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Discrete-Time Single Server Queues with Correlated Inputs

By B. GOPINATH and J. A. MORRISON

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A wide variety of queueing systems with a single server can be modeled by the equation $b_{n+1} = (b_n - 1)^+ + z_n$, where b_n denotes queue length and z_n the input. The usual assumption about the sequence $\{z_n\}$ is that it be a sequence of independent identically distributed (i. i. d.) random variables. However, in many applications, this is not really the case; $\{z_n\}$ is a sequence of correlated random variables. We show that with the help of a transformation, a $(k + 1)$-dimensional Markov process that suffices to describe the queueing system may be found, where k is the memory of the input process. We derive an equation for the steady-state generating function corresponding to the joint distribution of this vector process. We find that a simple set of equations can be obtained for the marginal distributions. In particular, the steady-state distribution of b_n, the queue length, can be obtained without solving for the joint distribution.

I. INTRODUCTION

Several computer systems and networks involve queueing models with single server queues. We consider a discrete-time queueing system, with service time normalized to unity, modeled by the equation

$$\begin{aligned} b_{n+1} &= b_n - 1 + z_n \text{ if } b_n \geq 1 \\ &= z_n \qquad \text{if } b_n = 0 \end{aligned}$$

or equivalently

$$b_{n+1} = (b_n - 1)^+ + z_n \tag{1}$$

Here b_n denotes queue length[1] and the nonnegative integer valued sequence z_n is the input.

A vast majority of literature in queueing theory deals with the case when $\{z_n\}$ is a sequence of independent identically distributed random variables. In this situation, when the average value $Ez_n < 1, b = \lim_{n\uparrow\infty}$* b_n is a well-defined random variable, and various authors have analyzed the distribution of b; see Ref. 1.

An interesting approach is due to Spitzer[2] who uses a simple consequence of eq. (1): when $b_0 = 0$ then

$$b_{n+1} = \max_r \left\{ \sum_{i=0}^{r} z_{n-i} - r \right\} \tag{2}$$

to derive an integral equation for the distribution of b. However, we will follow the approach that models $\{b_n\}$ as a Markov process as in Ref. 3. Here the theory of Markov chains can be used to derive formulas for the equilibrium distribution of b_n, that is, the distribution of b.

The literature dealing with models where $\{z_n\}$ are not necessarily independent is relatively scant. Recently Ali Khan[4] and Herbert[5] have analyzed the case when z_n is the state of a denumerable Markov chain. In this case (b_n,z_n) forms a Markov process, thus relaxing somewhat the condition that $\{z_n\}$ are independent identically distributed (i. i. d.) random variables.

The queueing process that motivated the work presented in this paper arose in a data communications system. Messages are temporarily stored in a buffer before they are sent across the communications network. It is assumed that the buffer transmits one packet, the basic unit of data, in a unit time interval, provided that it is not empty. In this context, then, z_n is the number of packets that arrive at the buffer in the time interval $(n,n + 1]$. It is assumed that the inputs are correlated and z_n is taken to be a sum of moving averages.

In order to illustrate the techniques, the particular example $z_n = x_n^1 + x_{n-2}^1 + x_n^2$ is first analyzed. This corresponds to the arrival of two kinds of messages. The first kind of message consists of two packets which are spread apart in time, the second packet being transmitted two units of time after the first packet. The number of such messages generated in the $(n + 1)$st time unit is denoted by x_n^1. The second kind of message consists of just one packet, and the number of such messages generated in the $(n + 1)$st time unit is denoted by x_n^2. It is assumed that (x_n^1,x_n^2), $n = 0, 1, 2, \ldots$, are independent identically distributed vector random variables. However, for each n, x_n^1 and x_n^2 may be dependent. In particular, if

$$E(t_1{}^{x_n^1} t_2{}^{x_n^2}) = \Phi[(1 - \rho)t_1 + \rho t_2]$$

with $0 \leq \rho \leq 1$ fixed, then the probability that a message is of the first kind is $1 - \rho$, and the probability that it is of the second kind is ρ.

