iu)| \leq e^{-\alpha u} \sup_t |g(t)| \quad \text{for} \quad u \geq 0. \tag{11}$$

There is the analogous theorem connecting functions $g(t)$ whose Fourier transforms vanish over (β,∞) and functions $g(\tau)$ analytic in the lower half plane. The specialization of Theorem 1 to functions whose Fourier transforms vanish over $(-\infty,\alpha)$ and (β,∞) is the following. (We assume that $-\infty < \alpha < \beta < \infty$ and according to the convention above qualify g to be continuous.)

Theorem 2. The Fourier transform of a continuous bounded function g vanishes outside $[\alpha,\beta]$ if and only if $g(t)$ is the restriction to the real line of an entire function $g(\tau)$, $\tau = t + iu$, satisfying

$$\begin{aligned} \sup_t |g(t + iu)| &\leq e^{-\alpha u} \sup_t |g(t)|, \quad u \geq 0 \\ &\leq e^{-\beta u} \sup_t |g(t)|, \quad u \leq 0. \end{aligned} \tag{12}$$

These theorems are essential to the theory of single-sideband systems for bounded signals which do not have ordinary Fourier transforms.

Actually we do not need a uniform bound on the rate of growth (decay) of $g(t + iu)$ to infer that the Fourier transform of $g(t)$ vanishes over

$(-\infty,\alpha)$. In fact, an asymptotic bound implies a uniform bound.

Theorem 3. If $g(\tau)$ is analytic in the upper half-plane and satisfies

$$\sup_t |g(t+iu)| < \infty \quad \text{for} \quad u \geq 0$$

then the asymptotic estimate

$$\sup_t |g(t+iu)| = O\{e^{-\alpha u}\} \quad \text{as } u \to \infty$$

implies

$$\sup_t |g(t+iu)| \leq e^{-\alpha u} \sup_t |g(t)| \quad \text{for} \quad u \geq 0.$$

Proofs of Theorems 1, 2, and 3 are given in Appendix A.

We note the following corollaries of Theorems 1 and 2, concerning the Fourier transforms of products.

Corollary 1. If the Fourier transform of g_1 vanishes over $(-\infty,\alpha)$ and the Fourier transform of g_2 vanishes over $(-\infty,\beta)$, then the Fourier transform of g_1g_2 vanishes over $(-\infty,\alpha+\beta)$.

Corollary 2. If the Fourier transform of g_1 vanishes outside $[\alpha_1,\beta_1]$ and the Fourier transform of g_2 vanishes outside $[\alpha_2,\beta_2]$, then the Fourier transform of g_1g_2 vanishes outside $[\alpha_1+\alpha_2,\beta_1+\beta_2]$.

2.4 Terminology

Functions whose Fourier transforms vanish outside a finite interval are called *bandlimited* functions. Generally, we think of the interval centered at the origin and refer to bandlimited functions also as *low-pass* functions.

Functions whose Fourier transforms vanish over an interval centered at the origin are called *high-pass* functions, and functions which are both high-pass and low-pass are called *band-pass* functions.

Functions (signals) whose Fourier transforms vanish over a half-line, usually $(-\infty,0)$, are generally called *analytic* signals.

2.5 The Hilbert transform and the analytic signal

We would like to map the space of real-valued bounded signals $x(t)$ of spectral support $[-\Omega,\Omega]$ into the space of complex-valued bounded signals $z(t)$ of spectral support $[0,\Omega]$. We would like the mapping to be linear and also have the property that translates of x map into translates of z, so that no "time stretching" is involved. The usual way of doing this is to take

$$z(t) = x(t) + iy(t) \tag{13}$$

where $y = \tilde{x}$, the Hilbert transform of x.

The Hilbert transform is defined by

$$\tilde{x}(t) = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{x(s)}{t-s} ds \tag{14}$$

where the cut in the integral sign indicates a Cauchy principal value at $s = t$. The difficulty we encounter is that an arbitrary bounded band-limited function does not have a Hilbert transform so we have to restrict x somehow.

We may regard the Hilbert transform as the limit as $a \to 0$ of convolution transforms

$$\tilde{x}_a(t) = \int_{-\infty}^{\infty} x(s) K_a(t-s) ds \tag{15}$$

with regular kernels K_a given by

$$K_a(t) = \frac{1}{\pi} \frac{t}{t^2 + a^2}, \; a > 0. \tag{16}$$

Now $K_a(t)$ belongs to L_p for every $p > 1$ and has a Fourier transform $\hat{K}_a(\omega)$ given by

$$\hat{K}_a(\omega) = -i(\text{sgn } \omega) e^{-a|\omega|}. \tag{17}$$

Thus if $x(t)$ has a Fourier transform $\hat{x}(\omega)$, the Hilbert transform $\tilde{x}(t)$ has a Fourier transform given by

$$\int_{-\infty}^{\infty} \tilde{x}(t) e^{-i\omega t} dt = -i(\text{sgn } \omega)\hat{x}(\omega) \tag{18}$$

and consequently the Fourier transform of $z(t)$ as defined in (13) vanishes over $(-\infty,0)$. Now $z(t)$ is the boundary value of the function $z(\tau)$, $\tau = t + iu$, defined by

$$z(\tau) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{x(s)}{\tau - s} ds, \quad u > 0. \tag{19}$$

In case x does not have an ordinary Fourier transform, but has a bounded Hilbert transform $\tilde{x}$, the function $z(\tau)$ is bounded and analytic in the upper half-plane and according to our definition and Theorem 1, the Fourier transform of $z(t)$ vanishes over $(-\infty,0)$. Also if the Fourier transform of $x(t)$ vanishes outside $[-\Omega,\Omega]$ then the Fourier transform of $z(t)$ vanishes outside $[0,\Omega]$.

The subclass of bounded functions which have bounded Hilbert transforms does not have a simple alternate description. In order for $x(t)$ to have a bounded Hilbert transform it is sufficient that $x(t)$ have a bounded derivative and a bounded integral (Ref. 15). If the Fourier transform of x vanishes outside $[-\Omega,\Omega]$ then $x(t)$ has a bounded derivative ("Bernstein's Theorem," cf. Theorem 11.12, Ref. 6) satisfying

$$\sup_t |x'(t)| \le \Omega \sup_t |x(t)|, \tag{20}$$

and there are various ways to restict $x(t)$ to have a bounded integral. For example, we may assume that $g(t)$ is an arbitrary bandlimited signal and set

$$x(t) = g'(t). \tag{21}$$

(Also any high-pass function has a bounded integral (Ref. 14).) Then we can determine $g(t)$ within an additive constant from the real part of $z(t)$. Assuming that the Fourier transform of g vanishes outside $[-\Omega,\Omega]$ we have the inequality (Theorem 11.4.3, Ref. 6) implied by (21),

$$\sup_t |\tilde{x}(t)| \le \Omega \sup_t |g(t)|. \tag{22}$$

An interesting subclass of bandlimited functions which have bounded Hilbert transforms are the band-pass functions. For these functions, there are equivalent Hilbert transform kernels which belong to L_1. If the Fourier transform of $x(t)$ vanishes outside the intervals $[-\Omega,-r\Omega]$ and $[r\Omega,\Omega]$ where $0 < r < 1$, then x has a bounded Hilbert transform satisfying (Ref. 7)

$$\sup_t |\tilde{x}(t)| \le \left\{A + \frac{2}{\pi}\log\frac{1}{r}\right\} \sup_t |x(t)| \tag{23}$$

where $A < 4/\pi$ and $2/\pi$ cannot be replaced by a smaller number (i.e., as r approaches zero).

Other subclasses worthy of mentioning may be generated by convolution transforms on L_∞. Thus if $|g| \le M$ and k is a kernel in L_1 which has a Hilbert transform also in L_1 then the class of functions of the form

$$x(t) = \int_{-\infty}^{\infty} g(s)k(t-s)ds \tag{24}$$

have bounded Hilbert transforms given by

$$\tilde{x}(t) = \int_{-\infty}^{\infty} g(s)\tilde{k}(t-s)ds. \tag{25}$$

For example, $x(t)$ may be the output of some crude sort of band-pass filter (like an a-c amplifier) as would be the case for

$$\begin{aligned} k(t) &= ae^{-at} - be^{-bt}, \quad t \ge 0 \\ &= 0, \qquad t < 0 \end{aligned} \tag{26}$$

where $a > b > 0$. Then

$$\int_0^\infty k(t)e^{-i\omega t}dt = \frac{a}{a+i\omega} - \frac{b}{b+i\omega} = \frac{(a-b)i\omega}{(a+i\omega)(b+i\omega)} \tag{27}$$

and

$$\int_{-\infty}^{\infty} \tilde{k}(t)e^{-i\omega t}dt = \frac{(a-b)|\omega|}{(a+i\omega)(b+i\omega)}. \tag{28}$$

It follows from (28) and (5) that $\tilde{k}$ belongs to L_1.

Another sufficient condition for a bandlimited function $x(t)$ to have a bounded Hilbert transform is the requirement that x belong to L_p where $1 \le p < \infty$. Then if the Fourier transform of x vanishes outside $[-\Omega,\Omega]$ we have (Ref. 14)

$$x(t) = \int_{-\infty}^{\infty} x(s)\frac{\sin \Omega(t-s)}{\pi(t-s)}ds. \tag{29}$$

Then x has a Hilbert transform given by

$$\tilde{x}(t) = \int_{-\infty}^{\infty} x(s)\frac{1-\cos \Omega(t-s)}{\pi(t-s)}ds \tag{30}$$

and thus by Hölder's inequality

$$|\tilde{x}(t)| \le \|x\|_p \left\{\int_{-\infty}^{\infty} \left|\frac{1-\cos \Omega t}{\pi t}\right|^q dt\right\}^{1/q} \quad \text{where } \frac{1}{p}+\frac{1}{q}=1. \tag{31}$$

However, the condition that x belong to L_p $(1 \le p < \infty)$ is not a satisfactory condition for communication signals.

Hereafter we will suppose that $x(t)$ is so restricted that $z(t) = x(t) + iy(t)$ is a bounded function whose Fourier transform vanishes outside $[0,\Omega]$. We should note that this assumption does not imply that y is the Hilbert transform of x (even within an additive constant). That is, the assumption does not imply that the integral

$$\int_{-T}^{T} \frac{x(t)-x(0)}{t}dt$$

tends to a limit as $T \to \infty$, so in effect we are allowing some functions that are not of the form $z(t) = x(t) + i\tilde{x}(t)$. In case we further restrict $z(t)$ to be a bounded function whose Fourier transform vanishes outside $[r\Omega,\Omega]$ where $0 < r < 1$, then we can assert that $z(t)$ is of the form $x(t) + i\tilde{x}(t)$ where the Fourier transforms of x and $\tilde{x}$ vanish outside $[-\Omega,-r\Omega]$ and $[r\Omega,\Omega]$. Conversely if x is a real-valued (band-pass) function whose Fourier transform vanishes outside these two intervals, then $x(t)$ has a bounded Hilbert transform $\tilde{x}(t)$ and $\{x(t) + i\tilde{x}(t)\}$ is a bounded function whose Fourier transform vanishes outside $[r\Omega,\Omega]$. In other words, we can always regard *any* real-valued band-pass signal $x(t)$ as the real part of an analytic signal $z(t)$. This is a special case of a representation theorem for high-pass signals (Ref. 13).

III. MODULATION AND EQUIVALENT BASEBAND TRANSMISSION

We suppose that $z(t)$ is a bounded bandlimited signal with spectrum confined to $[0,\Omega]$. Now let $f(z)$ be a function which is analytic over a region which includes the disk $|z| \leq m$, where $m = \sup |z(t)|$. We take $f(z)$ as a *modulation law* and generate

$$w(t) = f\{z(t)\} \tag{32}$$

which is the boundary value of a function $w(\tau)$ bounded and analytic in the uhp. Hence the Fourier transform of $w(t)$ vanishes over $(-\infty,0)$. Generally, the modulation process also includes translation of the spectrum of $w(t)$ leading to a transmitter output

$$\sigma(t) = Re\,\{w(t)e^{i\omega_c t}\}. \tag{33}$$

where $\omega_c > 0$ is the carrier frequency. The Fourier transform of $\sigma(t)$ then vanishes over $(-\omega_c,\omega_c)$ and hence $\sigma(t)$ is called a single-sideband signal, although the upper side-band may be infinite in extent.

In conventional single-sideband amplitude modulation (SSBAM) the modulation law is the linear law, $f(z) = z$, in which case the Fourier transform of $\sigma(t)$ vanishes outside the intervals $[\omega_c,\omega_c + \Omega]$ and $[-\omega_c - \Omega, -\omega_c]$, so that the bandwidth required for transmitting $\sigma(t)$ is (counting positive and negative frequencies) $2\Omega + \epsilon$ where ϵ is an arbitrarily small positive number. In other words, $\sigma(t)$ has a reproducing kernel in L_1 of bandwidth slightly larger than 2Ω. (Recall that the Fourier transform of a function of L_1 is continuous.) We will see that the spectral economy of SSBAM carries over to more general modulation laws $f(z)$. So we assume that $\sigma(t)$ is transmitted over a channel (characterized by an L_1 impulse response) which has unity transmission over the frequency bands $[\omega_c,\omega_c + \alpha]$ and $[-\omega_c - \alpha,-\omega_c]$ where $\alpha > \Omega$. The transmission may be zero outside slightly larger intervals.

We denote the received signal by $\sigma_R(t)$ and since its Fourier transform vanishes over $(-\omega_c,\omega_c)$ it has a Hilbert transform $\tilde{\sigma}_R(t)$. We assume that the carrier frequency and phase are known at the receiver so that we can form

$$w_\alpha(t) = e^{-i\omega_c t}\{\sigma_R(t) + i\tilde{\sigma}_R(t)\}. \tag{34}$$

In engineering parlance the real part of $w_\alpha(t)$ is obtained by in-phase synchronous demodulation of $\sigma_R(t)$, while the imaginary part of $w_\alpha(t)$ is obtained by quadrature synchronous demodulation of $\sigma_R(t)$. The Fourier transform of $w_\alpha(t)$ vanishes over $(-\infty,0)$ and we have

$$w_\alpha(t) = \int_{-\infty}^{\infty} w(s)K_\alpha(t-s)ds \tag{35}$$

where $K_\alpha(t)$ is the impulse of an equivalent baseband channel satisfying for some $\alpha > \Omega$

$$\int_{-\infty}^{\infty} K_\alpha(t) e^{-i\omega t} dt = 1, \quad \text{for } 0 \leq \omega \leq \alpha \tag{36}$$

$$\int_{-\infty}^{\infty} |K_\alpha(t)| dt < \infty. \tag{37}$$

Hereafter we will be concerned with recovering $z(t)$, and hence $x(t)$, from $w_\alpha(t)$ in the equivalent baseband transmission of $w(t)$ as given by (35).

IV. THE INVERSE FUNCTION AS A DETECTOR

We would like to solve (35) for $z(t)$ where $w(t) = f\{z(t)\}$. This is (superficially) similar to the problem of Landau and Miranker[11,12] where $w(t) = f\{x(t)\}$ and f is a real function of a real variable, $x(t)$ is a real-valued bandlimited function of L_2 whose Fourier transform vanishes outside $[-\Omega,\Omega]$, and $K_\alpha(t) = (\sin \Omega t)/\pi t$. In order for $x(t)$ to be recovered (by an iterative process) they require that f have an inverse over the range of x and that is essentially what we require. The inverse of an analytic f is more complicated, but the fact that the Fourier transforms of $z(t)$ and $f\{z(t)\}$ vanish over $(-\infty,0)$ simplifies the recovery problem.

Let us write

$$w = f(z) \tag{38}$$

and

$$z = \varphi(w) \tag{39}$$

for the inverse and think first of the problem of recovering $z(t)$ from $w(t)$. In case f maps $|z| \leq m$ one-one onto some region D^*, there is no problem since φ is single valued over D^*. In general φ is not single valued and we have to know something about $z(\tau)$ in order to decide what element of φ is "the" inverse. For example, suppose $f(z) = 2z + z^2$. Then given

$$w(t) = 2ae^{it} + a^2 e^{i2t}$$

we do not know whether $z(t) = ae^{it}$ or $z(t) = -2 - ae^{it}$ without some additional knowledge, such as for example, $\lim_{u\to\infty} z(t + iu) = 0$, or $|z(t)| \leq 1$. In this example the inverse function,

$$z = \varphi(w) = -1 + (1 + w)^{1/2}$$

is not a single-valued function of the complex variable w and one generally speaks of two branches of the inverse function. The branches have singularities at $w = -1$, the image of $z = -1$ where $f'(z) = 0$. Clearly in this case, if we require $|z(t)| \leq 1$ then we know

$$z(t) = -1 + \sqrt{1 + w(t)}$$

where $\sqrt{1} = 1$. Then $\varphi\{w(t + iu)\}$ is bounded for $u \geq 0$ and analytic for $u > 0$. If we relax the requirement to $|z(t + iu)| \leq 1$ for $u > b$ then $\varphi\{w(t + iu)\}$ will be analytic for $u > b$, but not necessarily for $u > 0$.

In general, *we require that $z(\tau)$ and $f(z)$ are so constrained that $\varphi\{w(\tau)\}$ is analytic for $u \geq u_0$.*

4.1 Received signal in the range of the inverse function

Now we are not given $w(t)$ but instead we have a filtered version $w_\alpha(t)$. Suppose $\varphi\{w(\tau)\}$ is analytic in the upper half-plane $u \geq 0$ and $w_\alpha(t)$ is sufficiently close to $w(t)$ that $\varphi\{w_\alpha(\tau)\}$ is also analytic in the uhp $u \geq 0$. We say then that $w_\alpha(t)$ *is in the range of the inverse function.* A simple sufficient condition for this is that φ be an entire function. Also the channel could have sufficiently large bandwidth for $w_\alpha(t)$ to be close enough to $w(t)$.

We assume then that

$$w_\alpha(t + iu) \in D^*, \quad u \geq 0 \tag{40}$$

where

$$\varphi(w) \text{ is analytic for } w \in D^* \tag{41}$$

$$|\varphi'(w)| \leqslant M \text{ for } w \in D^* \tag{42}$$

Then we may take the inverse of $w_\alpha(t)$ to obtain

$$z_\alpha(t) = \varphi\{w_\alpha(t)\}. \tag{43}$$

Now we will see that the Fourier transforms of $z_\alpha(t)$ and $z(t)$ agree over $(-\infty,\alpha)$.

First, it follows from (35)–(37) and Ex. 1 of Sec. 2.2 that the Fourier transforms of $w(t)$ and $w_\alpha(t)$ agree over $(-\infty,\alpha)$; i.e., the Fourier transform of $\{w(t) - w_\alpha(t)\}$ vanishes over $(-\infty,\alpha)$. Then from Theorem 1 we have

$$|w(t + iu) - w_\alpha(t + iu)| \leq e^{-\alpha u} \sup_t |w(t) - w_\alpha(t)|. \tag{44}$$

From (42) we have

$$|\varphi(w) - \varphi(w_\alpha)| \leq M|w - w_\alpha| \text{ for } w \in D^*, \; w_\alpha \in D^* \tag{45}$$

Thus $\{z(t) - z_\alpha(t)\}$ is the boundary value of a function bounded and analytic in the uhp satisfying

$$|z(t + iu) - z_\alpha(t + iu)| \leq Me^{-\alpha u} \sup_t |w(t) - w_\alpha(t)|. \tag{46}$$

Hence from Theorem 1,

$$z(t) - z_\alpha(t) = h_\alpha(t) \tag{47}$$

where the Fourier transform of $h_\alpha(t)$ vanishes over $(-\infty,\alpha)$. Since the Fourier transform of $z(t)$ vanishes outside $[0,\Omega]$ and $\alpha > \Omega$, (47) implies that we can bandlimit $z_\alpha(t)$ with an appropriate low-pass filter to obtain $z(t)$. Thus if $K_{\Omega,\alpha}(t)$ is any kernel of L_1 satisfying

$$\int_{-\infty}^{\infty} K_{\Omega,\alpha}(t)e^{-i\omega t}dt = \begin{cases} 1, & 0 \leq \omega \leq \Omega \\ 0, & \omega \geq \alpha \end{cases} \tag{48}$$

we have from Ex. 6, Sec. 2.2, with the convention that $z(t)$ is continuous,

$$z(t) = \int_{-\infty}^{\infty} z(s)K_{\Omega,\alpha}(t-s)ds \tag{49}$$

and since the Fourier transform of h_α vanishes over $(-\infty,\alpha)$ we have $(K_{\Omega,\alpha} \otimes h_\alpha)(t) \equiv 0$; i.e.,

$$z(t) = \int_{-\infty}^{\infty} z_\alpha(s)K_{\Omega,\alpha}(t-s)ds. \tag{50}$$

4.2 Pre-detection filtering

In case the received signal $w_\alpha(t)$ is not in the range of the inverse function, we may under suitable conditions recover $z(t)$ by appropriate filtering before (and after) detection. Here then we replace (40) with the condition

$$w(t+iu) \in D^* \quad \text{for } u \geq u_0 \qquad (\geq 0). \tag{51}$$

It follows from (44) and (42) that for sufficiently large b we have

$$w_\alpha(t+iu) \in D_1^* \quad \text{for } u \geq b \tag{52}$$

where D_1^* is slightly larger than D^* and

$$\varphi(w) \text{ is analytic for } w \in D_1^* \tag{53}$$

$$|\varphi'(w)| < M_1 \text{ for } w \in D_1^*. \tag{54}$$

Then we have $w_\alpha(t+ib)$ in the range of the inverse function.

Now the Poisson kernel with parameter u

$$P_u(t) = \frac{1}{\pi}\frac{u}{t^2+u^2}, \quad u > 0 \tag{55}$$

reproduces functions bounded and analytic in the uhp from their boundary values. (See the proof of Theorem 1 in Appendix A.) We have

$$w_\alpha(t + ib) = \int_{-\infty}^{\infty} w_\alpha(s)P_b(t - s)ds. \qquad (56)$$

That is, we may determine $w_\alpha(\tau)$ along a line $u = b$ parallel to the real axis by convolving $w_\alpha(t)$ with the Poisson kernel (parameter $u = b$). This operation we term *Poisson filtering.* Since, by assumption, $w_\alpha(t + ib)$ is in the range of the inverse we may take the inverse of $w_\alpha(t + ib)$ to obtain

$$z_\alpha(t + ib) = \varphi\{w_\alpha(t + ib)\} \qquad (57)$$

which is analytic in the uhp and then as argued before

$$|z(t + iu + ib) - z_\alpha(t + iu + ib)| \le M_2 e^{-\alpha u}. \qquad (58)$$

So the Fourier transforms of $z(t + ib)$ and $z_\alpha(t + ib)$ agree over $(-\infty,\alpha)$.

Thus if the conditions (51), (41), and (42) are met we may by suitable pre-detection filtering (Poisson filtering) obtain a function $z_\alpha(t + ib)$ which corresponds to replacing $z(t)$ at the transmitter by $z(t + ib)$; i.e., from the reproducing property of the Poisson kernel

$$w(t + iu) = \int_{-\infty}^{\infty} w(s)P_u(t - s)ds$$

$$= f\{z(t + iu)\} = \int_{-\infty}^{\infty} f\{z(s)\}P_u(t - s)ds. \qquad (59)$$

Then interchanging the order of convolutions with $P_b(t)$ and $K_\alpha(t)$ in (35) we have

$$w_\alpha(t + ib) = \int_{-\infty}^{\infty} f\{z(s + ib)\}K_\alpha(t - s)ds. \qquad (60)$$

This relation has been noted by Foschini.[8]

The Poisson kernel is a contraction operator; i.e., it averages the values of a function so that the range of the resultant is no larger than the range of the function. Also as a filter it has the frequency response

$$\int_{-\infty}^{\infty} P_u(t)e^{-i\omega t}dt = e^{-u|\omega|}, \quad (u > 0). \qquad (61)$$

We have

$$z(t + ib) = \int_{-\infty}^{\infty} z(s)P_b(t - s)ds \qquad (62)$$

and therefore for large b we would expect the range of $z(t + ib)$ to be appreciably less than the range of $z(t)$ for a wide class of $z(t)$ since for very large b it is only the low-frequency content of $z(t)$ that contributes appreciably to $z(t + ib)$. It is possible, though, for the low-frequency content to be such that, for every $u \geq 0$

$$z(t + iu) \sim \cos \sqrt{t} + i \sin \sqrt{t} \quad \text{as } t \to \infty.$$

If we require $z(t + iu)$ to tend uniformly to a limit z_0 as $u \to \infty$; i.e.,

$$|z(t + iu) - z_0| \leq \epsilon(u), \quad -\infty < t < \infty, \tag{63}$$

where $\epsilon(u) \to 0$ as $u \to \infty$ and in addition

$$f'(z_0) \neq 0, \tag{64}$$

then $\varphi(w)$ will be analytic in the neighborhood of $w_0 = f(z_0)$ and so (51) will be satisfied for sufficiently large u_0 with (41) and (42) holding.

The condition (63) is not a severe constraint. In fact, all the simple sufficient conditions, given in (21)–(29), for $x(t)$ to have a Hilbert transform imply (63) with $z_0 = 0$. However, $x(t)$ may have a Hilbert transform without (63) holding.

In connection with pre-detection filtering, we note that equivalent Poisson filtering can be effected at the carrier frequency (or an intermediate frequency) in the receiver before the synchronous demodulation of the received signal $\sigma_R(t)$ indicated in (34). That is, if the signal $\sigma_R(t)$ is passed through a filter whose frequency response $F_b(\omega)$ satisfies

$$F_b(\omega) = e^{-b(\omega - \omega_c)}, \quad \omega \geq \omega_c \tag{65}$$

$$F_b(-\omega) = \overline{F}_b(\omega)$$

a signal $\sigma_R(t;b)$ is obtained which we may identify as

$$\sigma_R(t;b) = Re\,\{e^{i\omega_c t} w_\alpha(t + ib)\}. \tag{66}$$

Then synchronous in-phase and quadrature detection of $\sigma_R(t;b)$ yields the real and imaginary parts of $w_\alpha(t + ib)$. Thus the Poisson filtering may be accomplished with a single equivalent frequency-translated filter, whereas the direct Poisson filtering of the complex signal $w_\alpha(t)$ requires two Poisson filters acting separately on the real and imaginary parts.

4.3 Post-detection filtering

When Poisson filtering is required to bring the received analytic signal within the range of the inverse function the low pass filtering after detection must be modified. The output of the detector is $z_\alpha(t + ib)$ and we need $z(t)$. Now the Fourier transforms of $z(t + ib)$ and $z_\alpha(t + ib)$ agree over $(-\infty,\alpha)$ and

$$z(t + ib) = \int_{-\infty}^{\infty} z(s) P_b(t - s)\,ds. \tag{67}$$

The Poisson filtering operation has an inverse for bandlimited functions; i.e.,

$$z(t) = \int_{-\infty}^{\infty} z(s + ib)Q_b(t - s)ds \tag{68}$$

where $Q_b(t)$ is any function of L_1 satisfying

$$\int_{-\infty}^{\infty} Q_b(t)e^{-i\omega t}dt = e^{b\omega}, \quad 0 \leq \omega \leq \Omega. \tag{69}$$

Since the Fourier transforms of $z(t + ib)$ and $z_\alpha(t + ib)$ agree over $(-\infty,\alpha)$, the Fourier transform of

$$z(t) - \int_{-\infty}^{\infty} z_\alpha(s + ib)Q_b(t - s)ds \tag{70}$$

vanishes over $(-\infty,\alpha)$, and since the Fourier transform of $z(t)$ vanishes outside $[0,\Omega]$, we have [cf. (47)–(50)]

$$z(t) = \int_{-\infty}^{\infty} z_\alpha(s + ib)k(t - s)ds \tag{71}$$

where $k(t) \equiv k(t;b,\Omega,\alpha)$ is any kernel in L_1 satisfying

$$\int_{-\infty}^{\infty} k(t)e^{-i\omega t}dt = e^{\omega b}, \quad 0 \leq \omega \leq \Omega \tag{72}$$

$$= 0, \qquad \omega \geq \alpha.$$

That is, the post-detection filtering must invert the pre-detection filtering over the band $[0,\Omega]$ and remove frequencies greater than α. (Of course, in a practical system we are interested in recovering only $x(t)$ so that only one post-detection filter is required, acting on the real part of $z_\alpha(t + ib)$.)

V. SPECIALIZATION TO BAND-PASS SIGNALS

In case the base-band signal $x(t)$ is of the band-pass type, i.e., a signal whose Fourier transform vanishes outside $[r\Omega,\Omega]$ and $[-\Omega,-r\Omega]$ where $0 < r < 1$, the detection theory may be modified so that no Poisson filtering is required. In this case the inverse function may be replaced by an entire function, in particular, a polynomial. All we require for the recovery of band-pass signals is that $f\{z(\tau)\}$ be analytic in the uhp and

$$f'(0) \neq 0. \tag{73}$$

Then

$$w = f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{for } |z| \text{ sufficiently small} \tag{74}$$

$$z = \varphi(w) = \sum_{k=1}^{\infty} b_k (w - a_0)^k \quad \text{for } |w - a_0| \text{ sufficiently small.} \tag{75}$$

Here $b_1 = (a_1)^{-1} = [f'(0)]^{-1}$.

Now for band-pass signals $x(t)$, the Fourier transform of the analytic signal $z(t)$ vanishes over $(-\infty, r\Omega)$ where $\Omega > 0$ and $0 < r < 1$. Thus by the Paley-Wiener Theorem for L_∞,

$$|z(t + iu)| \le e^{-r\Omega u} \sup_t |z(t)|, \quad u \ge 0. \tag{76}$$

Then we have

$$w(t + iu) - a_0 = \sum_{k=1}^{\infty} a_k \{z(t + iu)\}^k \quad \text{for sufficiently large } u. \tag{77}$$

It follows that the Fourier transform of $\{w(t) - a_0\}$ also vanishes over $(-\infty, r\Omega)$. Hence the Fourier transform of $\{w_\alpha(t) - a_0\}$ vanishes over $(-\infty, r\Omega)$.

Now let $\varphi^*(w)$ be any entire function of the form

$$\varphi^*(w) = \sum_{k=1}^{\infty} c_k (w - a_0)^k \tag{78}$$

where

$$c_k = b_k, \quad \text{for } k = 1, 2, \ldots, n \tag{79}$$

and n is an integer such that

$$nr \ge 1. \tag{80}$$

Defining

$$z_\alpha^*(t) = \varphi^*\{w_\alpha(t)\} \tag{81}$$

we have $z_\alpha^*(\tau)$ analytic in the uhp and for sufficiently large u

$$\begin{aligned} z(t + iu) - z_\alpha^*(t + iu) &= \varphi\{w(t + iu)\} - \varphi^*\{w_\alpha(t + iu)\} \\ &= \sum_{k=1}^{n} b_k [\{w(t + iu) - a_0\}^k - \{w_\alpha(t + iu) - a_0\}^k] \\ &\quad + \sum_{k=n+1}^{\infty} b_k \{w(t + iu) - a_0\}^k \\ &\quad - \sum_{k=n+1}^{\infty} c_k \{w_\alpha(t + iu) - a_0\}^k. \end{aligned} \tag{82}$$

Since the Fourier transforms of $\{w(t) - a_0\}$ and $\{w_\alpha(t) - a_0\}$ vanish over $(-\infty, r\Omega)$, the last two sums in (82) are of the order of $\exp\{-(n + 1)r\Omega u\}$.

Also, for $k \geq 1$

$$\begin{aligned} &|\{w(t+iu)-a_0\}^k - \{w_\alpha(t+iu)-a_0\}^k| \\ &= O\{|w(t+iu)-w_\alpha(t+iu)|\} \\ &= O(e^{-\alpha u}). \end{aligned} \tag{83}$$

So it follows from the Paley-Wiener Theorem (with Theorem 3) that the Fourier transforms of $z(t)$ and $z_\alpha^*(t)$ agree over $(-\infty, B)$ where

$$B = \min\{\alpha, (n+1)r\Omega\} > \Omega. \tag{84}$$

Therefore

$$z(t) = \int_{-\infty}^{\infty} z_\alpha^*(s) K_{\Omega,B}(t-s)\,ds \tag{85}$$

where $K_{\Omega,B}$ is any function of L_1 satisfying

$$\begin{aligned} \int_{-\infty}^{\infty} K_{\Omega,B}(t) e^{-i\omega t} dt &= 1, \quad 0 \leq \omega \leq \Omega \\ &= 0, \quad \omega \geq B. \end{aligned} \tag{86}$$

Thus for band-pass signals we have the option of replacing the inverse function $\varphi(w)$ by an equivalent entire function $\varphi^*(w)$ so that it does not matter whether or not the received analytic signal $w_\alpha(t)$ is in the range of the inverse function $\varphi(w)$. In particular, $\varphi^*(w)$ may be a polynomial of degree n where n is roughly the ratio of the upper and lower cut-off frequencies of the base-band signal.

VI. DETECTION OF EXPONENTIAL MODULATION

The exponential modulation law $f(z) = e^z$ offers the unique advantage of eliminating the need for preliminary in-phase and quadrature detection of the received single-sideband signal $\sigma_R(t)$. In this case we have

$$z = \varphi(w) = \log w \tag{87}$$

or using Log to denote the real part of the logarithm,

$$x(t) = \mathrm{Log}\,|w(t)| \tag{88}$$

$$y(t) = \arg\{w(t)\}. \tag{89}$$

We may regard either $x(t)$ or $y(t)$ as the signal to be recovered. The transmitted signal is

$$\sigma(t) = Re \exp\{i\omega_c t + z(t)\} \tag{90}$$

and

$$\sigma(t) + i\tilde{\sigma}(t) = \exp\{i\omega_c t + z(t)\} \tag{91}$$

where $\tilde{\sigma}(t)$ denotes the Hilbert transform of $\sigma(t)$. The envelope of $\sigma(t)$ is

$$E\{\sigma(t)\} = |\sigma(t) + i\tilde{\sigma}(t)| = e^{x(t)} \tag{92}$$

The instantaneous phase of $\sigma(t)$ is

$$A\{\sigma(t)\} = \arg\{\sigma(t) + i\tilde{\sigma}(t)\} = \omega_c t + y(t) \tag{93}$$

For perfect transmission; i.e., $\sigma_R(t) = \sigma(t)$, we have

$$x(t) = \text{Log}\, E\{\sigma(t)\} \tag{94}$$

and using an ideal discriminator (FM detector) we obtain

$$y'(t) = \frac{d}{dt} A\{\sigma(t)\} - \omega_c. \tag{95}$$

Replacing $\sigma(t)$ by $\sigma_R(t)$ we have

$$x_\alpha(t) = \text{Log}\, E\{\sigma_R(t)\} = \text{Log}\, |w_\alpha(t)| \tag{96}$$

$$y'_\alpha(t) = \frac{d}{dt} A\{\sigma_R(t)\} - \omega_c = \frac{d}{dt} \text{Arg}\,\{w_\alpha(t)\}. \tag{97}$$

Now if $w_\alpha(\tau)$ is zero-free in the uhp, then

$$z_\alpha(\tau) = \log w_\alpha(\tau) \tag{98}$$

is analytic in the uhp and by the previous theory the Fourier transforms of $z(t)$ and $z_\alpha(t)$ agree over $(-\infty,\alpha)$. In this case the Fourier transforms of $x_\alpha(t)$ and $x(t)$ agree over $(-\alpha,\alpha)$. Also the Fourier transforms of $y'_\alpha(t)$ and $y'(t)$ agree over $(-\alpha,\alpha)$. So if the received analytic signal $w_\alpha(t)$ is zero-free in the uhp, $x(t)$ may be recovered by taking the Log of the envelope of the received signal and then filtering with an ideal low-pass filter having unity transmission in the band $[-\Omega,\Omega]$ and zero transmission outside the band $[-\alpha,\alpha]$. Similarly $y'(t)$ may be recovered by filtering the output of an ideal discriminator acting on the received signal.

Later, in examining the bandwidth requirements of single-sideband exponential modulation we give sufficient conditions for $w_\alpha(t)$ to be zero-free in the uhp so that the simple detectors described above may be used.

The simple detectors can always be used with appropriate pre-detection and post-detection filters, since for sufficiently large b, $w_\alpha(\tau + ib)$ will be zero-free in the uhp. We can give an estimate for b under the condition

$$\sup_t |z(t)| \le m. \tag{99}$$

We have

$$e^{-m} \le |w(t)| \le e^m \tag{100}$$

and consequently

$$e^{-m} \leq |w(t + iu)| \leq e^{m}, \quad u \geq 0. \tag{101}$$

Also, $w(\tau)$ is zero-free in the uhp and hence [cf. (219)] if

$$|w(t + ib) - w_\alpha(t + ib)| < e^{-m} \tag{102}$$

then $w_\alpha(\tau + ib)$ will be zero-free in the uhp. From (44) we have

$$|w(t + ib) - w_\alpha(t + ib)| \leq e^{-\alpha b} \sup_t |w(t) - w_\alpha(t)| \tag{103}$$

and since

$$w_\alpha(t) = \int_{-\infty}^{\infty} w(s) K_\alpha(t - s) ds \tag{104}$$

we have

$$\sup_t |w_\alpha(t)| \leq \sup_t |w(t)| \cdot \|K_\alpha\|_1 \leq e^{m} \|K_\alpha\|_1 \tag{105}$$

and hence

$$\sup |w(t) - w_\alpha(t)| \leq \{1 + \|K_\alpha\|_1\} e^{m}. \tag{106}$$

Thus if

$$e^{-\alpha b}\{1 + \|K_\alpha\|_1\} < e^{-2m} \tag{107}$$

then (102) will be satisfied. That is, $w_\alpha(\tau + ib)$ will be zero-free in the uhp for

$$b > \frac{2m + \log \{1 + \|K_\alpha\|_1\}}{\alpha} \tag{108}$$

Thus if (99) is satisfied we may use a frequency-translated Poisson filter with parameter b satisfying (108) to obtain $\sigma_R(t;b)$ [cf. (66)] and then operate on the envelope and phase of $\sigma_R(t;b)$ as before. Then the appropriate post-detection filtering may be employed to recover $x(t)$ and $y'(t)$.

VII. NOTE ON THE FACTORIZATION OF CERTAIN POSITIVE FUNCTIONS

The detection theory for SSBEM has important application to the problem of factoring certain positive functions of exponential type, i.e., certain positive bandlimited functions.

Voelcker[22] has proposed a scheme for demodulating conventional single-sideband signals via envelope detection. Conventional SSBAM is characterized by linear modulation; i.e., $f(z) = z$. There is no bandwidth expansion so we assume that the Fourier transform of $z(t)$ vanishes outside $[0,\Omega]$ and the channel is such that the received signal is simply the transmitted signal; i.e.,

$$\sigma_R(t) = \sigma(t) = Re\, e^{i\omega_c t} z(t). \tag{109}$$

The envelope of the received signal is $|z(t)|$. Voelcker's scheme requires first that $z(\tau)$ be zero-free in the uhp in order that $z(t)$ may be recovered from $|z(t)|$. This is insured by requiring $Re\, z(t) = x(t) > 0$. We have

$$z(t + iu) = \int_{-\infty}^{\infty} z(s) P_u(t - s) ds \tag{110}$$

and since the Poisson kernel is positive,

$$Re\, z(t + iu) = \int_{-\infty}^{\infty} x(s) P_u(t - s) ds > 0. \tag{111}$$

Therefore $z(\tau)$ is zero-free in the uhp. The function $\log z(\tau)$ is analytic in the uhp and with some additional conditions on $z(\tau)$, e.g.,

$$\lim_{u \to \infty} z(t + iu) = 1, \tag{112}$$

the imaginary part of $\log z(t)$ can be determined from $\mathrm{Log}\, |z(t)|$ and hence $z(t)$ can be recovered from $|z(t)|$. In particular, if

$$x(t) = 1 + g(t), \tag{113}$$

where

$$g(t) > -1 \text{ and } g(t) \text{ belongs to } L_p \; (1 \leq p < \infty), \tag{114}$$

then $\mathrm{Log}\, |z(t)|$ will belong to L_p and will therefore have a Hilbert transform. A more attractive condition for recovering $x(t)$ is the condition

$$g(t) > -1 \text{ and } g(t) \text{ of band-pass type.} \tag{115}$$

Then if (115) is satisfied, the Fourier transform of $\{z(t) - 1\}$ vanishes outside $[r\Omega,\Omega]$, where $0 < r < 1$, and hence

$$w(\tau) = \log z(\tau) \tag{116}$$

is analytic in the uhp and satisfies

$$\begin{aligned} w(t + iu) &= \log\, [1 + \{z(t + iu) - 1\}] \\ &= O\{|z(t + iu) - 1|\} = O(e^{-r\Omega u}), \qquad u \to \infty. \end{aligned} \tag{117}$$

Therefore, if (115) is satisfied, the Fourier transform of $w(t)$ vanishes over $(-\infty,r\Omega)$ and hence the Fourier transforms of $\mathrm{Log}\, |z(t)|$ and $\arg\, \{z(t)\}$ vanish over $(-r\Omega,r\Omega)$. That is, *if (115) is satisfied, then the log of the envelope of $z(t)$ and the phase (arg) of $z(t)$ are high-pass functions.*

If either (114) or (115) are satisifed we have

$$\arg\{z(t)\} \equiv \varphi(t) = \fint_{-\infty}^{\infty} \frac{\text{Log}\,|z(s)|}{\pi(t-s)}\,ds \tag{118}$$

and then

$$z(t) = |z(t)|e^{i\varphi(t)}. \tag{119}$$

The practical problem encountered here is in approximating the Hilbert transform in (118). The function Log $|z(t)|$ is not band-limited† and the implementation of (118) requires a filter whose frequency characteristic is, formally,

$$H(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\pi t}\,dt = -i \int_{-\infty}^{\infty} \frac{\sin \omega t}{\pi t}\,dt = -i \,\text{sgn}\, \omega. \tag{120}$$

$$(-\infty < \omega < \infty)$$

These stringent filter requirements can be avoided, for we can, by proper application of the previous theory, ignore the frequency content of Log $|z(t)|$ outside the band $(-\alpha,\alpha)$ where $\alpha > \Omega$. Actually if (114) is satisfied we may take $\alpha = \Omega$.

The Hilbert transform problem is simplified if we begin with a filtered version of Log $|z(t)|$; viz.,

$$\lambda_\alpha(t) = \int_{-\infty}^{\infty} \text{Log}\,|z(s)|h_\alpha(t-s)ds \tag{121}$$

where $h_\alpha(t)$ is an even real-valued function whose Fourier transform satisfies

$$\hat{h}_\alpha(\omega) = \int_{-\infty}^{\infty} h_\alpha(t)e^{-i\omega t}dt = 1 \quad \text{for } -\alpha \le \omega \le \alpha. \tag{122}$$

We suppose further that $h_\alpha(t)$ is sufficiently smooth to have a Hilbert transform $\tilde{h}_\alpha(t)$. Then the Hilbert transform of $\lambda_\alpha(t)$ is given by

$$\tilde{\lambda}_\alpha(t) \equiv \varphi_\alpha(t) = \int_{-\infty}^{\infty} \text{Log}\,|z(s)|\tilde{h}_\alpha(t-s)ds \tag{123}$$

$$= \int_{-\infty}^{\infty} \varphi(s)h_\alpha(t-s)ds.$$

Then defining

$$w_\alpha(t) = \lambda_\alpha(t) + i\varphi_\alpha(t) \tag{124}$$

we have

$$w_\alpha(t) = \int_{-\infty}^{\infty} \log\{z(s)\}h_\alpha(t-s)ds. \tag{125}$$

† Unless $z(t)$ = constant. See Theorem 6 in Section IX for a stronger statement.