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* We mean here limit in distribution: for each j,

$$\lim_{n\uparrow\infty} Pr\,\{b_n \leq j\} = Pr\,\{b \leq j\}$$

There are several other examples where such a model for the input process z_n is more appropriate than the usual one. We give two examples. Consider a queueing system where each request for service may consist of a sequence of tasks to be completed by the same server. However, these tasks may not be available for completion in the same time interval; instead they are spread out in time. Hence the random variables corresponding to the number of tasks arriving at the server may be correlated as in the above example. This model may apply to a scheduler in a computer processing system. Another example, that of a dam fed by rivers that originate at geographically distant points, motivated the model considered by Herbert.[6] When rainfall occurs, affecting the flow in all of the rivers, the increase in flow to the dam is spread out in time since the origins of the rivers are at different distances from the dam. A discrete time model of the dam process, similar to the one in the packet network example above, can be solved by the method presented in this paper.

In general we assume that

$$z_n = \sum_{i=1}^{\ell} \sum_{j=0}^{k} \alpha_j^i x_{n-j}^i \tag{3}$$

where the nonnegative integer valued random variables in the sequence $\{(x_n^1, x_n^2, \ldots, x_n^\ell)\}$ are independent and identically distributed, and α_j^i are nonnegative integers with $\alpha_0^i > 0$ for each i. For each n the random variables $x_n^1, x_n^2, \ldots, x_n^\ell$ may be dependent on each other. Notice that z_n by itself is not necessarily a Markov process. As far as we know there is only one work dealing with a special case of eq. (3) which is related to ours. Herbert[6] considers the case when

$$z_n = \sum_{j=0}^{k} \alpha_j x_{n-j} \tag{4}$$

where $\{x_n\}$ are i. i. d. random variables and α_j are positive integers. In this case whenever $x_n \neq 0$, $b_{n+i} \neq 0$, $i = 1, \ldots, k+1$, hence b_{n+r} is linearly related to b_{n+1}, $r = 2, \ldots, k+2$ from eq. (1). From this property, formulas can be derived for the equilibrium distribution for b_n given $x_{n-1}, x_{n-2}, \ldots, x_{n-k}$. However, even in this special case our approach gives formulas for

$$b = \lim_{n \uparrow \infty} b_n$$

itself more simply than the method of Ref. 6.

In the general case b_n is not a Markov process, but it is shown that, with the help of a transformation, a $(k+1)$-dimensional Markov process that suffices to describe the queueing system may be found. The first component of this Markov process is just b_n. An equation is derived for the steady-state generating function corresponding to the joint distributions. This equation involves a multinomial, which corresponds to zero queue length. It is shown that a finite system of linear equations can be obtained to solve for the coefficients in this multinomial. A simple set of equations for the marginal distributions is then derived, leading to the calculation of the steady-state generating function of the queue length.

In Sec. II we review the case when z_n are i. i. d. random variables. An example for a system where z_n is a moving average is worked out in Sec. III to illustrate our method. In Sec. IV we introduce the model considered in this paper and describe the transformation that leads to the simplification in the solution. The generating function of the underlying vector Markov process is derived in Sec. V. The method of solving for certain parameters that occur in Sec. V is described in Sec. VI. The isolation of marginals and the derivation of a simple set of equations for them is the subject of Sec. VII. A pair of limiting cases of the input process is analyzed in Sec. VIII. Finally, for a special class of problems, some formulas relating the limiting cases are also derived in Sec. VIII. The terminology of Markov chains used in this paper is consistent with that of Ref. 3.

II. QUEUE WITH INDEPENDENT INPUTS

When $\{z_n\}$ is a sequence of independent identically distributed random variables, it follows that b_n is a Markov process. The number of packets waiting to be transmitted, b_n, serves as the state for a Markov chain S. The state space of S is the set of nonnegative integers. The transition probabilities for S are generated by eq. (1) as follows:

$$P_i^{n+1} \triangleq Pr\,\{b_{n+1} = i\} = \sum_{j\geq 0} Pr\,\{b_{n+1} = i \,|\, b_n = j\}\, P_j^n$$

$$= \sum_{j\geq 0} Pr\,\{z_n = i - (j-1)^+\}\, P_j^n \qquad (5)$$

Let $Pr\,\{z_n = i\} = p_i$ for $i = 0,1,\ldots$. Then, since z_n is a nonnegative integer,

$$P_i^{n+1} = p_i P_0^n + \sum_{j=1}^{i+1} p_{i-j+1} P_j^n \qquad (6)$$

When $1 > p_0 > 0$, S is irreducible and aperiodic. The following theorem gives conditions under which S is positive recurrent.