Then according to the previous theory

$$z(t) = \int_{-\infty}^{\infty} k(t-s) \exp \{w_\alpha(s)\} ds \tag{126}$$

where $k(t)$ is any function of L_1 satisfying

$$\int_{-\infty}^{\infty} k(t) e^{-i\omega t} dt = 1, \quad 0 \le \omega \le \Omega \tag{127}$$

$$= 0, \qquad \omega \ge \alpha.$$

In case (115) is satisfied the function $h_\alpha(t)$ need satisfy (122) only over the intervals $(r\Omega,\alpha)$ and $(-\alpha,-r\Omega)$, since the Fourier transform of Log $|z(t)|$ vanishes over $(-r\Omega,r\Omega)$. That is, $h_\alpha(t)$ can then be chosen so that the equivalent Hilbert transform kernel $\tilde{h}_\alpha(t)$ has a Fourier transform that is more easily approximated (within a linear phase factor $e^{-i\omega T}$) by practical filters.

VIII. NOTE ON LOGARITHMIC COMPANDING

Suppose $g(t)$ is a function belonging to L_p for some p satisfying $1 \le p < \infty$ and suppose the Fourier transform of $g(t)$ vanishes outside $[-\Omega,\Omega]$. Companding functions f are sometimes used to compress the range of $g(t)$ for transmission; i.e. $f\{g(t)\}$ is transmitted rather than $g(t)$. Landau and Miranker[11,12] showed that $g(t)$ (in L_2) can be recovered from the bandlimited version of $f\{g(t)\}$ with suitable conditions on f. The recovery is accomplished by an iterative scheme. Here we use the detection theory to give an explicit solution to the problem of Landau and Miranker for the case

$$f(x) = \frac{1}{2} \text{Log}\,(1+x), \quad x > -1. \tag{128}$$

Accordingly, we further require $g(t)$ to satisfy

$$g(t) > -1. \tag{129}$$

The function $f(x)$ given by (128) is not an odd function, as one might desire for companding purposes, but is interesting because the recovery problem is simple.

The fact that $g(t)$ is a bandlimited function belonging to L_p for some p satisfying $1 \le p < \infty$ implies [cf. (29)]

$$g(t) = \int_{-\infty}^{\infty} g(s) \frac{\sin \Omega(t-s)}{\pi(t-s)} ds \tag{130}$$

from which one can conclude with the aid of Hölder's inequality that

$$\lim_{t\to\pm\infty} g(t) = 0. \tag{131}$$

It follows from (129) and (131) and (128) that

$$f\{g(t)\} \text{ belongs to } L_p. \tag{132}$$

Since $g(t)$ is bounded and belongs to L_p it follows that $g(t)$ belongs to $L_{p'}$ for every p' satisfying $p \leq p' \leq \infty$. Hence $f\{g(t)\}$ also belongs to $L_{p'}$ for such p'.

Now we suppose we are given

$$\lambda_\Omega(t) = \int_{-\infty}^{\infty} f\{g(s)\} \frac{\sin \Omega(t-s)}{\pi(t-s)} ds \tag{133}$$

where the integral is absolutely convergent by Hölder's inequality. In fact (Ref. 16) $\lambda_\Omega(t)$ belongs to L_p, and therefore has a Hilbert transform. Furthermore, since $\lambda_\Omega(t)$ is bandlimited, its Hilbert transform is given by

$$\begin{aligned} \tilde{\lambda}_\Omega(t) \equiv \varphi_\Omega(t) &= \int_{-\infty}^{\infty} \lambda_\Omega(s) \frac{1-\cos \Omega(t-s)}{\pi(t-s)} ds \\ &= \int_{-\infty}^{\infty} f\{g(s)\} \frac{1-\cos \Omega(t-s)}{\pi(t-s)} ds. \end{aligned} \tag{134}$$

Defining

$$w_\Omega(t) = \lambda_\Omega(t) + i\varphi_\Omega(t) \tag{135}$$

we have

$$w_\Omega(t) = \int_{-\infty}^{\infty} f\{g(s)\} K_\Omega(t-s) ds \tag{136}$$

where

$$K_\Omega(t) = \frac{e^{i\Omega t} - 1}{i\pi t}. \tag{137}$$

So $w_\Omega(\tau)$ is an entire function which is bounded in the uhp.

Now $\{1 + g(t)\}$ is a positive bandlimited function which can be represented as (Theorem 7.5.1 with Theorem 6.4.5, Ref. 6)

$$1 + g(t) = \gamma(t)\overline{\gamma}(t) \tag{138}$$

where the Fourier transform of $\gamma(t)$ vanishes outside $[-\Omega/2, \Omega/2]$ and $\gamma(\tau)$ is zero-free in the uhp. Then $z(t)$ defined by

$$z(t) = \gamma(t)e^{i\Omega t/2} \tag{139}$$

is a function whose Fourier transform vanishes outside $[0, \Omega]$ and $z(\tau)$ is zero-free in the uhp. Thus we have

$$1 + g(t) = |z(t)|^2. \tag{140}$$

We may assume that [cf. (131)]

$$\lim_{t\to\pm\infty} z(t) = 1. \tag{141}$$

Then $z(t)$ is given by

$$z(t) = \exp w(t) \tag{142}$$

where

$$w(\tau) = \frac{i}{2}\int_{-\infty}^{\infty} \operatorname{Log}\{1 + g(s)\}\frac{ds}{\pi(\tau - s)} \tag{143}$$

and

$$\lim_{u\to 0+} w(t + iu) = \lambda(t) + i\varphi(t) \tag{144}$$

$$\lambda(t) = \frac{1}{2}\operatorname{Log}\{1 + g(t)\} \tag{145}$$

$$\varphi(t) = \tilde{\lambda}(t), \text{ the Hilbert transform of } \lambda(t). \tag{146}$$

We see from (136) and (143) that

$$w_\Omega(t) = \int_{-\infty}^{\infty} w(s)K_\Omega(t - s)ds \tag{147}$$

$$= \int_{-\infty}^{\infty} \log\{z(s)\}K_\Omega(t - s)ds.$$

Then the Fourier transform of $z_\Omega(t)$ defined by

$$z_\Omega(t) = \exp\{w_\Omega(t)\} \tag{148}$$

agrees over $(-\infty,\Omega)$ with the Fourier transform of $z(t)$. Since $\{z(t) - 1\}$ belongs to L_p and its Fourier transform vanishes outside $[0,\Omega]$ we have

$$z(t) - 1 = \int_{-\infty}^{\infty} \{z_\Omega(s) - 1\}\frac{\sin \Omega(t - s)}{\pi(t - s)}ds. \tag{149}$$

Writing $z(t) = x(t) + iy(t)$ we have

$$x(t) = 1 + \int_{-\infty}^{\infty} \{e^{\lambda_\Omega(s)}\cos\varphi_\Omega(s) - 1\}\frac{\sin \Omega(t - s)}{\pi(t - s)}ds \tag{150}$$

$$y(t) = \int_{-\infty}^{\infty} e^{\lambda_\Omega(s)}\sin\varphi_\Omega(s)\frac{\sin \Omega(t - s)}{\pi(t - s)}ds \tag{151}$$

Then we have

$$g(t) = x^2(t) + y^2(t) - 1. \tag{152}$$

Thus the recovery problem is solved by means of the Hilbert transform in (134) and the formulas (150)–(152).

Expressing the solution in terms of the bandlimiting operator $\mathcal{B}_\Omega$, defined for h in L_p, $1 \le p < \infty$, by†

$$\mathcal{B}_\Omega h(t) = \int_{-\infty} h(s) \frac{\sin \Omega(t-s)}{\pi(t-s)} ds, \tag{153}$$

the resulting class of functions denoted by $B_p(\Omega)$, we have

$$g(t) = |\mathcal{B}_\Omega \exp \{\lambda_\Omega(t) + i\tilde{\lambda}_\Omega(t)\}|^2 - 1 \tag{154}$$

where $\lambda_\Omega(t)$ is the given function

$$\lambda_\Omega(t) = \frac{1}{2} \mathcal{B}_\Omega \operatorname{Log} \{1 + g(t)\} \tag{155}$$

$$g \text{ in } B_p(\Omega), \quad g > -1,$$

and $\tilde{\lambda}_\Omega$ is the Hilbert transform of λ_Ω.

The solution (154) is deceptive in that it suggests that $\lambda_\Omega(t)$ may be any function of $B_p(\Omega)$, $1 \le p < \infty$, since $g(t)$ given by (154) is a function of $B_p(\Omega)$ satisfying $g > -1$. However the solution was obtained on the premise that $\lambda_\Omega(t)$ is a given function of the form (155). All functions in $B_p(\Omega)$ do not have the representation (155). The crucial point is that the function

$$z(t) = \mathcal{B}_\Omega \exp \{\lambda_\Omega(t) + i\tilde{\lambda}_\Omega(t)\}, \quad \{z(t) - 1\} \text{ in } B_p(\Omega), \tag{156}$$

whose Fourier transform vanishes outside $[0,\Omega]$ should extend as a function zero-free in the upper half-plane. Then, and only then, according to the general theory, will we have

$$\mathcal{B}_\Omega \log \{z(t)\} = \lambda_\Omega(t) + i\tilde{\lambda}_\Omega(t) \tag{157}$$

and hence

$$\mathcal{B}_\Omega \operatorname{Log} |z(t)| = \frac{1}{2} \mathcal{B}_\Omega \operatorname{Log} \{1 + g(t)\} = \lambda_\Omega(t), \quad g \text{ in } B_p(\Omega), \tag{158}$$

$$g > -1.$$

On the other hand, if (158) is known to hold, implying (157), then $z(t)$ must necessarily extend as a function zero-free in the upper half-plane.

We state this important result as

Theorem 4. Given a function $\lambda_\Omega(t)$ in $B_p(\Omega)$, for some p satisfying 1 ≤

† The operator $\mathcal{B}_\Omega$ can be extended to certain other classes of functions. For example, $\mathcal{B}_\Omega$ is an identity for the constant function, which fact is used in (154).

$p < \infty$, the equation (155) has a solution $g(t)$ in the same class $B_p(\Omega)$ satisfying

$$g(t) > -1$$

if and only if the function

$$z(t) = \mathscr{B}_\Omega \exp\{\lambda_\Omega(t) + i\tilde{\lambda}_\Omega(t)\}$$

where $\tilde{\lambda}_\Omega(t)$ is the Hilbert transform of $\lambda_\Omega(t)$, extends as a function zero-free in the upper half-plane. Then the solution of (155) is given by (154).

IX. BANDWIDTH REQUIREMENTS FOR EXPONENTIAL MODULATION

We have seen that for a wide class of analytic signals $z(t)$ and modulation laws $f(z)$ the bandwidth requirement for transmitting $f\{z(t)\}$ and recovering $z(t)$ is $\Omega + \epsilon$ (for any $\epsilon > 0$) where Ω is the bandwidth of $z(t)$, *provided* we allow the use of Poisson filtering at the receiver. In case the inverse function $z = \varphi(w)$ is an entire function there is no need for Poisson filtering. If we look at the overall system design, as contrasted to a detection problem, it is reasonable to ask for the bandwidth requirements for a given $f(z)$ and a *fixed* receiver, namely *the inverse function $\varphi(w)$ followed by a low-pass filter,* such that we recover all $z(t)$ whose Fourier transforms vanish outside $[0,\Omega]$ and satisfy some sort of norm constraint, say $|z(t)| \le m$. The problem then is to specify a channel of finite bandwidth, i.e., a function $K_{\alpha,\beta}(t)$ in L_1, with $\Omega < \alpha < \beta$, satisfying

$$\hat{K}_{\alpha,\beta}(\omega) = \int_{-\infty}^{\infty} K_{\alpha,\beta}(t)e^{-i\omega t}dt = 1, \quad 0 \le \omega \le \alpha \tag{159}$$

$$= 0, \qquad \omega > \beta$$

such that the received (bandlimited) analytic signal $w_{\alpha,\beta}(t)$, given by

$$w_{\alpha,\beta}(t) = \int_{-\infty}^{\infty} f\{z(s)\}K_{\alpha,\beta}(t-s)ds, \tag{160}$$

satisfies

$$\varphi\{w_{\alpha,\beta}(\tau)\} \text{ analytic in the uhp} \tag{161}$$

for all $z(t)$ whose Fourier transforms vanish outside $[0,\Omega]$ and which satisfy

$$|z(t)| \le m, \quad -\infty < t < \infty. \tag{162}$$

We assume of course that $f(z)$ is analytic for $|z| \le m$. We would like to make the channel bandwidth β as small as possible consistent with (161) and (162). Clearly, we may take $\Omega = 1$ with no loss in generality. We define the minimum bandwidth $\beta_0(m)$ as

$$\beta_0(m) = \inf \beta(m) \tag{163}$$

where the infimum is over all functions $K_{\alpha,\beta}(t)$ subject to (159), (161), and (162) with $\Omega = 1$. In taking the infimum we may allow $\alpha = 1$.

The determination of $\beta_0(m)$ is in general a very difficult problem. We give some estimates here for $\beta_0(m)$ for the case $f(z) = e^z$, $\varphi(w) = \log w$. In this case, (161) is satisfied if and only if

$$w_{\alpha,\beta}(\tau) \text{ is zero-free in the uhp.} \tag{164}$$

We have

$$w(t) = \exp\{z(t)\} \tag{165}$$

and with (162)

$$e^{-m} \leq |w(t)| \leq e^m. \tag{166}$$

If $w_{\alpha,\beta}(t)$ is sufficiently close to $w(t)$, (164) will be satisfied; i.e., a sufficient condition for (164) is

$$|w(t) - w_{\alpha,\beta}(t)| < e^{-m}. \tag{167}$$

It is intuitively obvious, with the freedom we have in defining $\hat{K}_{\alpha,\beta}$, that for sufficiently large β we can find a function $K_{\alpha,\beta}(t)$ such that $w_{\alpha,\beta}(t)$ given by (160), with $f(z) = e^z$, will satisfy (167). It is important to note in this connection that, although the definition of $\hat{K}_{\alpha,\beta}(\omega)$ for $\omega < 0$ does not affect $w_{\alpha,\beta}(t)$, we are free to define $\hat{K}_{\alpha,\beta}(\omega)$ for $\omega < 0$ (as well as for $\alpha < \omega < \beta$) in the most favorable way to obtain the estimate (167).

First we obtain lower bounds for $\beta_0(m)$.

9.1 Lower bounds for $\beta_0(m)$

We can obtain a lower bound for $\beta_0(m)$ by taking

$$z(t) = me^{it}. \tag{168}$$

We have

$$w(t) = \exp\{me^{it}\} = \sum_{k=0}^{\infty} \frac{m^k}{k!} e^{ikt}. \tag{169}$$

Now assume the channel cutoff frequency β satisfies

$$n < \beta \leq n + 1 \tag{170}$$

where $n \geq 1$ is an integer. Then

$$w_{\alpha,\beta}(t) = 1 + me^{it} + \sum_{k=2}^{n} a_k e^{ikt} \tag{171}$$

where the a_k depend on m and the definition of $\hat{K}_{\alpha,\beta}(\omega)$ for $1 < \omega \leq n$. We have

$$w_{\alpha,\beta}(t) = \prod_{k=1}^{n} \left(1 - \frac{\zeta}{\zeta_k}\right) \tag{172}$$

where $\zeta = e^{it}$ and

$$\sum_{k=1}^{n} \frac{1}{\zeta_k} = -m. \tag{173}$$

We require $w_{\alpha,\beta}(t)$ to be zero-free in the uhp; i.e.,

$$|\zeta_k| \geq 1. \tag{174}$$

Thus

$$m = \left|\sum_{k=1}^{n} \frac{1}{\zeta_k}\right| \leq \sum_{k=1}^{n} \frac{1}{|\zeta_k|} \leq n. \tag{175}$$

Therefore, we must have $\beta > n \geq m$ in order for $w_{\alpha,\beta}(t)$ to be zero-free in the uhp. Then

$$\beta_0(m) > [m]^+ \tag{176}$$

where $[m]^+$ is the smallest integer which is not less than m.

9.2 Lower bound for small m

We know that $\beta_0(m) > 1$ for any $m > 0$ but (176) does not say how much $\beta_0(m)$ must exceed 1 for $0 < m < 1$. For sufficiently small ϵ and correspondingly small m we can show that $\beta_0(m) > 1 + \epsilon$.

For small m, we have

$$w(t) = 1 + z(t) + \frac{z^2(t)}{2} + O(m^3). \tag{177}$$

Now by Corollary 2 of Theorem 2 the Fourier transform of $z^2(t)$ vanishes outside [0,2] and we would like to find a $z(t)$ such that a channel filter with a sharp cut-off, i.e., $\beta = 1 + \epsilon$, acting on a small $z^2(t)$ gives a large (negative) output at $t = 0$. For sufficiently small ϵ and fixed m we can accomplish this by taking

$$z(t) = -\frac{me^{it/2}}{\sqrt{2}}\{1 + iS_n(\epsilon t)\} \tag{178}$$

where $S_n(t)$ is a sine polynomial,

$$S_n(t) = \sum_{k=1}^{n} a_k \sin kt \tag{179}$$

with real coefficients a_k and

$$\max |S_n(t)| = 1. \tag{180}$$

Also we require

$$n\epsilon = \frac{1}{2} \tag{181}$$

so that the Fourier transform of $z(t)$ given by (178) vanishes outside [0,1]. Also we have

$$\max |z(t)| = m \tag{182}$$

and

$$w(t) = 1 - \frac{me^{it/2}}{\sqrt{2}}\{1 + iS_n(\epsilon t)\} + \frac{m^2}{4}e^{it}\{1 - S_n^2(\epsilon t) + 2iS_n(\epsilon t)\} + R_3(t), \tag{183}$$

where

$$R_3(t) = \sum_{k=3}^{\infty} \frac{z^k(t)}{k!} \tag{184}$$

$$|R_3(t)| \le \frac{m^3}{3!}\frac{1}{\left(1 - \frac{m}{4}\right)}, \qquad (m < 4). \tag{185}$$

We have

$$w_{\alpha,\beta}(t) = \int_{-\infty}^{\infty} w(\xi)K_{\alpha,\beta}(t-\xi)d\xi = 1 - \frac{me^{it/2}}{\sqrt{2}}\{1 + S_n(\epsilon t)\} + \frac{m^2}{4}\int_{-\infty}^{\infty} e^{i\xi}\{1 - S_n^2(\epsilon\xi) + 2iS_n(\epsilon\xi)\}K_{\alpha,\beta}(t-\xi)d\xi + \int_{-\infty}^{\infty} R_3(\xi)K_{\alpha,\beta}(t-\xi)d\xi \tag{186}$$

and $w_{\alpha,\beta}(t)$ is a polynomial of degree $2n$ in $\exp(i\epsilon t)$ where $\epsilon = 1/2n$. We have

$$2ie^{it}S_n(\epsilon t) = -e^{it}\sum_{k=1}^{n} a_k e^{-ik\epsilon t} + e^{it}\sum_{k=1}^{n} a_k e^{ik\epsilon t} \tag{187}$$

$$e^{it}S_n^2(\epsilon t) = e^{it}\sum_{k=-2n}^{2n} b_k e^{ik\epsilon t}. \tag{188}$$

So

$$w_{\alpha,\beta}(t) = 1 - \frac{me^{it/2}}{\sqrt{2}}\{1 + iS_n(\epsilon t)\}$$

$$+ \frac{m^2}{4}\left\{1 - e^{it}\sum_{k=-2n}^{0} b_k e^{ik\epsilon t} - e^{it}\sum_{k=1}^{n} a_k e^{-ik\epsilon t}\right\} \tag{189}$$

$$+ r_3(t)$$

where

$$r_3(t) = \int_{-\infty}^{\infty} R_3(\xi)K_{\alpha,\beta}(t - \xi)d\xi. \tag{190}$$

Since $R_3(t)$ is a periodic function of the form $\Sigma_0^\infty c_k e^{-ik\epsilon t}$ we may take for $K_{\alpha,\beta}(t)$ any function of L_1 whose Fourier transform satisfies

$$\begin{aligned} \hat{K}_{\alpha,\beta}(\omega) &= 1, \qquad 0 \le \omega \le 1 \\ &= 0, \qquad \omega \ge \beta = 1 + \epsilon \end{aligned} \tag{191}$$

It is shown in Appendix B that there exists a function $K_{\alpha,\beta}(t)$ whose Fourier transform satisfies (191) with

$$\int_{-\infty}^{\infty} |K_{\alpha,\beta}(t)|dt < 1 + \frac{1}{\pi}\log\left(1 + \frac{4}{3\epsilon}\right). \tag{192}$$

Thus

$$|r_3(t)| \le \max |R_3(t)| \int_{-\infty}^{\infty} |K_{\alpha,\beta}(t)|dt \tag{193}$$

$$< \frac{m^3}{3!\left(1 - \frac{m}{4}\right)}\left\{1 + \frac{1}{\pi}\log\left(1 + \frac{4}{3\epsilon}\right)\right\}.$$