Theorem 1: The Markov-chain S is positive recurrent when $Ez_n < 1$.

When S is positive recurrent then

$$\lim_{n\uparrow\infty} b_n = b$$

is a well-defined random variable and the equilibrium distribution of b_n, that is, the distribution of b, is such that

$$\lim_{n\uparrow\infty} Pr\,\{b_n = j\} = \pi_j > 0 \text{ for } j = 0,1,\ldots \tag{7}$$

Furthermore, if $P_j^n = \pi_j$, $j = 0,1,\ldots$, so is P_j^{n+1}, and π_j are the unique nonnegative solution to the infinite system of linear equations:

$$\sum_{j=0}^{\infty} \pi_j = 1$$

and, for $i = 0,1,\ldots,$

$$\pi_i = p_i\pi_0 + \sum_{j=1}^{i+1} p_{i-j+1}\pi_j \tag{8}$$

These are obtained from eq. (6) by substituting $P_i^n = P_i^{n+1} = \pi_i$.

For a proof of the above results see Karlin.[3]

In order to solve for the equilibrium distribution we will employ the method of generating functions. For any random variable x, the generating function of x, $\phi_x(s)$, is defined as

$$\phi_x(s) = Es^x,\ |s| \leq 1 \tag{9}$$

Let

$$\phi_n(s) = Es^{b_n} = \sum_{i=0}^{\infty} P_i^n\, s^i.$$

Then using eq. (1) and the independence of b_n, z_n we have

$$Es^{b_{n+1}} = Es^{(b_n-1)^+}\, Es^{z_n} \tag{10}$$

From the definition of ϕ_n it follows that

$$\phi_{n.1}(s) = (s^{-1}\phi_n(s) + (1 - s^{-1})P_0^n)\phi_z(s) \tag{11}$$

where $\phi_z(s) = Es^{z_n}$. Assuming that $Ez_n < 1$ and $1 > p_0 > 0$, let the generating function of

$$b = \lim_{n\uparrow\infty} b_n$$

be

$$\phi(s) = \sum_{i=0}^{\infty} \pi_i s^i$$

[see eq. (7)]. Then from above it is clear that if $\phi_n(s) = \phi(s)$ then $\phi_{n+1}(s) = \phi(s)$. So from eq. (11) we get

$$\phi(s) = \frac{(1 - s^{-1})\pi_0\phi_z(s)}{1 - s^{-1}\phi_z(s)} \tag{12}$$

To find π_0 we take expectations of both sides of eq. (1) and take the limit as $n \uparrow \infty$. Then

$$\pi_0 = 1 - Ez_n \tag{13}$$

So

$$\phi(s) = \frac{(1-s)\phi_z(s)(1-Ez_n)}{\phi_z(s) - s} \tag{14}$$

This gives the generating function of b in terms of $\phi_z(s)$. However, to get π_j, we need not invert the generating function $\phi(s)$. Treating ϕ_z, ϕ as formal power series, using Π_j to denote $\Sigma_{i=0}^{j} \pi_i$, and equating like powers of s in eq. (14), we can show:

$$\Pi_0 = \pi_0 = 1 - Ez_n$$

$$\Pi_1 = (\pi_0 p_1 + \Pi_0 - p_1 \Pi_0)/p_0$$

$$\vdots$$

$$\Pi_j = \left(\pi_0 p_j + \Pi_{j-1} - \sum_{i=1}^{j} p_i \Pi_{j-i}\right) \Big/ p_0 \tag{15}$$

Equations (15) give explicitly the formulas needed to solve for π_j or Π_j. Notice that any finite number of the π_j's can be determined by solving a finite number of linear equations. Informally we refer to such a situation as being finitely solvable.

III. AN EXAMPLE OF A QUEUING PROCESS WITH CORRELATED INPUTS

In the context of the application discussed in 1, there are instances when the data arriving at the buffer form a sequence of correlated random variables. For an example we consider here a case when there are two classes of sources that generate data. The first kind generates two packets whenever it transmits a message. However, these packets are not generated simultaneously; instead they are spread apart in time, the second packet being transmitted two seconds after the first one. The number of such messages generated in the $(n+1)$st second is denoted by x_n^1. The second class of sources generates messages of one packet each and the number of such messages generated in the $(n+1)$st second is denoted by x_n^2. (x_n^1, x_n^2), $n = 0, 1, 2, \ldots$, are assumed to be independent identically distributed vector random variables. Note that, for each n, x_n^1 and x_n^2 may be dependent. Then the number of packets arriving at the buffer in the $(n+1)$st second is

$$z_n = x_n^1 + x_{n-2}^1 + x_n^2 \tag{16}$$

So the number of packets in the buffer at the end of the $(n+1)$st second is given as in eq. (1) by

$$b_{n+1} = (b_n - 1)^+ + x_n^1 + x_{n-2}^1 + x_n^2 \tag{17}$$

It is clear b_n is not a Markov process. However, $(b_n, x_{n-1}^1, x_{n-2}^1, x_{n-1}^2, x_{n-2}^2)$ is a five-dimensional Markov process. We will derive another Markov process from eq. (17) that is only three-dimensional and suffices to describe the queueing process. Define

$$\begin{aligned} y_{0n} &= b_n \\ y_{1n} &= y_{0n} + x_{n-2}^1 \\ y_{2n} &= y_{1n} + x_{n-1}^1 \end{aligned} \tag{18}$$

Then from eq. (17) we have

$$\begin{aligned} y_{0,n+1} &= [(y_{0n} - 1)^+ - y_{0n}] + y_{1n} + x_n^1 + x_n^2 \\ y_{1,n+1} &= [(y_{0n} - 1)^+ - y_{0n}] + y_{2n} + x_n^1 + x_n^2 \\ y_{2,n+1} &= [(y_{0n} - 1)^+ - y_{0n}] + y_{2n} + 2x_n^1 + x_n^2 \end{aligned} \tag{19}$$

Let $v_{0n} = v_{1n} = x_n^1 + x_n^2$ and $v_{2n} = 2x_n^1 + x_n^2$. Then (v_{0n}, v_{1n}, v_{2n}) is independent of (y_{0j}, y_{1j}, y_{2j}) for $j \le n$ by assumptions about x_n^1, x_n^2. Hence (y_{0n}, y_{1n}, y_{2n}) is a three-dimensional Markov process. The state space of the corresponding Markov-chain S can naturally be indexed by a triple of nonnegative integers. Let

$$P_{i_0,i_1,i_2}^n = Pr\,\{y_{0n} = i_0, y_{1n} = i_1, y_{2n} = i_2\} \tag{20}$$

Then

$$P_{i_0,i_1,i_2}^{n+1} = \sum_{j_0,j_1,j_2} Pr\,\{y_{0,n+1} = i_0, y_{1,n+1} = i_1, y_{2,n+1} = i_2 | \; y_{0n} = j_0, y_{1n} = j_1, y_{2n} = j_2\}\, P_{j_0,j_1,j_2}^n \tag{21}$$

These form the equations for transition probabilities. Notice that not all states (i_0,i_1,i_2) communicate with (0, 0, 0). For example, we can show that when $i_0 = 0$, the only states that communicate with (0, 0, 0) are (0, 0, 0) and (0, 1, 1). Suppose $y_{0,n+1} = 0$, $y_{1,n+1} = i_1$ and $y_{2,n+1} = i_2$. Then $b_{n+1} = 0$. Hence, from eq. (17), $b_n \le 1$, $x_n^1 = 0$. But $x_n^1 = 0$ implies $y_{2,n+1} = y_{1,n+1}$. Also, $b_n \le 1$ implies $x_{n-1}^1 \le 1$. Further, $y_{0,n+1} = 0$ and $x_{n-1}^1 \le 1$ imply $y_{1,n+1} \le 1$. However, it can be shown that states that do not communicate with (0, 0, 0) are transient (see Sec. IV). So we will restrict the state space by allowing it to consist only of those states, denoted by $\mathcal{A}$, that communicate with zero. We will continue to denote by S the Markov chain on the restricted state space $\mathcal{A}$. Then S is irreducible and aperiodic (see Sec. IV). Notice that for every state at time n