We have from (189)

$$w_{\alpha,\beta}(0) = 1 - \frac{m}{\sqrt{2}} + \frac{m^2}{4}\left\{1 - \sum_{k=-2n}^{0} b_k - \sum_{k=1}^{n} a_k\right\} \tag{194}$$

$$+ r_3(0).$$

Now $S_n^2(t)$ is an even function and $S_n(0) = 0$. Thus

$$\sum_{k=-2n}^{2n} b_k = 0, \qquad b_{-k} = b_k \tag{195}$$

$$\sum_{k=-2n}^{-1} b_k = \sum_{k=1}^{2n} b_k = \frac{-b_0}{2} \tag{196}$$

$$\sum_{k=-2n}^{0} b_k = \frac{b_0}{2} \tag{197}$$

and

$$b_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_n^2(t)dt = \frac{1}{2} \sum_{k=1}^{n} a_k^2 < 1. \tag{198}$$

Now we may take $S_n(t)$ to be an approximation to sgn{sin t}. In particular we may take $S_n(t)$ to be the function found by Szegö[19] which maximizes $\Sigma_1^n a_k$ subject to (180). He gives the inequality

$$\sum_1^n a_k \leq M_n = \frac{2}{n+1} \sum_{\substack{1 \leq k \leq n \\ k \text{ odd}}} \cot\left(\frac{k\pi}{2(n+1)}\right) \sim \frac{2}{\pi} \log n, \quad n \to \infty. \tag{199}$$

Here we should not identify a_k with the terms in the second sum. For equality in (199) we must have

$$S_n\left(\frac{k\pi}{n+1}\right) = 1, \quad 1 \leq k \leq n, \quad k \text{ odd}. \tag{200}$$

Equality in (199) is attained for

$$S_n(t) = \sum_{\substack{1 \leq k \leq n \\ k \text{ odd}}} \left\{ I_n\left(t - \frac{k\pi}{n+1}\right) - I_n\left(t + \frac{k\pi}{n+1}\right) \right\} \tag{201}$$

where

$$I_n(t) = \left\{ \frac{\sin \frac{n+1}{2} t}{(n+1) \sin \frac{t}{2}} \right\}^2. \tag{202}$$

It is easy to show that

$$\sum_{\substack{k=-n \\ (k \text{ odd})}}^{n} I_n\left(t - \frac{k\pi}{n+1}\right) \equiv 1 \quad \text{for } n \text{ odd} \tag{203}$$

$$\sum_{\substack{k=-(n-1) \\ (k \text{ odd})}}^{n-1} I_n\left(t - \frac{k\pi}{n+1}\right) + \frac{1}{2}\{I_n(t-\pi) + I_n(t+\pi)\} \tag{204}$$

$$\equiv 1 \qquad \text{for } n \text{ even}$$

It follows from (203), (204), and (201) that $S_n(t)$ given by (201) satisfies

$$-1 \leq S_n(t) \leq 1. \tag{205}$$

From (201) and (202) we find that

$$a_k = \frac{4}{n+1}\left(1 - \frac{k}{n+1}\right)\frac{\sin^2 \dfrac{k\pi}{2}}{\sin \dfrac{k\pi}{n+1}}, \qquad \begin{array}{l} n \text{ odd} \\ k = 1,2,\ldots,n \end{array} \tag{206}$$

$$a_k = \frac{4}{n+1}\left(1 - \frac{k}{n+1}\right)\frac{\left(\sin \dfrac{nk\pi}{2(n+1)}\right)^2}{\sin \dfrac{k\pi}{n+1}}, \quad \begin{array}{l} n \text{ even} \\ k = 1,2,\ldots,n \end{array} \tag{207}$$

It is shown in Appendix C that M_n given by (199) satisfies

$$M_n > \frac{2}{\pi}\log n + \frac{2}{\pi}\left(\log\frac{4}{\pi} + \gamma\right) \tag{208}$$

where

$$\gamma = 0.5772\ldots \qquad \text{(Euler's constant)}. \tag{209}$$

So with $S_n(t)$ given by (201) and $z(t)$ given by (178) we have from (193), (194), (197), and (208)

$$\begin{aligned} w_{\alpha,\beta}(0) < 1 - \frac{m}{\sqrt{2}} + \frac{m^2}{4}\left\{1 - \frac{b_0}{2} - \frac{2}{\pi}\log n - \frac{2}{\pi}\left(\log\frac{4}{\pi} + \gamma\right)\right\} \\ + \frac{m^3}{3!\left(1 - \dfrac{m}{4}\right)}\left\{1 + \frac{1}{\pi}\log\left(1 + \frac{8n}{3}\right)\right\} \end{aligned} \tag{210}$$

where $1 < \alpha < \beta$,

$$\beta = 1 + \frac{1}{\epsilon} = 1 + \frac{1}{2n}$$

$$b_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_n^2(t)dt \sim 1 \text{ as } n \to \infty.$$

Now if we set

$$\frac{m^2}{2\pi}\log n = 1 \tag{211}$$

it is clear that

$$w_{\alpha,\beta}(0) < 0 \text{ for sufficiently large } n. \tag{212}$$

Since

$$w_{\alpha,\beta}(t) = \sum_{k=0}^{2n} c_k e^{ikt/2n} \tag{213}$$

where the c_k are real and $c_0 > 0$, we have $w_{\alpha,\beta}(\mathrm{iu})$ real and

$$\lim_{u \to \infty} w_{\alpha,\beta}(\mathrm{iu}) = c_0 > 0. \tag{214}$$

So (212) and (214) imply that $w_{\alpha,\beta}(t)$ has at least one zero on the positive imaginary axis; i.e. for sufficiently large n, and m given by (211), we must have $\beta > 1 + 1/2n$ (to pick up at least another harmonic) in order for $w_{\alpha,\beta}(t)$ to be zero free in the uhp. Thus

$$\beta_0 \left(\sqrt{\frac{2\pi}{\log n}} \right) > 1 + \frac{1}{2n} \text{ for sufficiently large } n. \tag{215}$$

In connection with obtaining lower bounds for β_0 the idea comes to mind that we might be able to find a $z(t)$ satisfying $|z(t)| \leq m$ (for sufficiently large m) such that the Fourier transform of $w(t)$ would vanish over a large interval $(1,\beta)$. Then if $w_{\alpha,\beta}(t)$ were not zero free in the uhp we would have $\beta_0 > \beta(m)$. The idea is to obtain a $w_{\alpha,\beta}(t)$ which would be independent of the choice of $K_{\alpha,\beta}(t)$. However, we cannot make the Fourier transform of $w(t)$ vanish over large intervals unless $z(t) \equiv$ constant.

Theorem 5. Suppose $z(t)$ is a bounded continuous function whose Fourier transform vanishes outside $[0,\Omega]$, and suppose that the Fourier transform of $w(t)$, where

$$w(t) = \exp\{z(t)\},$$

vanishes over (a,b) where

$$a \geq 0 \quad \text{and} \quad b - a > \Omega.$$

Then

$$z(t) \equiv \text{constant}.$$

A similar result holds for the logarithmic function.

Theorem 6. Suppose $z(t)$ is a bounded continuous function whose Fourier transform vanishes outside $[0,\Omega]$ and suppose its analytic continuation $z(\tau)$ satisfies

$$|z(t + iu)| \geq \epsilon > 0 \quad \text{for } u \geq 0$$
$$-\infty < t < \infty.$$

Suppose further that the Fourier transform of $w(t)$, where

$$w(t) = \log\{z(t)\}$$

vanishes over (a,b) where

$$a \geq 0, \quad b - a > \Omega.$$

Then

$$z(t) \equiv \text{constant}.$$

As an application of the last theorem, we may take $\Omega = n$ and

$$z(t) = \prod_{k=1}^{n} (1 - \lambda_k e^{ikt})$$

Then, assuming $|\lambda_k| < 1$, we have

$$\log z(t) = - \sum_{m=1}^{\infty} \frac{\mu_m e^{imt}}{m}$$

where

$$\mu_m = \sum_{k=1}^{n} (\lambda_k)^m$$

Then the following is true.

Corollary. If $\{\lambda_k\}$, $k = 1,2,\ldots,n$, *is any set of n complex numbers and*

$$\sum_{k=1}^{n} (\lambda_k)^m = 0 \text{ for } m = p, p+1, p+2, \ldots p+n-1$$

where p is a positive integer, then $\lambda_k = 0$, $k = 1,2,\ldots,n$.

Proofs of Theorems 5 and 6 are given in Appendices D and E.

9.3 Upper bound for $\beta_0(m)$

We have

$$w(t) = e^{z(t)} \quad \text{and} \quad |z(t)| \le m \tag{216}$$

so $\{w(t)\}^{-1}$ is bounded and analytic in the uhp. Thus the quotient

$$\frac{w_{\alpha,\beta}(t)}{w(t)} = 1 + \frac{w_{\alpha,\beta}(t) - w(t)}{w(t)} \tag{217}$$

is bounded and analytic in the uhp, and is reproduced by the Poisson kernel from its values on the real line. Then if

$$\left| \frac{w(t) - w_{\alpha,\beta}(t)}{w(t)} \right| < 1 \text{ for } -\infty < t < \infty \tag{218}$$

the function $w_{\alpha,\beta}(t)$ is necessarily zero-free in the uhp. Thus a sufficient condition for $w_{\alpha,\beta}(t)$ to be zero-free in the uhp is

$$|w(t) - w_{\alpha,\beta}(t)| < e^{-m}. \tag{219}$$

For lack of something better we will use this condition to obtain an upper bound for $\beta_0(m)$. To meet this condition for large m will require considerably more bandwidth than the lower bound for $\beta_0(m)$. In fact for the case $z(t) = me^{it}$ we have

$$w(t) = \sum_{k=0}^{\infty} \frac{m^k}{k!} e^{ikt} \tag{220}$$

and

$$w_{\alpha,\beta}(t) = 1 + me^{it} + \sum_{2}^{[\beta]} a_k e^{ikt} \tag{221}$$

where the a_k depend on the definition of $\hat{K}_{\alpha,\beta}(\omega)$ for $\omega > 1$. Since

$$\frac{m^k}{k!} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{w(t) - w_{\alpha,\beta}(t)\} e^{-ikt} dt, \quad k \geq \beta \tag{222}$$

we have

$$\frac{m^k}{k!} \leq \max |w(t) - w_{\alpha,\beta}(t)| \quad \text{where } k \geq \beta. \tag{223}$$

Hence in order to satisfy (219) for the case $z(t) = me^{it}$ we must have

$$\max_{k \geq \beta} \left\{ \frac{m^k}{k!} \right\} < e^{-m}. \tag{224}$$

For large m we must have β large since

$$k! < \sqrt{2\pi k}\, k^k e^{-k} e^{1/12k}. \tag{225}$$

We find for large m that

$$\beta > \rho m + o(m) \tag{226}$$

where ρ is the root of

$$\rho = e^{1+1/\rho} \doteq 3.591121477.$$

Thus for large m the upper bound we obtain for $\beta_0(m)$ from the condition (219) must be something like 3.6 times as large as the lower bound ($\sim m$) we obtained previously. We can in fact obtain an upper bound for $\beta_0(m)$ that is close to ρm for large m and, as it turns out, is close to the lower bound for $\beta_0(m)$ as $m \to 0$.

To do this we suppose that

$$\alpha = n < \beta \tag{227}$$

where n is a positive integer. We take

$$\begin{aligned} \hat{K}_{\alpha,\beta}(\omega) &= 1, & 0 \leq \omega \leq n \\ &= \frac{\beta - \omega}{\beta - n}, & n < \omega \leq \beta \\ &= 0, & \omega > \beta. \end{aligned} \tag{228}$$

$\hat{K}_{\alpha,\beta}(\omega)$ is defined for $\omega < 0$ in such a way (see Appendix B) that

$$\int_{-\infty}^{\infty} |K_{\alpha,\beta}(t)|dt < \frac{1}{\pi} \log \left\{1 + \frac{4n}{3(\beta - n)}\right\} + 1. \tag{229}$$

Now we write

$$w(t) = e^{z(t)} = P_n\{z(t)\} + \sum_{k=n+1}^{\infty} \frac{\{z(t)\}^k}{k!} \tag{230}$$

where

$$P_n\{z(t)\} = \sum_{k=0}^{n} \frac{\{z(t)\}^k}{k!}. \tag{231}$$

Since by Corollary 2 the Fourier transform of $P_n\{z(t)\}$ vanishes outside $[0,n]$, we have

$$\begin{aligned} w_{\alpha,\beta}(t) &= \int_{-\infty}^{\infty} w(s)K_{\alpha,\beta}(t-s)ds \\ &= P_n\{z(t)\} + R_{n+1}(t) \end{aligned} \tag{232}$$

where

$$R_{n+1}(t) = \int_{-\infty}^{\infty} \left\{\sum_{k=n+1}^{\infty} \frac{z^k(s)}{k!}\right\} K_{\alpha,\beta}(t-s)ds \tag{233}$$

$$|R_{n+1}(t)| < \left\{\sum_{k=n+1}^{\infty} \frac{m^k}{k!}\right\} \int_{-\infty}^{\infty} |K_{\alpha,\beta}(t)|dt. \tag{234}$$

Thus

$$|w(t) - w_{\alpha,\beta}(t)| < \left\{\sum_{k=n+1}^{\infty} \frac{m^k}{k!}\right\} \left\{1 + \int_{-\infty}^{\infty} |K_{\alpha,\beta}(t)|dt\right\}. \tag{235}$$

Therefore if

$$1 + \int_{-\infty}^{\infty} |K_{\alpha,\beta}(t)|dt < \frac{e^{-m}}{\sum_{k=n+1}^{\infty} \frac{m^k}{k!}} \equiv Q_n(m) \tag{236}$$

$w_{\alpha,\beta}(t)$ will be zero-free in the uhp. Then from (229) and (236) we can get an upper bound on β for each choice of n. Since

$$\int_{-\infty}^{\infty} K_{\alpha,\beta}(t)e^{-i\omega t}dt = 1 \quad \text{for } 0 \le \omega \le \alpha \tag{237}$$

it follows that

$$\int_{-\infty}^{\infty} |K_{\alpha,\beta}(t)|dt > 1. \tag{238}$$

So the inequality (236) can hold only if m and n are such that the right-hand member of (236) is greater than 2. Then setting

$$A_n(m) = \max\{0,[Q_n(m) - 2]\} \tag{239}$$

we see from (229) and (236) that

$$\frac{1}{\pi}\log\left\{1 + \frac{4n}{3[\beta_n(m) - n]}\right\} = A_n(m) \tag{240}$$

defines for each positive integer n an upper bound for $\beta_0(m)$, viz.,

$$\beta_n(m) = n + \frac{\frac{4}{3}n}{e^{\pi A_n(m)} - 1}. \tag{241}$$

We take $\beta_n(m) = \infty$ for $A_n(m) = 0$. For fixed m, we have $A_n(m) > 0$ for sufficiently large n. Then

$$\beta_0(m) \le B(m) \equiv \min_n \beta_n(m) \quad n = 1,2,3,\ldots \tag{242}$$

Since $\beta_n(m) > n$, it is clear that the minimum in (242) will be $\beta_1(m)$ for sufficiently small m. We have

$$A_1(m) = 2m^{-2} + O(m^{-1}), \quad m \to 0. \tag{243}$$

So the upper bound $\beta_1(m)$ for small m compares favorably with the lower bound (215). At least we have the dominant exponential behavior pinned down as $m \to 0$; i.e.,

$$\lim_{m\to 0} m^2 \log\{\beta_0(m) - 1\} = -2\pi. \tag{244}$$

The function $A_n(m)$ defined in (239) behaves like $(n + 1)!/m^{n+1}$ as $m \to 0$ and decreases to zero at $m = m_n$ where

$$m_n \approx \frac{(n+1)}{\rho} + \frac{1}{2(1+\rho)}\log(n+1) + \frac{1}{1+\rho}\left\{\frac{1}{2}\log 2\pi + \log\frac{\rho - 1}{2\rho}\right\} \tag{245}$$

and $\rho \doteq 3.591121477$ is defined in (226),

$$\frac{1}{\rho} \doteq .278464543$$

$$\frac{1}{2(1+\rho)} \doteq .1089058525$$

$$\frac{1}{1+\rho}\left\{\frac{1}{2}\log 2\pi + \log\frac{\rho-1}{2\rho}\right\} \doteq -.0219080253.$$

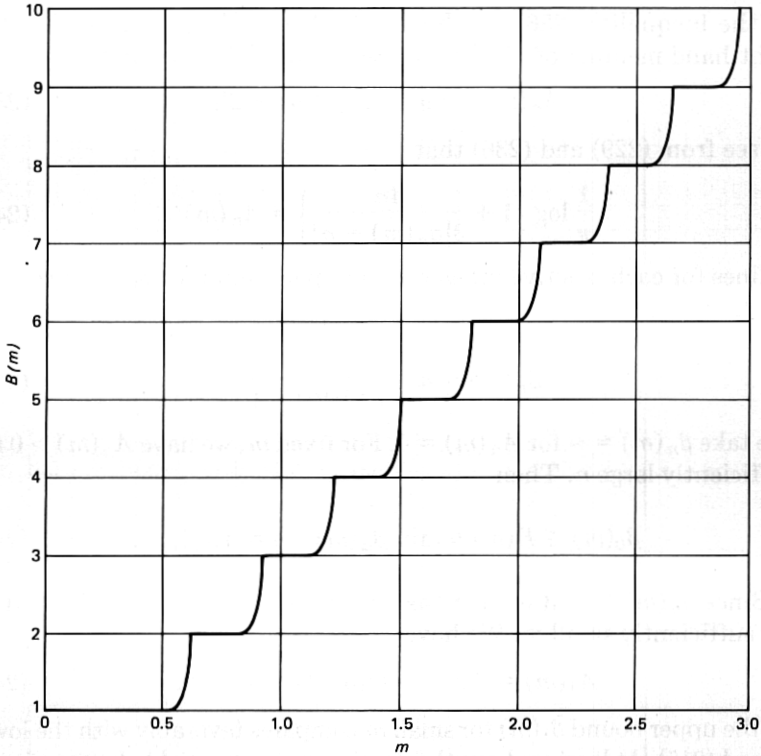

Fig. 1—Upper bound for transmission bandwidth required in ESSB for simple detection of signals $z(t)$ whose Fourier transforms vanish outside [0,1] and satisfy $|z(t)| \leq m$.

The behavior of $A_n(m)$ is such that $\beta_n(m)$ defined in (241) for $0 \leq m \leq m_n$ is very close to n over most of this range and increases suddenly as $m \to m_n$. Consequently, the upper bound $B(m)$ defined in (242) is roughly a staircase function as shown in Figure 1. For $0 \leq m \leq .62$ we have $B(m) = \beta_1(m)$. In Figure 2 a graph of $\log_{10}\{B(m) - 1\}$ is plotted for $.48 \leq m \leq .62$. It is seen that only .1% increase in transmission bandwidth is required for $m \leq .48$, and 10% increase suffices for $m \leq .57$. We know that $\beta_0(m) > 2$ for $m > 1$, so without Poisson filtering SSBEM is interesting perhaps only for $|z| < .6$.

X. A POLYNOMIAL PROBLEM

To each polynomial

$$P_n(\zeta) = 1 + \sum_{k=1}^{n} a_k \zeta^k \tag{246}$$

we can assign a positive integer ν which is the smallest integer $\geq n$ such

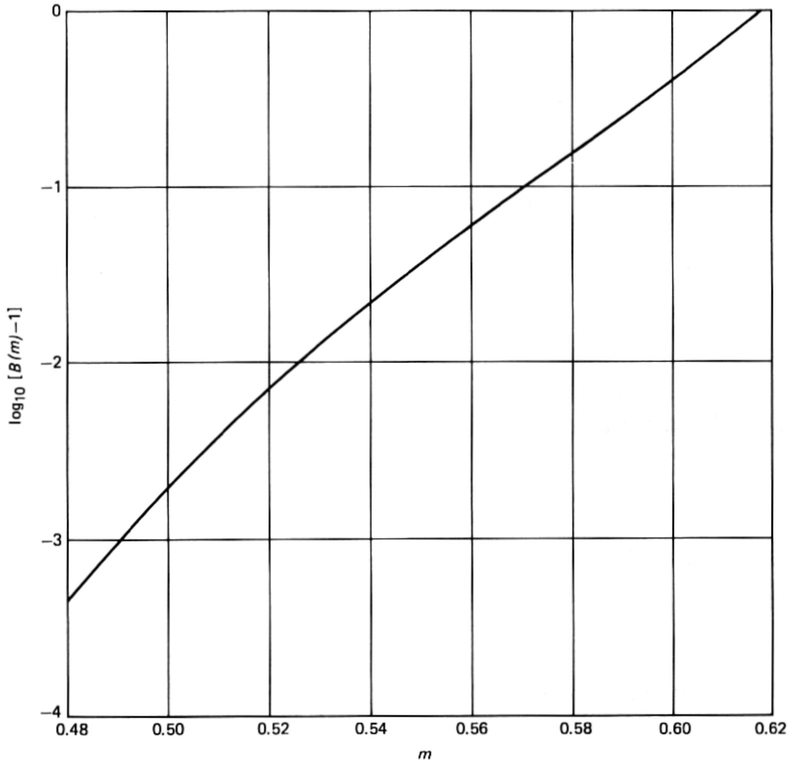

Fig. 2—Logarithmic expansion of Fig. 1 for small m.

that for some choice of a_k for $k = n + 1, n + 2, \ldots, \nu$, the polynomial

$$P_{n,\nu}(\zeta) = 1 + \sum_{k=1}^{n} a_k \zeta^k + \sum_{n+1}^{\nu} a_k \zeta^k \tag{247}$$

is zero free for $|\zeta| < 1$. The integer ν is some complicated function of the coefficients a_k, $k = 1,2, \ldots n$. The fact that ν is finite is a rather remarkable fact that follows from the previous theory. To see this we set

$$\zeta = re^{it}, \quad r > 0. \tag{248}$$

For sufficiently small r we have $P_n(re^{it})$ zero-free in the uhp. Then taking

$$Q(re^{it}) = \log P_n(re^{it}) \tag{249}$$

$$= \sum_{1}^{\infty} b_k r^k e^{ikt}$$

and

$$Q_n(re^{it}) = \sum_1^n b_k r^k e^{ikt} \tag{250}$$

we have

$$\exp\{Q_n(e^{it})\} = 1 + \sum_1^\infty c_k e^{ikt} \tag{251}$$

where

$$c_k = a_k, \quad k = 1,2,\ldots,n. \tag{252}$$

Now we identify $Q_n(e^{it/n})$ with $z(t)$ in the previous section where we obtained upper bounds on $\beta_0(m)$. We know we can bandlimit exp $\{Q_n(e^{it})\}$ to obtain a function of the form

$$1 + \sum_1^n c_k e^{ikt} + \sum_{n+1}^N d_k e^{ikt}$$

which for sufficiently large N is zero free in the uhp. In particular, we have shown that this is possible for

$$N \le nB(m) \tag{253}$$

where $B(m)$ is defined in (242) and

$$m = \max_t |Q_n(e^{it})|. \tag{254}$$

Thus

$$\nu \le N \le nB(m). \tag{255}$$

Given the a_k, or equivalently the b_k, for $k = 1,2,\ldots,n$, we are interested in obtaining a lower bound for ν.

Writing

$$P_{n,\nu}(\zeta) = \prod_{k=1}^{\nu} (1 - \lambda_k \zeta) \tag{256}$$

where $|\lambda_k| \le 1$

we have

$$\log P_{n,\nu}(\zeta) = \sum_{k=1}^\infty b_k \zeta^k, \quad |\zeta| < 1 \tag{257}$$

where

$$b_k = -\frac{1}{k} \sum_{j=1}^{\nu} (\lambda_j)^k. \tag{258}$$

Since $|\lambda_k| \leq 1$ we have $|b_k| \leq \nu/k$ and hence

$$\nu \geq \max_k |kb_k|, \quad k = 1,2, \ldots, n. \tag{259}$$

Consider, for example, the case

$$P_n(\zeta) = 1 + m\zeta^n, \quad m \geq 0 \tag{260}$$

$$P_{n,\nu}(\zeta) = 1 + m\zeta^n + a_{n+1}\zeta^{n+1} + \ldots a_\nu \zeta^\nu. \tag{261}$$

We have

$$\log P_{n,\nu}(\zeta) = m\zeta^n + b_{n+1}\zeta^{n+1} + \ldots \tag{262}$$

So for the case (260) we have

$$\nu \geq nm. \tag{263}$$

This inequality is clearly best possible in case m is a positive integer. Now suppose $m = 1 + \epsilon$, where $0 < \epsilon < 1/n$. Since ν is an integer we conclude from (263) that $\nu \geq n + 1$. However, we can show (see Appendix F) by another method that $m > 1$ in (260) implies $\nu \geq 2n$. Then it is apparent from the example

$$P_{n,2n}(\zeta) = 1 + m\zeta^n + a_{2n}\zeta^{2n}$$

(with appropriate choice of a_{2n}) that $\nu = 2n$ for $1 < m \leq 2$. It is conjectured that this large jump in ν at $m = 1$ also occurs at all integer values of m, but we have not been able to show, for example, that $m > 2$ implies $\nu \geq 3n$.

In order to improve the lower bounds obtained for $\beta_0(m)$ we are interested in maximizing the ratio ν/n subject to the constraint

$$\max_t \left| b_o + \sum_{rf1}^{n} b_k e^{ikt} \right| \leq m. \tag{264}$$

For any choice of b_k, $k = 1,2, \ldots n$, we are free to choose b_o so as to minimize the maximum modulus of the sum. That is, in the bandwidth problem of the previous section we take $\Omega = n$ and

$$z(t) = \sum_{k=0}^{n} b_k e^{ikt} \tag{265}$$

with the constraint (257). Then assuming $n \leq \alpha \leq n + 1$, and $\nu < \beta \leq \nu + 1$, we have

$$w_{\alpha,\beta}(t) = e^{b_o} \left\{ 1 + \sum_{k=1}^{n} a_k e^{ikt} + \sum_{k=n+1}^{\nu} a_k e^{ikt} \right\} \tag{266}$$

$$= e^{b_o} \prod_{k=1}^{\nu} (1 - \lambda_k e^{it}), \quad |\lambda_k| \leq 1.$$

Now we want lower bounds on ν implied by (258), subject to (264), giving us $\beta_0(m) > \nu/n$. For a given n (large) we would like to choose the b_k so as to maximize ν. For a particular choice of b_k we can in principle determine the minimum value of ν required to satisfy (258). In order to do this we may assign values to the $(n - \nu)$ coefficients a_k, $k = n + 1, \ldots, \nu$, in (264) and see if it is possible to make $|\lambda_k| \leq 1$ for $k = 1,2, \ldots, \nu$. Recall that the first n coefficients are determined by

$$\exp\left\{\sum_{k=1}^{n} b_k e^{ikt}\right\} = 1 + a_1 e^{it} + \ldots + a_n e^{int} + \ldots \tag{267}$$

Perhaps a computer study could shed some light on this very difficult and challenging problem.

APPENDIX A

Proofs of Theorems 1, 2, and 3

In view of Ex. 2 in Sec. 2.2 it is sufficient to prove Theorem 1 for the interval $(-\infty,0)$. So we assume first that $g(t)$ is a bounded function whose Fourier transform vanishes over $(-\infty,0)$ and we wish to show that $g(t)$ is the boundary value of a function bounded and analytic in the upper half-plane. For this purpose we define

$$g_u(t) = \int_{-\infty}^{\infty} g(s) P_u(t - s) ds, \quad u > 0 \tag{268}$$

where

$$P_u(t) = \frac{u}{\pi} \frac{1}{t^2 + u^2}. \tag{269}$$

We have

$$|g_u(t)| \leq \int_{-\infty}^{\infty} |g(s)||P_u(t - s)| ds.$$

Hence

$$\sup_t |g_u(t)| \leq \sup_t |g(t)|. \tag{270}$$

Also

$$\lim_{u \to 0} g_u(t) = g(t) \quad \text{for almost all } t. \tag{271}$$

Now all we have to show is that $g_u(t)$ is an analytic function of $\tau = t + iu$. Since the Fourier transform of $g(t)$ vanishes over $(-\infty,0)$ we may replace $P_u(t)$ in (268) by any function of L_1 whose Fourier transform agrees (for each $u > 0$) over $(0,\infty)$ with that of $P_u(t)$. In particular, we may replace $P_u(t)$ by the analytic kernel,

$$K(t+iu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega(t+iu)}d\omega, \quad u > 0 \tag{272}$$

$$\begin{aligned} \text{where } F(\omega) &= 1 \quad \text{for} \quad \omega \geq 0 \\ &= \omega + 1 \quad \text{for} \quad -1 \leq \omega < 0 \\ &= 0 \quad \text{for} \quad \omega < -1. \end{aligned}$$

The definition of $F(\omega)$ for $\omega < 0$ is important only to the extent that the integral in (272) converges for $u > 0$ and implies

$$\int_{-\infty}^{\infty} |K(t+iu)|dt < \infty \quad \text{for} \quad u > 0. \tag{273}$$

It is sufficient for (273) that $e^{-\omega u}F(\omega)$ belong to L_2 and have a derivative in L_2 [see (5)].

Thus we have

$$g_u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)K(t+iu-s)ds = G(t+iu) \tag{274}$$

where $G(\tau)$ is analytic in the uhp and from (270) and (271)

$$|G(\tau)| \leq \sup_t |g(t)| \tag{275}$$

$$\lim_{u\to 0} G(t+iu) = g(t) \quad \text{for almost all } t. \tag{276}$$

This proves the first half of the theorem and we may as well write

$$G(t+iu) = g(t+iu) = g_u(t). \tag{277}$$

Now for the second half of the theorem we wish to establish that if $g(\tau)$ is bounded and analytic in the uhp, then

$$\int_{-\infty}^{\infty} g(t)h(-t)dt = 0 \tag{278}$$

for all functions h of L_1 whose Fourier transforms vanish over $(0,\infty)$, or equivalently

$$\int_{-\infty}^{\infty} g(t)h(t)dt = 0 \tag{279}$$

for all functions h of L_1 whose Fourier transforms vanish over $(-\infty,0)$. To do this we need some lemmas concerning analytic functions belonging to L_1 on lines parallel to the real axis.

Lemma 1. If $h(t)$ belongs to L_1 and its Fourier transform vanishes over

$(-\infty,0)$ then $h(t)$ has an analytic continuation $h(t+iu)$ in the upper half-plane $u > 0$ satisfying

$$\int_{-\infty}^{\infty} |h(t+iu)|dt \le \int_{-\infty}^{\infty} |h(t)|dt. \tag{280}$$

Proof. We have

$$h(t) = \int_0^{\infty} \hat{h}(\omega)e^{i\omega t}d\omega \tag{281}$$

with the Fourier integral providing the analytic continuation

$$h(t+iu) = \int_0^{\infty} \hat{h}(\omega)e^{i\omega(t+iu)}d\omega, \quad u > 0. \tag{282}$$

Since $\hat{h}(\omega) = 0$ for $\omega \le 0$ we may write

$$h(t+iu) = \int_{-\infty}^{\infty} \hat{h}(\omega)e^{-u|\omega|}e^{i\omega t}d\omega, \quad u > 0 \tag{283}$$

and hence conclude that

$$h(t+iu) = \int_{-\infty}^{\infty} h(s)P_u(t-s)ds \tag{284}$$

where $P_u(t)$ is the Poisson kernel defined in (269). Then (280) follows from (284), since the L_1 norm of the Poisson kernel is 1 for every $u >$ 0.

Lemma 2. Suppose $h(\tau)$, $\tau = t + iu$, is analytic in the strip $0 < u < b$ and satisfies

$$\int_{-\infty}^{\infty} |h(t+iu)|dt < \infty \quad \text{for} \quad 0 \le u < b. \tag{285}$$

Then

$$h(t+iu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{h}(\omega)e^{i\omega(t+iu)}d\omega, \quad 0 \le u < b \tag{286}$$

where

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt. \tag{287}$$

Proof: Defining

$$\hat{h}(\omega,u) = \int_{-\infty}^{\infty} h(t+iu)e^{-i\omega t}dt, \quad 0 \le u < b \tag{288}$$

we wish to show that

$$\hat{h}(\omega;u) = e^{-\omega u}\hat{h}(\omega). \tag{289}$$

We have

$$h(t+iu) = \lim_{\Omega\to\infty} \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \left\{1 - \frac{|\omega|}{\Omega}\right\} \hat{h}(\omega;u) e^{i\omega t} d\omega. \tag{290}$$

Also, from the analyticity of $h(\tau)$, we have

$$\frac{\partial}{\partial t} h(t+iu) = \frac{1}{i} \frac{\partial}{\partial u} h(t+iu), \quad 0 < u < b. \tag{291}$$

We would like to establish (289) from (291) by differentiating inside the integral of (290) but at this point we do not know enough about $\hat{h}(\omega;u)$ to justify the differentiation. Therefore, we will define

$$g(t+iu) = \int_{-\infty}^{\infty} k(s) h(t+iu-s) ds \tag{292}$$

where $k(t)$ is a function of L_1 whose Fourier transform $\hat{k}(\omega)$ does not vanish for any argument and such that $\omega\hat{k}(\omega)$ belongs to L_1. We would also like $k'(t)$ to belong to L_1. We may take

$$k(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}. \tag{293}$$

Then $g(\tau)$ is analytic in the strip and

$$\int_{-\infty}^{\infty} |g(t+iu)| dt \le \int_{-\infty}^{\infty} |h(t+iu)| dt. \tag{294}$$

We have

$$g(t+iu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{k}(\omega) \hat{h}(\omega;u) e^{i\omega t} d\omega, \quad 0 \le u < b, \tag{295}$$

and since $\omega\hat{k}(\omega)$ is in L_1,

$$\frac{\partial}{\partial t} g(t+iu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} i\omega \hat{k}(\omega) \hat{h}(\omega;u) e^{i\omega t} d\omega, \quad 0 \le u < b. \tag{296}$$

Now

$$g'_u(t) \equiv \frac{\partial}{\partial t} g(t+iu) \equiv \frac{\partial}{i\partial u} g(t+iu) \tag{297}$$

belongs to L_1 for $0 < u < b$ since

$$g'_u(t) = \int_{-\infty}^{\infty} h(s+iu) k'(t-s) ds \tag{298}$$

and k' belongs to L_1. Hence the function of t

$$\frac{\partial}{\partial u} g(t+iu)$$

has a Fourier transform for $0 < u < b$. Thus from (295)

$$\frac{\partial}{\partial u} g(t+iu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{k}(\omega) \frac{\partial}{\partial u} \hat{h}(\omega;u) e^{i\omega t} d\omega, \quad 0 < u < b. \quad (299)$$

Since $\hat{k}(\omega) \neq 0$ for $-\infty < \omega < \infty$, we conclude from (296), (297), and (299) that

$$\frac{1}{i} \frac{\partial}{\partial u} \hat{h}(\omega;u) = i\omega \hat{h}(\omega;u). \quad (300)$$

Then (289) follows from (300).

A corollary of Lemma 2 is the following

Corollary. If $g(\tau)$, $\tau = t + iu$, is analytic in the strip $a < u < b$ and satisfies

$$\int_{-\infty}^{\infty} |g(t+iu)| dt < \infty \quad \text{for} \quad a < u < b, \quad (301)$$

then

$$\int_{-\infty}^{\infty} g(t+iu) dt = \text{constant}, \quad \text{for} \quad a < u < b. \quad (302)$$

The corollary follows by applying Lemma 2 to the function $g(t + ia + i\epsilon)$ for arbitrarily small positive ϵ.

Lemma 3. If $g(\tau)$, $\tau = t + iu$, is analytic in the upper half plane $u > 0$ and satisfies

$$\int_{-\infty}^{\infty} |g(t+iu)| dt < \infty \quad \text{for} \quad u \geq 0$$

then the asymptotic estimate

$$\int_{-\infty}^{\infty} |g(t+iu)| dt = O\{e^{-au}\} \quad \text{as} \quad u \to \infty \quad (303)$$

implies

$$\int_{-\infty}^{\infty} |g(t+iu)| dt \leq e^{-au} \int_{-\infty}^{\infty} |g(t)| dt \quad \text{for} \quad u \geq 0 \quad (304)$$

and

$$g(t+iu) = \int_{-\infty}^{\infty} \hat{g}(\omega) e^{i\omega(t+iu)} d\omega \quad \text{for} \quad u > 0 \quad (305)$$

where

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} g(t) dt = 0 \quad \text{for} \quad \omega \leq a. \quad (306)$$

Proof: From Lemma 2 we have the representation (305) so that

$$\hat{g}(\omega)e^{-\omega u} = \int_{-\infty}^{\infty} g(t+iu)e^{-i\omega t}dt. \tag{307}$$

Thus

$$|\hat{g}(\omega)e^{-\omega u}| \le \int_{-\infty}^{\infty} |g(t+iu)|dt. \tag{308}$$

Then (303) and (308) imply

$$\hat{g}(\omega) = 0 \quad \text{for} \quad \omega < a \tag{309}$$

and since $\hat{g}(\omega)$ is continuous, (309) implies

$$\hat{g}(\omega) = 0 \quad \text{for} \quad \omega \le a. \tag{310}$$

Thus we may write

$$g(t+iu) = \frac{e^{-au}}{2\pi}\int_{-\infty}^{\infty} e^{-u|\omega-a|}\hat{g}(\omega)e^{i\omega t}dt. \tag{311}$$

Then since

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-u|\omega-a|}e^{i\omega t}dt = e^{iat}P_u(t) \tag{312}$$

where $P_u(t)$ is the Poisson kernel of (269), we have

$$g(t+iu) = e^{-au}\int_{-\infty}^{\infty} g(s)e^{ia(t-s)}\,P_u(t-s)ds \tag{313}$$

and (304) follows from (313), and Lemma 3 is proved.

Now we are prepared to prove the second half of the Paley-Wiener Theorem for L_∞. We have $g(\tau)$ analytic for $u > 0$ and

$$\sup_t |g(t+iu)| \le M \quad \text{for} \quad u \ge 0. \tag{314}$$

Now suppose $h(t)$ is any function of L_1 whose Fourier transform vanishes over $(-\infty,0)$. We have from Lemma 1 that $h(t)$ is the boundary value of a function $h(\tau)$ analytic in the upper half-plane $u > 0$, satisfying

$$\int_{-\infty}^{\infty} |h(t+iu)dt \le \int_{-\infty}^{\infty} |h(t)|dt, \quad u \ge 0. \tag{315}$$

Now we consider the function

$$f(\tau) = g(\tau)h(\tau) \tag{316}$$

which is analytic in the uhp and satisfies

$$\int_{-\infty}^{\infty} |f(t+iu)|dt \le M\int_{-\infty}^{\infty} |h(t)|dt. \tag{317}$$

Then from Lemma 3

$$\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = 0 \quad \text{for} \quad \omega \le 0. \tag{318}$$

In particular

$$\int_{-\infty}^{\infty} f(t)dt = \int_{-\infty}^{\infty} g(t)h(t)dt = 0 \tag{319}$$

which completes the proof of the first (one-sided) Paley-Wiener Theorem.

Theorem 2. It is sufficient to prove the two-sided Paley-Wiener Theorem for functions $g(t)$ whose Fourier transforms vanish outside $[-a,a]$. We show that $g(t)$ is the boundary value of an entire function of exponential type by defining

$$G_1(t+iu) = \int_{-\infty}^{\infty} g(s)e^{au}\frac{ue^{-ia(t-s)}}{\pi\{(t-s)^2+u^2\}}ds, \quad u>0 \tag{320}$$

$$G_2(t+iu) = \int_{-\infty}^{\infty} g(s)e^{-au}\frac{|u|e^{ia(t-s)}}{\pi\{(t-s)^2+u^2\}}ds, \quad u<0. \tag{321}$$

We have

$$|G_1(t+iu)| \le e^{au}\sup_t |g(t)|, \quad u>0 \tag{322}$$

$$|G_2(t+iu)| \le e^{-au}\sup_t |g(t)|, \quad u<0 \tag{323}$$

and since $g(t)$ is continuous

$$\lim_{u\to 0} G_1(t+iu) = g(t) \quad \text{for all } t \tag{324}$$

$$\lim_{u\to 0} G_2(t+iu) = g(t) \quad \text{for all } t. \tag{325}$$

Now we define

$$G_3(\tau) = \int_{-\infty}^{\infty} K(\tau-s)g(s)ds \tag{326}$$

where

$$K(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{K}(\omega)e^{i\omega\tau}d\omega$$

and

$$\hat{K}(\omega) = 1, \quad -a \le \omega < a$$
$$= 2\left(1 - \frac{\omega}{2a}\right), \quad a \le \omega \le 2a$$
$$= 0, \quad \omega \ge 2a$$
$$\hat{K}(-\omega) = \hat{K}(\omega). \tag{327}$$

Then $K(\tau)$ is an entire function belonging to L_1 along each line parallel to the real axis. So $G_3(\tau)$ is an entire function bounded on each line parallel to the real axis. Since the Fourier transform of $g(t)$ vanishes outside $[-a,a]$ we may replace the convolution kernels in (320) and (321) by $K(t + iu)$ since in each case their Fourier transforms agree over $[-a,a]$.