$$y_{0n} \le y_{1n} \le y_{2n} \tag{22}$$

Later in this paper we will show that S is positive recurrent when $Ez_n < 1$. For now we will assume this is so. Interpreting the sums over j_0, j_1, j_2 to extend only over $\mathcal{A}$ we have from eqs. (19) and (21), and the definitions of v_{0n}, v_{1n}, v_{2n},

$$P^{n+1}_{i_0,i_1,i_2} = \sum'_{j_1-i_0=j_2-i_1} Pr\,\{v_{0n} = i_0 - j_1,\ v_{2n} = i_2 - j_2\}\, P^n_{0,j_1,j_2}$$

$$+ \sum'_{j_0>0, j_1-i_0=j_2-i_1} Pr\,\{v_{0n} = 1 + i_0 - j_1,\ v_{2n} = 1 + i_2 - j_2\}\, P^n_{j_0,j_1,j_2} \quad (23)$$

The equilibrium distribution of S:

$$\lim_{n\uparrow\infty} P^n_{i_0,i_1,i_2} = P_{i_0,i_1,i_2}$$

has the property that if $P^n_{i_0,i_1,i_2} = P_{i_0,i_1,i_2}$ for $(i_0, i_1, i_2) \in \mathcal{A}$, so does $P^{n+1}_{i_0,i_1,i_2}$. So P_{i_0,i_1,i_2} satisfies:

$$P_{i_0,i_1,i_2} = \sum'_{j_1-i_0=j_2-i_1} Pr\,\{v_{0n} = i_0 - j_1,\ v_{2n} = i_2 - j_2\}\, P_{0,j_1,j_2}$$

$$+ \sum'_{j_0>0, j_1-i_0=j_2-i_1} Pr\,\{v_{0n} = 1 + i_0 - j_1,\ v_{2n} = 1 + i_2 - j_2\}\, P_{j_0,j_1,j_2} \quad (24)$$

$$\sum_{(i_0,i_1,i_2)\in\mathcal{A}} P_{i_0,i_1,i_2} = 1$$

P_{i_0,i_1,i_2} is the unique nonnegative solution of eq. (24) (see Ref. 3). In principle, solving the infinite system of linear eq. (24) determines P_{i_0,i_1,i_2}, hence the equilibrium distribution of (y_{0n}, y_{1n}, y_{2n}). However we will see a much simpler way to find equilibrium distributions of the components y_{0n}, y_{1n}, y_{2n}, without computing P_{i_0,i_1,i_2}. Denote $Es^{y_{in}}$ by $\phi_{ni}(s)$ and $Es^{v_{in}}$ by $\phi_{iv}(s)$. Then from eq. (19) we can derive the following equations paralleling eq. (11):

$$\begin{aligned}\phi_{n+1,0}(s) &= [s^{-1}\phi_{n1}(s) + (1 - s^{-1})c_{1n}(s)]\phi_{0v}(s)\\ \phi_{n+1,1}(s) &= [s^{-1}\phi_{n2}(s) + (1 - s^{-1})c_{2n}(s)]\phi_{1v}(s)\\ \phi_{n+1,2}(s) &= [s^{-1}\phi_{n2}(s) + (1 - s^{-1})c_{2n}(s)]\phi_{2v}(s)\end{aligned} \quad (25)$$

Here

$$c_{in}(s) = \sum_{j\geq 0} Pr\,\{y_{0n} = 0,\ y_{in} = j\}s^j,\ i = 1,2 \quad (26)$$

For any n the only admissable states in $\mathcal{A}$ that have $y_{0n} = 0$ are $(0, 0, 0)$ and $(0, 1, 1)$. So $c_{in}(s)$, $i = 1,2$ are polynomials of degree 1, and $c_{1n}(s) =$

$c_{2n}(s)$. Let $\phi_i(s)$ denote the generating function of

$$y_i = \lim_{n\uparrow\infty} y_{in}$$

and

$$c_i(s) = \lim_{n\uparrow\infty} c_{in}(s)$$

for $i = 1,2$. Then

$$\begin{aligned}\phi_0(s) &= [s^{-1}\phi_1(s) + (1 - s^{-1})c_1(s)]\phi_{0v}(s)\\ \phi_1(s) &= [s^{-1}\phi_2(s) + (1 - s^{-1})c_1(s)]\phi_{1v}(s)\\ \phi_2(s) &= [s^{-1}\phi_2(s) + (1 - s^{-1})c_1(s)]\phi_{2v}(s)\end{aligned} \tag{27}$$