Thus

$$G_1(t + iu) = G_3(t + iu), \quad u > 0 \tag{328}$$

$$G_2(t + iu) = G_3(t + iu), \quad u < 0. \tag{329}$$

Hence $g(t)$ is the restriction to the real line of an entire function $G_3(t + iu) \equiv g(t + iu)$ satisfying

$$|g(t + iu)| \le e^{a|u|} \sup_t |g(t)|. \tag{330}$$

Now for the second half of the theorem we suppose that $g(\tau)$ is an entire function satisfying

$$\sup_t |g(t + iu)| \le e^{a|u|} \sup_t |g(t)| \tag{331}$$

and wish to conclude that

$$\int_{-\infty}^{\infty} g(t)h(t)dt = 0 \tag{332}$$

for all functions h in L_1 whose Fourier transforms vanish over $(-a,a)$ (and hence over $[-a,a]$). From the one-sided Paley-Wiener Theorem we have

$$\int_{-\infty}^{\infty} g(t)h_1(t)dt = 0 \tag{333}$$

for all functions h_1 in L_1 whose Fourier transforms are supported on $(-\infty,-a)$ and

$$\int_{-\infty}^{\infty} g(t)h_2(t)dt = 0 \tag{334}$$

for all function h_2 in L_1 whose Fourier transforms are supported on (a,∞). The difficulty we encounter in establishing (332) is that an arbitrary function h in L_1 whose Fourier transform vanishes over $[-a,a]$ cannot be decomposed as

$$h(t) = h_-(t) + h_+(t) \tag{335}$$

where h_- and h_+ belong to L_1, and the Fourier transform of h_- is supported on $(-\infty,-a)$, and the Fourier transform of h_+ is supported on (a,∞). In order to deduce (332) from (333) and (334) we have to approximate the test function h in (332) with bandlimited functions h_b. We may take

$$h_b(t) = \int_{-\infty}^{\infty} bK(bs)h(t-s)ds, \quad b > 0 \tag{336}$$

where

$$K(t) = \frac{2}{\pi}\frac{\sin^2 \frac{t}{2}}{t^2}. \tag{337}$$

Since

$$\int_{-\infty}^{\infty} K(t)dt = \int_{-\infty}^{\infty} |K(t)|dt = b\int_{\infty}^{\infty} |K(bt)|dt = 1,$$

we have

$$\int_{-\infty}^{\infty} |h_b(t)|dt \le \int_{-\infty}^{\infty} |h(t)|dt. \tag{338}$$

Also we may write

$$h(t) - h_b(t) = \int_{-\infty}^{\infty} bK(bs)\{h(t) - h(t-s)\}ds \tag{339}$$

which gives

$$\int_{-\infty}^{\infty} |h(t) - h_b(t)|dt \le \int_{-\infty}^{\infty} bK(bs)\mu_1(s;h)ds \tag{340}$$

where

$$\mu_1(s;h) = \int_{-\infty}^{\infty} |h(t) - h(t-s)|ds. \tag{341}$$

The function $\mu_1(s)$ is called the modulus of continuity of h. It is an even, *continuous*, bounded function of s (see Ref. 3) and

$$\begin{cases} \mu_1(0;h) = 0 \\ \mu_1(s;h) \le 2\int_{-\infty}^{\infty} |h(t)|dt = 2\|h\|_1. \end{cases} \tag{342}$$

Then writing

$$\int_{-\infty}^{\infty} bK(bs)\mu_1(s;h)ds = \int_{-\infty}^{\infty} K(t)\mu_1\left(\frac{t}{b}\,;h\right)dt \tag{343}$$

$$\leq \int_{-\sqrt{b}}^{\sqrt{b}} K(t)\mu_1\left(\frac{t}{b}\,;h\right)dt + 2\|h\|_1 \int_{|t|>\sqrt{b}} K(t)dt$$

it is clear that given $\epsilon > 0$ we can choose b so large ($a < b < \infty$) that

$$\int_{-\infty}^{\infty} |h(t) - h_b(t)|dt < \epsilon. \tag{344}$$

Now the Fourier transform of h_b is supported on the intervals $(-b,-a)$ and (a,b), so h_b does have the decomposition

$$h_b(t) = h_-(t) + h_+(t) \tag{345}$$

desired in (335). This follows from the existence† of a function $K_{a,b}(t)$ in L_1 whose Fourier transform satisfies

$$\begin{aligned} \hat{K}_{a,b}(\omega) &= 1, \quad a \leq \omega \leq b \\ &= 0, \quad \omega \leq -a \end{aligned} \tag{346}$$

so that

$$h_+(t) = \int_{-\infty}^{\infty} h_b(s)K_{a,b}(t-s)ds \tag{347}$$

and

$$\|h_+\|_1 \leq \|h_b\|_1 \cdot \|K_{a,b}\|_1. \tag{348}$$

We have

$$h_-(t) = h_b(t) - h_+(t). \tag{349}$$

So h_- also belongs to L_1.

Returning to (332) we have

$$\int_{-\infty}^{\infty} g(t)h(t)dt = \int_{-\infty}^{\infty} g(t)h_b(t)dt + \int_{-\infty}^{\infty} g(t)\{h(t) - h_b(t)\}dt. \tag{350}$$

Since

$$\begin{aligned} \int_{-\infty}^{\infty} g(t)h_b(t)dt &= \int_{-\infty}^{\infty} g(t)h_-(t)dt + \int_{-\infty}^{\infty} g(t)h_+(t)dt \\ &= 0 + 0 \end{aligned} \tag{351}$$

† See Appendix B for a good choice of $K_{a,b}(t)$.

we have

$$\left| \int_{-\infty}^{\infty} g(t)h(t)dt \right| \le \sup_t |g(t)| \int_{-\infty}^{\infty} |h(t) - h_b(t)|dt$$

$$\le \epsilon \sup_t |g(t)|. \tag{352}$$

Since we may choose b sufficiently large to make ϵ arbitrarily small we conclude that

$$\int_{-\infty}^{\infty} g(t)h(t)dt = 0 \tag{353}$$

for all h in L_1 whose Fourier transforms vanish over $(-a,a)$. This completes the proof of the two-sided Paley-Wiener Theorem.

Theorem 3. Here we wish to show that if $g(\tau)$ is analytic in the uhp and bounded on the real line as well as every line parallel to the real axis in the uhp, then the asymptotic estimate

$$\sup_t |g(t + iu)| = O\{e^{-au}\} \quad \text{as } u \to \infty \tag{354}$$

implies

$$\sup_t |g(t + iu)| \le e^{-au} \sup_t |g(t)|, \quad u \ge 0. \tag{355}$$

Now if x is any real number and y is any positive number, the function

$$h(\tau) = \frac{y}{\pi} \frac{\{g(\tau)e^{-ia\tau} - g(x + iy)e^{-ia(x+iy)}\}}{(\tau - x)^2 + y^2} \tag{356}$$

where we think of x and y fixed, is analytic in the upper half-plane $u > 0$ and satisfies

$$\int_{-\infty}^{\infty} |h(t + iu)|dt < \infty, \quad u \ge 0 \tag{357}$$

and

$$\int_{-\infty}^{\infty} |h(t + iu)|dt = O(1) \quad \text{as } u \to \infty. \tag{358}$$

It follows from Lemma 3 that

$$\int_{-\infty}^{\infty} h(t)dt = 0. \tag{359}$$

Hence

$$\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{g(t)e^{-iat}dt}{(t-x)^2+y^2}=g(x+iy)e^{-ia(x+iy)}. \tag{360}$$

Thus

$$|g(x+iy)|\le e^{-ay}\sup_t |g(t)|. \tag{361}$$

This inequality holds for any $-\infty < x < \infty$ and any $y > 0$. Then (355) follows from (361), if we note that (355) holds trivially for $u = 0$.

APPENDIX B

Reproducing Kernels of Small L_1 Norm

We would like to find a kernel $K_{\alpha,\beta}(t)$ of minimum L_1 norm whose Fourier transform satisfies

$$\begin{aligned}\hat{K}_{\alpha,\beta}(\omega) &= 1, \quad 0\le\omega\le\alpha\\ &= 0, \quad \omega>\beta\end{aligned} \tag{362}$$

where $0 < \alpha < \beta$.

Replacing $K_{\alpha,\beta}(t)$ by $1/\alpha K_{\alpha,\beta}(t/\alpha)$ we see that the minimum norm is a function of β/α, or if we like, a function of $\alpha/(\beta-\alpha)$. It is sufficient to consider functions $K_\lambda(t)$ whose Fourier transforms satisfy

$$\begin{aligned}\hat{K}_{\lambda}(\omega) &= 0, \quad \omega<0\\ &= 1, \quad 1\le\omega\le 1+\lambda\end{aligned} \tag{363}$$

where we make the identification

$$\lambda=\frac{\alpha}{\beta-\alpha}. \tag{364}$$

We will not treat the minimization problem here. Instead we give a construction for a particular function $\hat{K}_\lambda(\omega)$ which can be shown† to be the solution for the case $\lambda = n$, $n = 1,2,3,\ldots$. The construction provides an interpolation between the minimal norm values in case $n < \lambda < n + 1$.

We write

$$\lambda = n+\theta \tag{365}$$

where $n = [\lambda]$ is the largest integer contained in λ and $0\le\theta<1$. Then we set

$$K_\lambda(t) = 2\pi\{F_\lambda(t)\}^2 \tag{366}$$

† The details will be given in a future paper.

where the Fourier transform of $F_\lambda(t)$ vanishes for negative argument, and otherwise is defined by

$$\begin{aligned}\hat{F}_\lambda(\omega) &= a_k, \quad k \le \omega < k+1 \le n+1 \\ &= a_{n+1}, \quad n+1 \le \omega < n+1+\theta \\ &= 0, \quad \omega \ge n+1+\theta = \lambda+1 \end{aligned} \tag{367}$$

where the a_k are defined by

$$\frac{1}{\sqrt{1-z}} = \sum_0^\infty a_k z^k \tag{368}$$

i.e.,

$$\begin{aligned} a_k = \frac{(1/2)_k}{k!} &= \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\cdots\left(\frac{1}{2}+k-1\right)}{k!} \\ &= \frac{\Gamma\left(\frac{1}{2}+k\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(1+k)} \end{aligned} \tag{369}$$

We then have

$$\hat{K}_\lambda(\omega) = \int_0^\omega \hat{F}_\lambda(x)\hat{F}_\lambda(\omega - x)dx \tag{370}$$

which is a piecewise linear function satisfying

$$\hat{K}_\lambda(m) = \sum_{k=0}^{m-1} a_k a_{m-1-k} = 1 \tag{371}$$

for $m = 1,2,\ldots,n+1$. Thus

$$\begin{aligned}\hat{K}_\lambda(\omega) &= 1 \quad \text{for} \quad 1 \le \omega \le n+1 \\ &= \omega \quad \text{for} \quad 0 \le \omega \le 1. \end{aligned} \tag{372}$$

For $n+1 \le \omega \le n+1+\theta$ the convolution in (370) is independent of the definition of $\hat{F}_\lambda(x)$ for $x > n+1+\theta$; i.e.

$$\hat{K}_\lambda(\omega) = \hat{K}_{n+1}(\omega) \quad \text{for} \quad 0 \le \omega \le n+1+\theta. \tag{373}$$

So

$$\hat{K}_\lambda(\omega) = 1, \quad n+1 \le \omega \le n+1+\theta. \tag{374}$$

Thus

$$\begin{aligned}\hat{K}_\lambda(\omega) &= 1 \quad \text{for} \quad 1 \le \omega \le n+1+\theta \\ &= \omega \quad \text{for} \quad 0 \le \omega \le 1. \end{aligned} \tag{375}$$

We have

$$\mu(\lambda) \equiv \int_{-\infty}^{\infty} |K_\lambda(t)|dt = 2\pi \int_{-\infty}^{\infty} |F_\lambda(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} |\hat{F}_\lambda(\omega)|^2 d\omega$$

$$= \sum_0^n a_k^2 + \theta a_{n+1}^2 \quad (376)$$

which is a piecewise linear function of λ, i.e.,

$$\mu(\lambda) = (n+1-\lambda)\mu(n) + (\lambda - n)\mu(n+1), \quad n \le \lambda \le n+1. \quad (377)$$

We have

$$\mu(0) = 1, \quad \mu(1) = \frac{5}{4}, \quad \mu(2) = \frac{89}{64}. \quad (378)$$

The remainder of this appendix is devoted to estimating $\mu(n)$ for large n. We need an upper bound.

For convenience we set

$$\gamma(x) = \frac{\Gamma\left(\frac{1}{2} + x\right)}{\Gamma\,(1+x)} \quad (379)$$

and then

$$\mu(n) = \frac{1}{\pi} \sum_{k=0}^{n} \gamma^2(k). \quad (380)$$

First we estimate $\gamma(x)$. We have the representation for the Beta function

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{y-1}(1-t)^{x-1}dt \ (Re\, x > 0, Re\, y > 0). \quad (381)$$

Then

$$\gamma(x) = \frac{1}{\sqrt{\pi}} \int_0^1 \frac{(1-t)^x}{\sqrt{t(1-t)}} dt. \quad (382)$$

Setting $1 - t = e^{-s}$ we obtain

$$\gamma(x) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-s(x+1/2)}}{\sqrt{1-e^s}} ds$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-s(x+1/4)}}{\sqrt{2\sinh s/2}} ds. \quad (383)$$

Since

$$1 - e^{-s} < s$$

and $2 \sinh s/2 > s$ we obtain from (383)

$$\frac{1}{\sqrt{x + \frac{1}{2}}} < \gamma(x) < \frac{1}{\sqrt{x + \frac{1}{4}}}. \tag{384}$$

We can obtain a simple upper bound for $\mu(n)$ from (384). We have

$$\begin{aligned} \mu(n) &= 1 + \frac{1}{\pi} \sum_{k=1}^{n} \gamma^2(k) \\ &< 1 + \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{k + \frac{1}{4}}. \end{aligned} \tag{385}$$

Since t^{-1} is convex, we have

$$\int_T^{T+1} \frac{dt}{t} > \frac{1}{T + \frac{1}{2}}, \quad T > 0 \tag{386}$$

and thus

$$\mu(n) < 1 + \frac{1}{\pi} \int_{3/4}^{n+3/4} \frac{dt}{t} = 1 + \frac{1}{\pi} \log\left(1 + \frac{4n}{3}\right), \quad (n \geq 1). \tag{387}$$

Since $\log(1 + 4\lambda/3)$ is a concave function of λ and since $\mu(\lambda)$ is piecewise linear between integers, we conclude from (387) that

$$\mu(\lambda) < 1 + \frac{1}{\pi} \log\left(1 + \frac{4\lambda}{3}\right), \quad \lambda > 0. \tag{388}$$

A sharper estimate of $\mu(\lambda)$ for large λ is obtained as follows. We are interested in the constant term in the asymptotic expansion.

From (383) we have

$$\gamma\left(x - \frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-sx} ds}{\sqrt{1 - e^{-s}}}. \tag{389}$$

Then from the convolution theorem for Laplace transforms,

$$\gamma^2\left(x - \frac{1}{2}\right) = \int_0^\infty e^{-sx} \varphi(s) ds \tag{390}$$

where

$$\varphi(s) = \frac{1}{\pi} \int_0^s \frac{1}{\sqrt{1 - e^{-t}}} \frac{dt}{\sqrt{1 - e^{-(s-t)}}}. \tag{391}$$

Setting $e^{-t} = u$ in (391) we obtain

$$\varphi(s) = \frac{1}{\pi} \int_v^1 \frac{1}{\sqrt{1-u}\,\sqrt{u-v}} \frac{du}{\sqrt{u}} \tag{392}$$

where

$$v = e^{-s}.$$

Then with the substitution

$$1 - u = (1 - v)t$$

(392) becomes

$$\varphi(s) = \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{t}\,\sqrt{1-t}} \frac{dt}{\sqrt{1-(1-v)t}}. \tag{393}$$

Identifying $\varphi(s)$ with the hypergeometric function which has the representation,

$$_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt$$

$$(Re\, c > Re\, b > 0) \tag{394}$$

we see that

$$\varphi(s) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - e^{-s}\right). \tag{395}$$

Thus

$$\gamma^2\left(x - \frac{1}{2}\right) \equiv \frac{\Gamma^2(x)}{\Gamma^2\left(x + \frac{1}{2}\right)} = \int_0^\infty e^{-sx}{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - e^{-s}\right) ds \tag{396}$$

and

$$\begin{aligned}
\mu(n) = \frac{1}{\pi} \sum_{k=0}^{n} \gamma^2(k) &= \frac{1}{\pi} \int_0^\infty \frac{1 - e^{-s(n+1)}}{1 - e^{-s}} e^{-s/2}\varphi(s)ds \\
&= \frac{1}{\pi} \int_0^\infty \{e^{-s/2} - e^{-s(n+3/2)}\} \left\{\frac{\varphi(s)}{1 - e^{-s}} - \frac{1}{s}\right\} ds \\
&\quad + \frac{1}{\pi} \int_0^\infty \frac{e^{-s/2} - e^{-s(n+3/2)}}{s} ds \\
&= \frac{1}{\pi} \log(2n + 3) + \frac{1}{\pi} \int_0^\infty e^{-s/2} \left\{\frac{\varphi(s)}{1 - e^{-s}} - \frac{1}{s}\right\} ds \\
&\quad - \frac{1}{\pi} \int_0^\infty e^{-s(n+3/2)} \left\{\frac{\varphi(s)}{1 - e^{-s}} - \frac{1}{s}\right\} ds.
\end{aligned} \tag{397}$$

From (395) and the series

$$
{}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \tag{398}
$$

it is clear that

$$
\varphi(s) \geq \varphi(0) = 1, \quad s > 0 \tag{399}
$$

and hence that

$$
\frac{\varphi(s)}{1 - e^{-s}} - \frac{1}{s} \geq 0. \tag{400}
$$

Thus from (397) we have

$$
\mu(n) < \frac{1}{\pi} \log (2n + 3) + \frac{1}{\pi} \int_0^{\infty} e^{-s/2} \left\{ \frac{\varphi(s)}{1 - e^{-s}} - \frac{1}{s} \right\} ds \tag{401}
$$

and in fact

$$
\lim_{n \to \infty} \left\{ \mu(n) - \frac{1}{\pi} \log (2n + 3) \right\} = \frac{1}{\pi} \int_0^{\infty} e^{-s/2} \left\{ \frac{\varphi(s)}{1 - e^{-s}} - \frac{1}{s} \right\} ds. \tag{402}
$$

Now we can evaluate the integral in (402) by an indirect route.

We have

$$
\begin{aligned}
\mu(n) - \frac{1}{\pi} \sum_0^n \frac{1}{k+1} &= \sum_0^n \left\{ \frac{(1/2)_k (1/2)_k}{k!k!} - \frac{1}{\pi (k+1)} \right\} \\
&= \frac{1}{\pi} \sum_0^n \left\{ \gamma^2(k) - \frac{1}{k+1} \right\}.
\end{aligned} \tag{403}
$$

From the estimate (384) we see that the sum on the right converges as $n \to \infty$; i.e.

$$
\lim_{n \to \infty} \left\{ \mu(n) - \frac{1}{\pi} \sum_0^n \frac{1}{k+1} \right\} = \frac{1}{\pi} \sum_0^{\infty} \left\{ \gamma^2(k) - \frac{1}{k+1} \right\}. \tag{404}
$$

Now we can write

$$
\begin{aligned}
A \equiv \frac{1}{\pi} \sum_0^{\infty} \left\{ \gamma^2(k) - \frac{1}{k+1} \right\} &= \lim_{x \to 1} \left\{ \sum_{k=0}^{\infty} \frac{(1/2)_k (1/2)_k}{k!k!} x^k - \frac{x^k}{\pi (k+1)} \right\} \\
&= \lim_{x \to 1} \left\{ {}_2F_1 \left(\frac{1}{2}; \frac{1}{2}; 1; x \right) - \frac{1}{\pi x} \log \frac{1}{1-x} \right\}
\end{aligned} \tag{405}
$$

Then using (394) we have

$$\pi A = \lim_{x\to 1}\int_0^1 \left\{\frac{1}{\sqrt{t(1-t)(1-xt)}} - \frac{1}{1-xt}\right\} dt$$

$$= \lim_{x\to 1}\int_0^1 \frac{dt}{1-xt}\left\{\frac{\sqrt{1-xt}}{\sqrt{t(1-t)}} - 1\right\}$$

$$= \lim_{x\to 1}\int_0^1 \frac{dt}{1-xt}\left\{\frac{1}{\sqrt{t}} - 1\right\} + \lim_{x\to 1}\int_0^1 \frac{dt}{(1-xt)\sqrt{t}}\left\{\frac{\sqrt{1-xt}}{\sqrt{1-t}} - 1\right\}$$

$$= \int_0^1 \frac{dt}{(1+\sqrt{t})\sqrt{t}} + \lim_{x\to 1}\int_0^1 \frac{dt}{(1-xt)\sqrt{t}}\left\{\frac{\sqrt{1-xt}}{\sqrt{1-t}} - 1\right\}$$

$$= 2\log 2 + 2\log 2. \quad (406)$$

Some care is required in evaluating the last limit. An alternative way of obtaining the result is worth noting, as it makes use of an interesting series obtained from (396). A change of variables gives

$$\frac{\Gamma^2(x)}{\Gamma^2\left(x+\frac{1}{2}\right)} = \int_0^1 (1-t)^{x-1}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;t\right) dt$$

$$= \int_0^1 (1-t)^{x-1}\left\{\sum_0^\infty \frac{(1/2)_k(1/2)_k}{k!k!}t^k\right\} dt$$

$$= \frac{1}{x} + \frac{1}{x(x+1)}\left\{\frac{1}{2}\right\}^2 + \frac{2!}{x(x+1)(x+2)}\left\{\frac{\frac{1}{2}\cdot\frac{3}{2}}{2}\right\}^2 + \ldots \quad (407)$$

Then setting

$$F(x) = \frac{\Gamma^2\left(x+\frac{1}{2}\right)}{\Gamma^2(x+1)}$$

$$\left\{\frac{(1/2)_k}{k!}\right\}^2 = \left\{\frac{\Gamma\left(\frac{1}{2}+k\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(k+1)}\right\}^2 = \frac{1}{\pi}F(k)$$

we have

$$F\left(x-\frac{1}{2}\right) = \frac{1}{\pi}\sum_{k=0}^\infty \frac{k!}{(x)_{k+1}}F(k) \quad (408)$$

which is an interesting formula. In particular $\langle F(0), F(1), F(2), \ldots \rangle$ is an eigenvector of a certain infinite matrix. Also an interesting series for π is obtained by setting $x = n + 1/2$ (large n) in (408). Returning to (407) and recalling $a_k = (1/2)_k/k!$ we have

$$\frac{x\Gamma^2(x)}{\Gamma^2\left(x+\frac{1}{2}\right)} = \frac{\Gamma^2(x+1)}{x\Gamma^2\left(x+\frac{1}{2}\right)} = 1 + \frac{a_1^2}{x+1} + \frac{2!}{(x+1)(x+2)}a_2^2 + \dots \tag{409}$$

Then using the series

$$\frac{1}{x} = \int_0^1 (1-t)^{x-1}dt = \int_0^1 (1-t)^x(1+t+t^2+\dots)dt$$

$$= \frac{1}{x+1} + \frac{1}{(x+1)(x+2)} + \frac{2!}{(x+1)(x+2)(x+3)} + \dots \quad (x > 0)$$

we may write

$$\frac{1}{x}\left\{\frac{\Gamma^2(x+1)}{\Gamma^2\left(x+\frac{1}{2}\right)} - \frac{\Gamma^2(1)}{\Gamma^2\left(\frac{1}{2}\right)}\right\}$$

$$= 1 + \frac{1}{x+1}\left\{a_1^2 - \frac{1}{\pi}\right\} + \frac{2!}{(x+1)(x+2)}\left\{a_2^2 - \frac{1}{2\pi}\right\}$$

$$+ \dots \frac{k!}{(x+1)_k}\left\{a_k^2 - \frac{1}{k\pi}\right\} + \dots \tag{410}$$

Since

$$\frac{1}{\pi\left(k+\frac{1}{2}\right)} < a_k^2 < \frac{1}{\pi\left(k+\frac{1}{4}\right)}$$

we may let $x \to 0$ with the result

$$1 + \sum_{k=1}^{\infty}\left\{a_k^2 - \frac{1}{k\pi}\right\} = \frac{d}{dx}\frac{\Gamma^2(x+1)}{\Gamma^2\left(x+\frac{1}{2}\right)}\Bigg|_{x=0} = \frac{4}{\pi}\log 2 \tag{411}$$

and since $a_0^2 = 1$, this sum is the same as the sum on the right in (404). Hence the limit in (404) is

$$\lim_{n\to\infty}\left\{\mu(n) - \frac{1}{\pi}\sum_0^n \frac{1}{k+1}\right\} = \frac{4}{\pi}\log 2 \tag{412}$$

and since

$$\lim_{n\to\infty}\left\{\sum_{1}^{n}\frac{1}{k}-\log n\right\}=C=0.577215\ldots\text{ (Euler's constant)}\quad(413)$$

we have from (412), (413), and (402),

$$\lim_{n\to\infty}\left\{\mu(n)-\frac{1}{\pi}\log(2n+3)\right\}=\frac{1}{\pi}\int_0^\infty e^{-s/2}\left\{\frac{\varphi(s)}{1-e^{-s}}-\frac{1}{s}\right\}ds$$

$$=\frac{3}{\pi}\log 2+\frac{C}{\pi}.\quad(414)$$

Then from (401) and (414) we have

$$\mu(n)<\frac{1}{\pi}\log(2n+3)+\frac{3}{\pi}\log 2+\frac{C}{\pi}\quad(415)$$

and by the same argument used in establishing (388),

$$\mu(\lambda)<\frac{1}{\pi}\log(2\lambda+3)+\frac{3}{\pi}\log 2+\frac{C}{\pi},\quad \lambda>0.\quad(416)$$

We find from (397) and (414) that

$$\pi\mu(n)\sim\log(2n+3)+3\log 2+C$$

$$-\frac{3}{2(2n+3)}-\frac{43}{48(2n+3)^2}-\frac{7}{16(2n+3)^3}$$

$$+O(n^{-4}).\quad(417)$$

For comparing the estimates (389) and (415) we have

$$1+\frac{1}{\pi}\log\frac{4}{3}\doteq 1.0915720476\quad(418)$$

$$\frac{1}{\pi}\{4\log 2+C\}\doteq 1.066275853\quad(419)$$

and for use in (415) and (417)

$$\frac{1}{\pi}\{3\log 2+C\}\doteq\frac{2.656657207}{\pi}$$

$$\doteq .8456402533\quad(420)$$

In the following tabulations the estimates (388) and (415) and the asymptotic formula (417) are compared with the true value of $\mu(n)$.

n	$\mu(n)$	Asymptotic formula	Error
1	1.25	1.249927097	7.29 (−5)
2	1.390625	1.390607982	1.70 (−5)
3	1.48828125	1.488275479	5.77 (−6)
4	1.563049316	1.563046870	2.45 (−6)
5	1.623611450	1.623610248	1.20 (−6)
6	1.674500465	1.674499809	6.56 (−7)
7	1.718379259	1.718378872	3.87 (−7)
8	1.756944605	1.756944363	2.42 (−7)
9	1.791343941	1.791343782	1.59 (−7)
10	1.822389342	1.822389234	1.08 (−7)

n	Upper bound (388)	Upper bound (415)
1	1.269703286	1.357940252
2	1.413574619	1.465042692
3	1.512299999	1.545038559
4	1.587544884	1.608914025
5	1.648359655	1.662088992
6	1.699398305	1.707639405
7	1.743372924	1.747480071
8	1.782003284	1.782884290
9	1.816448738	1.814741844
10	1.847528028	1.843699061

APPENDIX C

Estimates for M_n

$$M_n = \frac{2}{n+1} \sum_{\substack{1 \le k \le n \\ k \text{ odd}}} \cot \frac{k\pi}{2(n+1)} \tag{421}$$

In order to express the sum as an integral we note first that

$$\begin{aligned} \pi \cot \pi x &= \frac{d}{dx} \log \sin \pi x \\ &= \frac{d}{dx} \log \frac{\pi}{\Gamma(x)\Gamma(1-x)} \\ &= \psi(1-x) - \psi(x) \end{aligned} \tag{422}$$

where

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \int_0^1 \left\{ -\frac{1}{\log t} - \frac{t^{x-1}}{1-t} \right\} dt, \quad x > 0. \tag{423}$$

So

$$\pi \cot \pi x = \int_0^1 \{t^{x-1} - t^{-x}\} \frac{dt}{1-t} \tag{424}$$

$$= \int_0^\infty \{e^{-u(x-1)} - e^{ux}\} \frac{e^{-u}}{1-e^{-u}} du, \quad 0 < x < 1.$$

Then for $0 < \theta < 1/\nu$, where ν is an odd integer,

$$\pi\theta \sum_{\substack{k=1 \\ k \text{ odd}}}^{\nu} \cot k\pi\theta = \theta \int_0^\infty \left\{ e^{u} \frac{(e^{-u\theta} - e^{-(\nu+2)u\theta})}{1 - e^{-2u\theta}} - \frac{(e^{u\theta} - e^{(\nu+2)u\theta})}{1 - e^{2u\theta}} \right\} \frac{e^{-u}}{1-e^{-u}} du \tag{425}$$

or

$$\pi\theta \sum_{\substack{k=1 \\ k \text{ odd}}}^{\nu} \cot k\pi\theta$$

$$= \int_0^\infty \left\{ e^{at} \frac{(e^{-t} - e^{-(\nu+2)t})}{1-e^{-2t}} + \frac{(e^{-t} - e^{\nu t})}{1-e^{-2t}} \right\} \cdot \frac{e^{-at}}{1-e^{-at}} dt$$

$$= \int_0^\infty \frac{e^{-t} - e^{-(\nu+2)t}}{1-e^{-2t}} dt - \int_0^\infty \frac{\{1 - e^{-(\nu+1)t}\}^2}{1-e^{-2t}} \frac{e^{(\nu-a)t}}{1-e^{-at}} dt \tag{426}$$

where $a = 1/\theta$. For n odd we take $\nu = n$, $a = 2(n+1)$. Then

$$M_n = \frac{4}{\pi} \int_0^\infty \frac{e^{-t} - e^{-(n+2)t}}{1-e^{-2t}} dt - \frac{4}{\pi} \int_0^\infty \frac{\{1 - e^{-(n+1)t}\}^2}{1-e^{-2t}} \frac{e^{-(n+2)t}}{1-e^{-2(n+1)t}} dt$$

$$= \frac{4}{\pi} \int_0^\infty \frac{e^{-t} - e^{-(n+2)t}}{1-e^{-2t}} dt - \frac{2}{\pi(n+1)} \int_0^\infty \frac{1-e^{-t}}{\sinh \dfrac{t}{n+1}} \frac{dt}{1+e^t} \qquad n \text{ odd} \tag{427}$$

The first integral is just the sum of the reciprocals of the odd integers from 1 through n. We have

$$\int_0^\infty \frac{e^{-t} - e^{-(n+2)t}}{t} dt = \log(n+2). \tag{428}$$

So

$$2\int_0^\infty \frac{e^{-t} - e^{-(n+2)t}}{1-e^{-2t}} dt = \log(n+2) + \int_0^\infty e^{-t} \left\{ \frac{2}{1-e^{-2t}} - \frac{1}{t} \right\} dt - \int_0^\infty e^{-(n+2)t} \left\{ \frac{2}{1-e^{-2t}} - \frac{1}{t} \right\} dt. \tag{429}$$

We have

$$\int_0^\infty e^{-t}\left\{\frac{2}{1-e^{-2t}}-\frac{1}{t}\right\}dt = \int_0^1\left\{\frac{2}{1-u^2}+\frac{1}{\log u}\right\}du$$

$$= \int_0^1\left\{\frac{1}{1-u}+\frac{1}{1+u}+\frac{1}{\log u}\right\}du$$

$$= \int_0^1 \frac{du}{1+u}+\int_0^1\left\{\frac{1}{1-u}+\frac{1}{\log u}\right\}du$$

$$= \log 2 - \psi(1)$$

$$= \log 2 + \gamma \tag{430}$$

where

$$\gamma = .577215\ldots \quad \text{(Euler's constant).} \tag{431}$$

Thus

$$M_n = \frac{2}{\pi}\log(n+2) + \frac{2}{\pi}(\log 2+\gamma) - \frac{2}{\pi}\int_0^\infty e^{-(n+2)t}\left\{\frac{2}{1-e^{-2t}}-\frac{1}{t}\right\}dt - \frac{2}{\pi(n+1)}\int_0^\infty \frac{1-e^{-t}}{\sinh\dfrac{t}{n+1}}\,\frac{dt}{1+e^t} \qquad (n \text{ odd}) \tag{432}$$

We may write

$$\int_0^\infty e^{-(n+2)t}\left\{\frac{2}{1-e^{-2t}}-\frac{1}{t}\right\}dt = \int_0^\infty e^{-(n+1)t}\left\{\frac{1}{\sinh t}-\frac{e^{-t}}{t}\right\}dt$$

$$= \int_0^\infty e^{-(n+1)t}\left\{\frac{1}{\sinh t}-\frac{1}{t}\right\}dt + \int_0^\infty e^{-(n+1)t}(1-e^{-t})\frac{dt}{t}$$

$$= \frac{1}{n+1}\int_0^\infty e^{-t}\left\{\frac{1}{\sinh\dfrac{t}{n+1}}-\frac{n+1}{t}\right\}dt + \log\frac{n+2}{n+1}. \tag{433}$$

Also

$$\frac{1}{n+1}\int_0^\infty \frac{1-e^{-t}}{1+e^t}\,\frac{dt}{\sinh\dfrac{t}{n+1}}$$

$$= \frac{1}{n+1} \int_0^\infty \frac{1-e^{-t}}{1+e^t} \left\{ \frac{1}{\sinh \dfrac{t}{n+1}} - \frac{n+1}{t} \right\} dt + \int_0^\infty \frac{1-e^{-t}}{1+e^t} \frac{dt}{t} \tag{434}$$

and (Ref. 9, p. 327)

$$\int_0^\infty \frac{1-e^{-t}}{1+e^t} \frac{dt}{t} = \log \frac{\pi}{2}. \tag{435}$$

Thus we find from (432)–(435),

$$M_n = \frac{2}{\pi} \log (n+1) + \frac{2}{\pi} \left(\log \frac{4}{\pi} + \gamma \right) + \frac{4}{\pi(n+1)} \int_0^\infty \left\{ \frac{n+1}{t} - \frac{1}{\sinh \dfrac{t}{n+1}} \right\} \frac{dt}{1+e^t} \qquad n \text{ odd.} \tag{436}$$

Clearly the integral in (436) is positive. In order to obtain the asymptotic series for M_n, n odd, we use the generating function for the Bernoulli polynomials (Ref. 1, formula 23.1.1)

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^\infty B_k(x) \frac{t^k}{k!} \qquad |t| < 2\pi \tag{437}$$

to obtain

$$\frac{t}{2 \sinh \dfrac{t}{2}} = \sum_{k=0}^\infty B_k \left(\frac{1}{2} \right) \frac{t^k}{k!} \qquad |t| < 2\pi \tag{438}$$

where (Ref. 1, formula 23.1.21)

$$B_k \left(\frac{1}{2} \right) = -(1 - 2^{1-k}) B_k, \quad B_k \equiv B_k(0). \tag{439}$$

Thus

$$\frac{t}{\sinh t} = - \sum_{k=0}^\infty (2^k - 2) B_k \frac{t^k}{k!} \qquad |t| < \pi. \tag{440}$$

We have (Ref. 1, formula 23.1.19)

$$B_{2k+1} = 0, \quad k = 1,2, \ldots \tag{441}$$

and (Ref. 1, Table 23.2)

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}$$

$$B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730},$$

$$B_{14} = \frac{7}{6}. \tag{442}$$

So

$$\frac{1}{t} - \frac{1}{\sinh t} = \sum_{k=1}^{\infty} (2^{2k} - 2) \frac{B_{2k} t^{2k-1}}{(2k)!} \tag{443}$$

$$= \frac{t}{6} - \frac{7}{360} t^3 + \frac{31}{15120} t^5 + \dots \quad |t| < \pi.$$

From Ref. 9, p. 325, we have

$$\int_0^\infty \frac{x^{2k-1}}{e^x + 1} dx = (2^{2k} - 2)|B_{2k}| \frac{\pi^{2k}}{4k}, \quad k = 1,2, \dots \tag{444}$$

Also, (Ref. 1, formula 23.1.18)

$$B_{2n} = B_{2n}(0) = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}}, \quad n = 1,2, \dots \tag{445}$$

so

$$B_{2k} = (-1)^{k+1} |B_{2k}|. \tag{446}$$

Thus we obtain the asymptotic series

$$\frac{1}{n+1} \int_0^\infty \left\{ \frac{n+1}{t} - \frac{1}{\sinh \dfrac{t}{n+1}} \right\} \frac{dt}{1+e^t}$$

$$\sim \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (2^{2k} - 2)^2 B_{2k}^2}{4k(2k)!} \left(\frac{\pi}{n+1} \right)^{2k}$$

$$\approx \frac{1}{72} \left(\frac{\pi}{n+1} \right)^2 - \frac{49}{43200} \left(\frac{\pi}{n+1} \right)^4 + \dots \tag{447}$$

and hence

$$M_n \sim \frac{2}{\pi} \log (n+1) + \frac{2}{\pi} \left(\log \frac{4}{\pi} + \gamma \right) + \frac{\pi}{18(n+1)^2} - \frac{49\,\pi^3}{10800(n+1)^4}$$

$$+ \dots + \frac{(-1)^{k+1}}{\pi} \frac{(2^{2k} - 2)^2 B_{2k}^2}{k(2k)!} \left(\frac{\pi}{n+1} \right)^{2k} + \dots \quad (n \text{ odd}). \tag{448}$$

Now we would like to show that

$$M_n > \frac{2}{\pi}\log(n+1) + \frac{2}{\pi}\left(\log\frac{4}{\pi} + \gamma\right)$$

$$< \frac{2}{\pi}\log(n+1) + \frac{2}{\pi}\left(\log\frac{4}{\pi} + \gamma\right) + \frac{\pi}{18(n+1)^2}$$

$$> \frac{2}{\pi}\log(n+1) + \frac{2}{\pi}\left(\log\frac{4}{\pi} + \gamma\right) + \frac{\pi}{18(n+1)^2} - \frac{49\pi^3}{10800(n+1)^4}$$

$$\text{etc.,} \qquad (n \text{ odd}) \qquad (449)$$

i.e., that the error in truncating the asymptotic series has the same sign as the next term of the series. To do this, we show that for $t > 0$

$$\frac{1}{t} - \frac{1}{\sinh t} > 0$$

$$< \frac{t}{6}$$

$$> \frac{t}{6} - \frac{7}{360}t^3$$

$$< \frac{t}{6} - \frac{7}{360}t^3 + \frac{31}{15120}t^5$$

$$\text{etc.} \qquad (450)$$

We have (Ref. 9, p. 23)

$$\frac{t}{\sinh t} = 1 + 2t^2 \sum_{k=1}^{\infty} (-1)^k \frac{1}{t^2 + k^2\pi^2} \qquad (451)$$

or

$$\frac{1}{t^2}\left\{1 - \frac{t}{\sinh t}\right\} = 2\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{t^2 + k^2\pi^2}. \qquad (452)$$

Now consider the polynomial

$$P_{2n}(t;k) = 2\frac{1 - \left(\frac{t}{ik\pi}\right)^{2n+2}}{t^2 + k^2\pi^2}. \qquad (453)$$

We have

$$P_{2n}(t) \equiv \sum_{k=1}^{\infty} (-1)^{k+1} P_{2n}(t;k) \qquad (454)$$

$$= \frac{1}{t^2}\left\{1 - \frac{t}{\sinh t}\right\} - 2(-1)^{n+1}t^{2n+2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k\pi)^{2n+2}(t^2 + k^2\pi^2)}$$

or

$$\frac{1}{t^2}\left\{1 - \frac{t}{\sinh t}\right\} - P_{2n}(t)$$
$$= 2(-1)^{n+1}t^{2n+2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{2n+2}(t^2 + k^2\pi^2)}. \qquad (455)$$

It follows that $P_{2n}(t)$ is a polynomial of degree $2n$ which agrees with the first $(2n + 1)$ terms of the Taylor series of $t^{-2}\{1 - t/\sinh t\}$. The sign of the difference in (455) is $(-1)^{n+1}$, as the sum is clearly positive. Then (450), and hence (449), follows from (455).

For n an even integer, the asymptotic expansion of M_n is not so readily obtained. In this case we set $\nu = (n - 1)$ in (426) and keep $a = 2(n + 1)$. Thus

$$M_n = \frac{4}{\pi}\int_0^\infty \frac{e^{-t} - e^{-(n+1)t}}{1 - e^{-2t}}\, dt$$
$$- \frac{4}{\pi}\int_0^\infty \frac{(1 - e^{-nt})^2}{1 - e^{-2t}} \frac{e^{-(n+3)t}}{1 - e^{-2(n+1)t}}\, dt, \qquad n \text{ even.} \qquad (456)$$

We have

$$2\int_0^\infty \frac{e^{-t} - e^{-(n+1)t}}{1 - e^{-2t}}\, dt = \log (n + 1)$$
$$+ \int_0^\infty \{e^{-t} - e^{-(n+1)t}\}\left\{\frac{2}{1 - e^{-2t}} - \frac{1}{t}\right\} dt. \qquad (457)$$

Now we would like to express the second integral in (456) as an asymptotic series in $(n + 1)^{-1}$. For convenience we set $e^{-t} = x$. Then

$$\frac{(1 - e^{-nt})^2}{1 - e^{-2(n+1)t}} = \frac{(1 - x^n)^2}{1 - x^{2n+2}} = \frac{\{1 - x^{n+1} - x^n(1 - x)\}^2}{(1 - x^{n+1})(1 + x^{n+1})}$$
$$= \left\{1 - \frac{x^n(1 - x)}{1 - x^{n+1}}\right\}\left\{\frac{1 - x^{n+1}}{1 + x^{n+1}} - \frac{x^n(1 - x)}{1 + x^{n+1}}\right\}$$
$$= \frac{1 - x^{n+1}}{1 + x^{n+1}} - \frac{x^n(1 - x)}{1 + x^{n+1}} - \frac{x^n(1 - x)}{1 + x^{n+1}} + \frac{x^{2n}(1 - x)^2}{1 - x^{2n+2}}.$$

Therefore

$$\int_0^\infty \frac{(1 - e^{-nt})^2}{1 - e^{-2t}} \frac{e^{-(n+3)t}}{1 - e^{-2(n+1)t}}\, dt = \int_0^\infty \frac{1 - e^{-(n+1)t}}{1 + e^{-(n+1)t}} \frac{e^{-(n+3)t}}{1 - e^{-2t}}\, dt$$
$$- 2\int_0^\infty \frac{e^{-nt}(1 - e^{-t})}{1 + e^{-(n+1)t}} \frac{e^{-(n+3)t}}{1 - e^{-2t}}\, dt$$
$$+ \int_0^\infty \frac{e^{-2nt}(1 - e^{-t})^2}{1 - e^{-2(n+1)t}} \frac{e^{-(n+3)t}}{1 - e^{-2t}}\, dt$$

$$= \int_0^\infty \frac{1 - e^{-(n+1)t}}{1 + e^{(n+1)t}} \frac{dt}{e^{2t} - 1}$$

$$- 2 \int_0^\infty \frac{e^{-2(n+1)t}}{1 + e^{-(n+1)t}} \frac{dt}{e^t + 1}$$

$$+ \int_0^\infty \frac{e^{-3(n+1)t}}{1 - e^{-2(n+1)t}} \frac{1 - e^{-t}}{1 + e^{-t}} dt. \quad (458)$$

We can combine a part of the second integral with the last by noting that

$$\frac{1}{e^t + 1} = \frac{1}{2} - \frac{1}{2} \tanh \frac{t}{2}.$$

Then

$$-2 \int_0^\infty \frac{e^{-2(n+1)t}}{1 + e^{-(n+1)t}} \frac{dt}{e^t + 1} + \int_0^\infty \frac{e^{-3(n+1)t}}{1 - e^{-2(n+1)t}} \tanh \frac{t}{2} dt$$

$$= - \frac{1}{(n+1)} \int_0^\infty \frac{e^{-2t}}{1 + e^{-t}} dt + \frac{1}{(n+1)} \int_0^\infty \frac{e^{-2t}}{1 - e^{-2t}} \tanh \frac{t}{2(n+1)} dt.$$

By making use of (430) and (435) we obtain

$$M_n = \frac{2}{\pi} \log(n+1) + \frac{2}{\pi} \left(\log \frac{4}{\pi} + \gamma \right)$$

$$- \frac{2}{\pi} \int_0^\infty e^{-(n+1)t} \left\{ \frac{2}{1 - e^{-2t}} - \frac{1}{t} \right\} dt$$

$$+ \frac{2}{\pi(n+1)} \int_0^\infty \frac{1 - e^{-t}}{1 + e^t} \left\{ \frac{n+1}{t} - \frac{2}{\exp\left(\frac{2t}{n+1}\right) - 1} \right\} dt$$

$$+ \frac{4}{\pi(n+1)} \int_0^\infty \frac{e^{-2t}}{1 + e^{-t}} dt$$

$$- \frac{4}{\pi(n+1)} \int_0^\infty \frac{e^{-2t}}{1 - e^{-2t}} \tanh \frac{t}{2(n+1)} dt, \qquad n \text{ even.} \quad (459)$$