From eq. (27)

$$\phi_2(s) = \frac{(1 - s^{-1})c_1(s)\phi_{2v}(s)}{1 - s^{-1}\phi_{2v}(s)} \tag{28}$$

Since ϕ_{0v}, ϕ_{1v}, ϕ_{2v} are known directly from the distribution of x_n^1, x_n^2, eq. (27) gives ϕ_0,ϕ_1 in terms of $c_1(s)$, the only unknown. Let $c_1(s) = k_0 + k_1 s$. Then

$$\begin{aligned}c_1(1) &= Pr\,\{y_0 = 0, y_1 = 0\} + Pr\,\{y_0 = 0, y_1 = 1\}\\ &= Pr\,\{y_0 = 0\} = k_0 + k_1\end{aligned}$$

As in eq. (13)

$$\begin{aligned}k_0 + k_1 &= 1 - E(x_n^1 + x_{n-2}^1 + x_n^2)\\ &= 1 - Ez_n\end{aligned} \tag{29}$$

In order to derive another equation for k_0,k_1 we go back to the original equations for $P_{i_0 i_1 i_2}$, eq. (24). From eq. (24) we can derive the following: for $i_0 = i_1 = i_2 = 0$, since v_{in} are nonnegative, $P_{000} = Pr\,\{v_{0n} = 0, v_{2n} = 0\}\,P_{000} + Pr\,\{v_{0n} = 0, v_{2n} = 0\}\,P_{111}$. However, $v_{2n} = 2x_n^1 + x_n^2 = 0$ implies $x_n^1 = x_n^2 = 0$, so $v_{0n} = 0$. Therefore

$$k_0 = P_{000} = Pr\,\{v_{0n} = 0\}\,(P_{000} + P_{111}) \tag{30}$$

Similarly

$$\begin{aligned}k_1 &= P_{011} = Pr\,\{v_{0n} = 0\}\,P_{112}\\ P_{112} &= Pr\,\{v_{0n} = 1, v_{2n} = 2\}\,P_{111}\\ &\quad + Pr\,\{v_{0n} = 1, v_{2n} = 2\}\,P_{000}\end{aligned}$$

Hence

$$k_1 = Pr\,\{v_{0n} = 0\}\,Pr\,\{v_{0n} = 1, v_{2n} = 2\}\,(P_{000} + P_{111}) \tag{31}$$

Notice that the various probabilities occurring on the right-hand sides of eqs. (30), (31) can be calculated from the distribution of (x_n^1, x_n^2). For example:

$$Pr\,\{v_{0n} = 1,\, v_{2n} = 2\} = Pr\,\{x_n^1 = 1,\, x_n^2 = 0\}$$

Therefore using eq. (29) we can determine k_0, k_1, hence $c_1(s)$. From eq. (27), therefore, it is easy to derive the formula for $\phi_0(s)$, namely

$$\phi_0(s) = (1 - s^{-1})c_1(s)\left[\frac{s^{-2}\phi_{2v}(s)\phi_{1v}(s)\phi_{0v}(s)}{1 - s^{-1}\phi_{2v}(s)} + s^{-1}\phi_{1v}(s)\phi_{0v}(s) + \phi_{0v}(s)\right]$$

$$= (1 - s^{-1})c_1(s)\phi_{0v}(s)\left[1 + \frac{s^{-1}\phi_{1v}(s)}{1 - s^{-1}\phi_{2v}(s)}\right] \tag{32}$$

To solve for the equilibrium distribution of b_n, i.e., distribution of y_0, we do not have to invert $\phi_0(s)$. It turns out that eq. (27) can be translated to linear recursions for marginal distributions for y_0, y_1, y_2. Hence, as in Sec. II, the distributions of y_0, y_1, y_2 are finitely solvable. That this is so, in the general case, is shown in Sec. VII.