From (437), (441), and (442) we have

$$\frac{2}{e^{2t} - 1} - \frac{1}{t} = \sum_{k=1}^{\infty} 2^k \frac{B_k t^{k-1}}{k!} = -1 + \sum_{k=1}^{\infty} 2^{2k} \frac{B_{2k} t^{2k-1}}{(2k)!}, \qquad |t| < \pi \quad (460)$$

$$\frac{2}{1 - e^{-2t}} - \frac{1}{t} = 1 + \sum_{k=1}^{\infty} 2^{2k} \frac{B_{2k} t^{2k-1}}{(2k)!}, \qquad |t| < \pi \quad (461)$$

i.e.,

$$\frac{2}{1-e^{-2t}}-\frac{1}{t}-1=\frac{2}{e^{2t}-1}-\frac{1}{t}+1=\sum_{k=1}^{\infty}2^{2k}\frac{B_{2k}t^{2k-1}}{(2k)!}. \qquad (462)$$

Thus

$$\frac{2}{\pi(n+1)}\int_0^{\infty}\frac{1-e^{-t}}{1+e^{t}}\left\{\frac{n+1}{t}-\frac{2}{\exp\left(\frac{2t}{n+1}\right)-1}\right\}dt$$

$$-\frac{2}{\pi}\int_0^{\infty}e^{-(n+1)t}\left\{\frac{2}{1-e^{-2t}}-\frac{1}{t}\right\}dt$$

$$=\frac{2}{\pi(n+1)}\int_0^{\infty}\frac{1-e^{-t}}{1+e^{t}}\left\{\frac{n+1}{t}-\frac{2}{\exp\left(\frac{2t}{n+1}\right)-1}-1\right\}dt$$

$$-\frac{2}{\pi(n+1)}\int_0^{\infty}e^{-t}\left\{\frac{2}{1-\exp\left(\frac{-2t}{n+1}\right)}-\frac{n+1}{t}-1\right\}dt$$

$$+\frac{2}{\pi(n+1)}\int_0^{\infty}\left\{\frac{1-e^{-t}}{1+e^{t}}-e^{-t}\right\}dt$$

$$=\frac{2}{\pi(n+1)}\int_0^{\infty}\left\{\frac{1-e^{-t}}{1+e^{t}}+e^{-t}\right\}\left\{\frac{n+1}{t}-\frac{2}{\exp\left(\frac{2t}{n+1}\right)-1}-1\right\}dt$$

$$-\frac{4}{\pi(n+1)}\int_0^{\infty}\frac{e^{-t}}{1+e^{t}}dt.$$

Then from the above and (459) we have

$$M_n=\frac{2}{\pi}\log(n+1)+\frac{2}{\pi}\left(\log\frac{4}{\pi}+\gamma\right)$$

$$+\frac{4}{\pi(n+1)}\int_0^{\infty}\left\{\frac{n+1}{t}-\frac{2}{\exp\left(\frac{2t}{n+1}\right)-1}-1\right\}\frac{dt}{1+e^{t}}$$

$$-\frac{2}{\pi(n+1)}\int_0^{\infty}\frac{e^{-t}}{1-e^{-t}}\tanh\frac{t}{4(n+1)}dt, \qquad n \text{ even} \qquad (463)$$

which is a form suitable for asymptotic expansion. We have

$$\tanh x = \sum_{k=1}^{\infty} 2^{2k}(2^{2k}-1)B_{2k}\frac{x^{2k-1}}{(2k)!} \tag{464}$$

$$\begin{aligned}\int_0^\infty t^k \frac{dt}{1+e^t} &= \int_0^\infty t^k(e^{-t} - e^{-2t} + e^{-3t} + \ldots)\,dt \\ &= k!\left(1 - \frac{1}{2^{k+1}} + \frac{1}{3^{k+1}} - \frac{1}{4^{k+1}} + \ldots\right) \\ &= k!\left(1 - \frac{1}{2^k}\right)\zeta(k+1)\end{aligned} \tag{465}$$

where

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}, \quad n > 1. \tag{466}$$

Also

$$\begin{aligned}\int_0^\infty \frac{e^{-t}}{1-e^{-t}} t^k\,dt &= \int_0^\infty t^k(e^{-t} + e^{-2t} + e^{-3t}\ldots)\,dt \\ &= k!\left(1 + \frac{1}{2^{k+1}} + \frac{1}{3^{k+1}}\ldots\right) \\ &= k!\,\zeta(k+1).\end{aligned} \tag{467}$$

From (460) and (465) we have

$$\begin{aligned}&\frac{4}{\pi(n+1)}\int_0^\infty \left\{\frac{n+1}{t} - \frac{2}{\exp\left(\dfrac{2t}{n+1}\right) - 1} - 1\right\}\frac{dt}{1+e^t} \\ &\quad\approx -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{B_{2k}2^{2k}(2k-1)!}{(2k)!\,(n+1)^{2k}}\left(1 - \frac{1}{2^{2k-1}}\right)\zeta(2k) \\ &\quad\approx -\frac{2}{\pi}\sum_{k=1}^{\infty}(2^{2k}-2)\frac{B_{2k}}{k}\frac{\zeta(2k)}{(n+1)^{2k}}\end{aligned} \tag{468}$$

and from (467) and (464)

$$\begin{aligned}&-\frac{2}{\pi(n+1)}\int_0^\infty \frac{e^{-t}}{1-e^{-t}}\tanh\frac{t}{4(n+1)}\,dt \\ &\quad\sim -\frac{2}{\pi}\sum_{k=1}^{\infty} 2^{2k}(2^{2k}-1)\frac{B_{2k}}{(2k)!4^{2k-1}(n+1)^{2k}}(2k-1)!\,\zeta(2k) \\ &\quad\sim -\frac{2}{\pi}\sum_{k=1}^{\infty}(2-2^{1-2k})\frac{B_{2k}}{k}\frac{\zeta(2k)}{(n+1)^{2k}}.\end{aligned} \tag{469}$$

From (445) we have

$$B_{2k} = (-1)^{k+1} 2 \frac{(2k)!}{(2\pi)^{2k}} \zeta(2k). \tag{470}$$

Adding (468) and (469), using (470), we obtain

$$\begin{aligned} M_n \sim \frac{2}{\pi} \log (n+1) + \frac{2}{\pi} \left(\log \frac{4}{\pi} + \gamma \right) \\ - \frac{7}{36} \frac{\pi}{(n+1)^2} + \frac{127}{21600} \frac{\pi^3}{(n+1)^4} + \dots \\ + \frac{1}{\pi} (-1)^k (2^{4k} - 2) \frac{B_{2k}^2}{k(2k)!} \frac{\pi^{2k}}{(n+1)^{2k}} + \dots, \qquad n \text{ even.} \end{aligned} \tag{471}$$

In the same manner as before, we can establish that

$$\begin{aligned} M_n < \frac{2}{\pi} \log (n+1) + \frac{2}{\pi} \left(\log \frac{4}{\pi} + \gamma \right) \\ > \frac{2}{\pi} \log (n+1) + \frac{2}{\pi} \left(\log \frac{4}{\pi} + \gamma \right) - \frac{7}{36} \frac{\pi}{(n+1)^2} \\ < \frac{2}{\pi} \log (n+1) + \frac{2}{\pi} \left(\log \frac{4}{\pi} + \gamma \right) - \frac{7}{36} \frac{\pi}{(n+1)^2} + \frac{127}{21600} \frac{\pi^3}{(n+1)^4} \\ \text{etc.,} \qquad \text{for } n \text{ even.} \end{aligned} \tag{472}$$

In this case we have two functions to consider in the polynomial approximation problem. First we note that

$$\frac{2}{e^{2x} - 1} + 1 = \coth x \tag{473}$$

and

$$\begin{aligned} \coth x = \frac{d}{dx} \log \sinh x = \frac{d}{dx} \log \left\{ x \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2\pi^2} \right) \right\} \\ = \frac{1}{x} + 2x \sum_{k=1}^{\infty} \frac{1}{x^2 + k^2\pi^2}. \end{aligned} \tag{474}$$

Clearly, for $x > 0$,

$$\frac{2}{e^{2x} - 1} - \frac{1}{x} < 0 \tag{475}$$

and

$$\frac{2}{e^{2x} - 1} - \frac{1}{x} + 1 = \coth x - \frac{1}{x} = 2x \sum_{k=1}^{\infty} \frac{1}{x^2 + k^2\pi^2} > 0. \tag{476}$$

Now we have

$$\frac{\coth x - \frac{1}{x}}{x} = 2\sum_{k=1}^{\infty} \frac{1}{x^2 + k^2\pi^2}. \tag{477}$$

Defining as before

$$P_{2n}(x;k) = \frac{1 - \left(\frac{x}{ik\pi}\right)^{2n+2}}{x^2 + k^2\pi^2}, \qquad k = 1,2,3,\ldots \tag{478}$$

and then

$$\begin{aligned} Q_{2n}(x) &\equiv 2\sum_{k=1}^{\infty} P_{2n}(x;k) \\ &= \frac{\coth x - \frac{1}{x}}{x} - (-1)^{n+1}\left(\frac{x}{\pi}\right)^{2n+2} \sum_{k=1}^{\infty} \frac{2}{k^{2n+2}(x^2 + k^2\pi^2)} \end{aligned} \tag{479}$$

we have

$$\frac{\coth x - \frac{1}{x}}{x} - Q_{2n}(x) = (-1)^{n+1}\left(\frac{x}{\pi}\right)^{2n+2} \sum_{k=1}^{\infty} \frac{2}{k^{2n+2}(x^2 + k^2\pi^2)}. \tag{480}$$

So $Q_{2n}(x)$ is a polynomial of degree $2n$ which agrees with the first $(2n + 1)$ terms of the Taylor series of $x^{-1}\{\coth x - x^{-1}\}$ and the sign of the difference is $(-1)^{n+1}$.

For the second function we have

$$\tanh x = \frac{d}{dx} \log \cosh x = 2x \sum_{k=1}^{\infty} \frac{1}{x^2 + (2k-1)^2 \frac{\pi^2}{4}}. \tag{481}$$

Now we define

$$p_{2n}(x;k) = \frac{1 - \left(\frac{2x}{i(2k-1)\pi}\right)^{2n+2}}{x^2 + (2k-1)^2 \frac{\pi^2}{4}}, \qquad k = 1,2,\ldots \tag{482}$$

and

$$q_{2n}(x) = 2 \sum_{k=1}^{\infty} p_{2n}(x;k)$$

$$= \frac{\tanh x}{x} - (-1)^{n+1} \left(\frac{2x}{\pi}\right)^{2n+2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2n+2} \left\{x^2 + (2k-1)^2 \frac{\pi^2}{4}\right\}}. \tag{483}$$

Then the alternating sign of the error

$$\left\{\frac{\tanh x}{x} - q_{2n}(x)\right\}$$

follows; i.e.,

$$\begin{aligned} \tanh x &< x \\ &> x - \frac{x^3}{3} \\ &< x - \frac{x^3}{3} + \frac{2}{15} x^5 \\ &\text{etc.,} \quad \text{for } x > 0. \end{aligned} \tag{484}$$

The inequalities (472) then follow from (463), (476), (480), and (484).

Then for n even or odd we certainly have

$$M_n > \frac{2}{\pi} \log (n+1) + \frac{2}{\pi} \left(\log \frac{4}{\pi} + \gamma\right) - \frac{7}{36} \frac{\pi}{(n+1)^2} \tag{485}$$

or, giving away a little,

$$M_n > \frac{2}{\pi} \log n + \frac{2}{\pi} \left(\log \frac{4}{\pi} + \gamma\right), \qquad n \geq 1. \tag{486}$$

Obviously (486) is true for n odd since for n odd we may replace n in (486) by $(n + 1)$, and for n even we consider

$$\log (n+1) - \frac{7}{72} \frac{\pi^2}{(n+1)^2} = \log n + \log \left(1 + \frac{1}{n}\right) - \frac{7}{72} \frac{\pi^2}{(n+1)^2}.$$

Now

$$\log (1 + x) = \int_0^x \frac{dt}{1+t} > \int_0^x (1-t) dt = x - \frac{x^2}{2}, \quad x > 0.$$

So

$$\log\left(1+\frac{1}{n}\right)-\frac{7}{72}\frac{\pi^2}{(n+1)^2}>\frac{1}{n}-\frac{1}{2n^2}-\frac{1}{n^2}>0 \quad \text{for } n \geq 2.$$

Thus (486) is valid for $n \geq 1$.

We note that the actual computation of M_n is simplified, particularly for n odd, by making use of the identity

$$\cot\theta=\frac{1+\cos 2\theta}{\sin 2\theta}.$$

Then for $\theta=\pi/2(n+1)$,

$$M_n=\frac{2}{n+1}\sum_{\substack{1\leq k\leq n\\ k\text{ odd}}}\cot k\theta=\frac{2}{n+1}\sum_{\substack{1\leq k\leq n\\ k\text{ odd}}}\frac{1+\cos 2k\theta}{\sin 2k\theta}. \tag{487}$$

For n odd, $k=1,3,5,\ldots,n$,

$$\cos\frac{2k\pi}{2(n+1)}=-\cos\frac{2(n+1-k)\pi}{2(n+1)}$$
$$\sin\frac{2k\pi}{2(n+1)}=\sin\frac{2(n+1-k)\pi}{2(n+1)}.$$

So

$$M_n=\frac{2}{n+1}\sum_{\substack{1\leq k\leq n\\ k\text{ odd}}}\frac{1}{\sin\dfrac{k\pi}{n+1}}, \quad n\text{ odd}. \tag{488}$$

We find

$$M_1=1$$
$$M_2=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}=1.1547005\ldots$$
$$M_3=\sqrt{2}=1.4142135\ldots$$

$$M_4=\frac{2}{5}\left[\frac{1}{\sin\frac{\pi}{5}}+\frac{2}{\sin\frac{2\pi}{5}}\right]=\frac{4}{5\sin\frac{2\pi}{5}}[1+\cos\pi/5]$$
$$=1.5216904\ldots$$

$$M_5=\frac{5}{3}=1.6666666\ldots$$

For use in the asymptotic formulas we have

$$\frac{2}{\pi} = 0.636619772367\ldots$$

$$\log\frac{4}{\pi} = 0.241564475270\ldots$$

$$\gamma = 0.577215664901\ldots$$

$$\log\frac{4}{\pi} + \gamma = 0.818780140172\ldots$$

$$\frac{2}{\pi}\left(\log\frac{4}{\pi} + \gamma\right) = 0.521251626\ldots$$

$$\frac{\pi}{18} = 0.174532925\ldots$$

$$\frac{7\pi}{36} = 0.610865238\ldots$$

$$\frac{49\pi^3}{10800} = 0.140676625\ldots$$

$$\frac{127\pi^3}{21600} = 0.182305423\ldots$$

The inequalities (449) and (472) give

$$\begin{aligned} M_1 &> 0.96252282\ldots \\ &< 1.00615605\ldots \\ &> 0.99736376\ldots \end{aligned}$$

$$\begin{aligned} M_2 &< 1.2206499\ldots \\ &> 1.1527760\ldots \\ &< 1.15502669\ldots \end{aligned}$$

$$\begin{aligned} M_3 &> 1.40379402\ldots \\ &< 1.41470233\ldots \\ &> 1.41415281\ldots \end{aligned}$$

$$\begin{aligned} M_4 &< 1.54585162\ldots \\ &> 1.52141701\ldots \\ &< 1.52170870 \end{aligned}$$

$$\begin{aligned} M_5 &> 1.66192113\ldots \\ &< 1.66676926\ldots \\ &> 1.66666071\ldots \end{aligned}$$

APPENDIX D

Proof of Theorem 5

We are given

$$w(t) = e^{z(t)} \tag{489}$$

where the Fourier transform of $z(t)$ vanishes outside $[0,\Omega]$. Also the Fourier transform of $w(t)$ vanishes over (a,b) where $0 \le a < b$ and

$$b - a > \Omega \tag{490}$$

and we wish to show that $w(t) = \text{constant}$.

We may write

$$w(t) = g(t) + h(t) \tag{491}$$

where the Fourier transform of $g(t)$ vanishes outside $[0,a]$ and the Fourier transform of $h(t)$ vanishes over $(-\infty,b)$. We have

$$\begin{aligned} w'(t) &= z'(t)e^{z(t)} = z'(t)\{g(t) + h(t)\} \\ &= g'(t) + h'(t). \end{aligned} \tag{492}$$

Now the Fourier transform of $z'(t)$ vanishes outside $[0,\Omega]$ and the Fourier transform of $g'(t)$ vanishes outside $[0,a]$. By Corollary 2 of Theorem 2 the Fourier transform of $z'(t)g(t)$ vanishes outside $[0,a + \Omega]$. The Fourier transform of $h'(t)$ vanishes over $(-\infty,b)$ and by Corollary 1 of Theorem 1 the Fourier transform of $z'(t)h(t)$ also vanishes over $(-\infty,b)$. Thus if $K_{a,b}$ is any kernel of L_1 satisfying

$$\begin{aligned} \int_{-\infty}^{\infty} K_{a,b}(t)e^{-i\omega t}dt &= 1, \quad 0 \le \omega \le a \\ &= 0, \qquad \omega \ge b \end{aligned} \tag{493}$$

we have

$$\int_{-\infty}^{\infty} w'(s)K_{a,b}(t - s)ds = z'(t)g(t) = g'(t). \tag{494}$$

Now $z(t)$ and $g(t)$ are the restrictions to the real line of entire functions of exponential type; so

$$\frac{g'(\tau)}{g(\tau)} = z'(\tau). \tag{495}$$

Hence $g(\tau)$ is zero free in the entire plane and is, therefore, of the form (Theorem 2.7.1, Ref. 6)

$$g(\tau) = Ae^{i\lambda\tau}. \tag{496}$$

Hence from (495) and (496), $z'(\tau) = i\lambda$, and since $z(\tau)$ is bounded on the real axis it follows that $\lambda = 0$; i.e., $z(t) =$ constant and hence

$$w(t) = \text{constant.} \tag{497}$$

APPENDIX E

Proof of Theorem 6

We are given

$$w(t) = \log \{z(t)\} \tag{498}$$

where the Fourier transform of $z(t)$ vanishes outside $[0,\Omega]$ and for some positive ϵ

$$|z(t + iu)| \geq \epsilon \quad \text{for } u \geq 0$$

$$-\infty < t < \infty \tag{499}$$

Thus $w(\tau)$ is bounded and analytic in the uhp; so by Theorem 1, the Fourier transform of $w(t)$ vanishes over $(-\infty,0)$. Also we are given that the Fourier transform of $w(t)$ vanishes over (a,b) where $0 \leq a < b$ and

$$b - a > \Omega \tag{500}$$

and wish to show that $z(t) =$ constant.

We proceed as in the proof of Theorem 5 and write

$$w(t) = g(t) + h(t) \tag{501}$$

where the Fourier transform of $g(t)$ vanishes outside $[0,a]$ and the Fourier transform of $h(t)$ vanishes over $(-\infty,b)$. We have

$$w'(t) = \frac{z'(t)}{z(t)} = g'(t) + h'(t) \tag{502}$$

or

$$z'(t) = z(t)g'(t) + z(t)h'(t). \tag{503}$$

Now the Fourier transform of $h'(t)$ vanishes over $(-\infty,b)$ so by Corollary 1 of Theorem 1 the Fourier transform of $z(t)h'(t)$ also vanishes over $(-\infty,b)$. By Corollary 2 of Theorem 2, the Fourier transform of $z(t)g'(t)$ vanishes outside $[0,\alpha + \Omega]$. Since $\alpha + \Omega < b$ we conclude as in the proof of Theorem 5 that

$$z'(t) = z(t)g'(t), \quad \text{or } g'(t) = \frac{z'(t)}{z(t)} \tag{504}$$

and hence that $z(t) = Ae^{i\lambda t}$ and since $g(t)$ is bounded, $\lambda = 0$. Therefore

$$z(t) = \text{constant.} \tag{505}$$

APPENDIX F

Lower Bound on the Degree of Certain Polynomials

Suppose a polynomial of degree ν is of the form

$$P_\nu(z) = 1 + a_n z^n + a_{n+1} z^{n+1} + \ldots a_\nu z^\nu \tag{506}$$

where $|a_n| > 1$, and $P_\nu(z)$ is zero free for $|z| < 1$. Then

$$\nu \geq 2n. \tag{507}$$

To prove this assertion we assume

$$\nu < 2n. \tag{508}$$

Then assuming that $P_\nu(z)$ is zero free for $|z| < 1$, the function

$$\frac{\bar{a}_\nu + \bar{a}_{\nu-1} z + \ldots \bar{a}_n z^{\nu-n} + z^\nu}{1 + a_n z^n + a_{n+1} z^{n+1} + \ldots + a_\nu z^\nu} = f(z) \tag{509}$$

is analytic for $|z| \leq 1$ and

$$|f(e^{i\theta})| = 1, \quad -\pi \leq \theta \leq \pi. \tag{510}$$

Then

$$f(z) = \sum_0^\infty b_k z^k, \quad |z| \leq 1 \tag{511}$$

where

$$b_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-ik\theta} d\theta. \tag{512}$$

Thus from (510) and (512),

$$|b_k| \leq 1. \tag{513}$$

However, with the assumption $\nu < 2n$ we see from (509) that

$$b_k = \bar{a}_{\nu-k} \quad \text{for } 0 \leq k \leq \nu - n < n. \tag{514}$$

In particular

$$b_{\nu-n} = \bar{a}_n. \tag{515}$$

But $|a_n| > 1$, so (515) contradicts (513) and therefore (508) is false.

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