IV. QUEUEING PROCESSES WITH MOVING AVERAGE INPUTS

The most general input process that we will consider in this paper is a finite sum of moving averages, i.e.,

$$z_n = \sum_{i=1}^{\ell} \sum_{j=0}^{k} \alpha_j^i x_{n-j}^i \tag{33}$$

Equation (1) in this setting is

$$b_{n+1} = (b_n - 1)^+ + \sum_{i=1}^{\ell} \sum_{j=0}^{k} \alpha_j^i x_{n-j}^i \tag{34}$$

The integer k is referred to as memory of the input process z_n. Under the assumptions below, the $(k\ell + 1)$ dimensional vector process $(b_n, x_{n-1}^1, x_{n-2}^1, \ldots, x_{n-k}^1, x_{n-1}^2, \ldots, x_{n-k}^2, \ldots, x_{n-1}^\ell, \ldots, x_{n-k}^\ell)$ is Markov as in the example of Sec. III.

However, by a transformation we will find a $(k + 1)$ dimensional Markov process that suffices to describe the queueing system. Define:

$$y_{0n} = b_n$$

and, for $r = 0, 1, \ldots, k - 1$,

$$y_{r+1,n} = y_{rn} + \sum_{i=1}^{\ell} \sum_{j=r+1}^{k} \alpha_j^i x_{n-j+r}^i \tag{35}$$

Let

$$\sum_{j=0}^{r} \alpha_j^i = \mu_r^i \text{ and } \sum_{i=1}^{\ell} \mu_r^i x_n^i = v_{rn}$$

for $r = 0, 1, \ldots, k$. Then using eq. (34) we can verify:

$$y_{r,n+1} = [(y_{0n} - 1)^+ - y_{0n}] + y_{r+1,n} + v_{rn}, r = 0, 1, \ldots, k-1 \quad (36)$$

$$y_{k,n+1} = [(y_{0n} - 1)^+ - y_{0n}] + y_{kn} + v_{kn} \quad (37)$$

We make the following assumptions for the rest of this paper: The α_j^i are assumed to be nonnegative integers and, for each i, $\alpha_0^i > 0$. We will assume that the vector, nonnegative integer valued random variables $(x_n^1, x_n^2, \ldots, x_n^\ell)$ are independent and identically distributed, though for each n, $x_n^1, x_n^2, \ldots, x_n^\ell$ will be allowed to be dependent on each other. We will also assume that $Pr\{\mathbf{v}_n = 0\} > 0$ and $Pr\{v_{rn} > 1\} > 0$ for some r.

From the assumptions about x_n^i, $(v_{0n}, v_{1n}, \ldots, v_{kn})^t \equiv \mathbf{v}_n$ is independent of $\mathbf{y}_j \equiv (y_{0j}, y_{1j}, \ldots, y_{kj})^t$ for $j \leq n$. Hence $\mathbf{y}_n$ is a $(k+1)$ dimensional Markov process. The state space corresponding to this Markov process is indexed naturally by a $(k+1)$ triple of nonnegative integers. Furthermore by definition of y_{in}, $i = 0, 1, \ldots, k$, $n = 0, 1, 2, \ldots$, $y_{0n} \leq y_{1n} \leq y_{2n} \leq \cdots \leq y_{kn}$. Hence we can assume that if $(i_0, i_1, \ldots, i_k)$ denotes a state then

$$i_0 \leq i_1 \leq i_2 \leq \cdots \leq i_k \quad (38)$$

Let $\mathcal{A}'$ denote the set of vectors satisfying (38) and S' the Markov chain with state space $\mathcal{A}'$. Of the states in $\mathcal{A}'$ let $\mathcal{A}$ denote the set of states that communicate with the state $\mathbf{0} = (0, 0, \ldots, 0)^t$. Using the following theorem, we will be able to restrict our attention to only those states that are in $\mathcal{A}$, and to the irreducible Markov chain S, with state space $\mathcal{A}$, derived from S'.

Theorem 2:

(*i*) Every state in $\mathcal{A}'$ of the form $(m, m, \ldots, m)^t$ belongs to $\mathcal{A}$.

(*ii*) Every state of S' transitions to a state belonging to $\mathcal{A}$ in at most k steps.

(*iii*) Every state in $\mathcal{A}$ is accessible from a state of the form $(m, m, \ldots, m)^t$ in at most k steps.

(*iv*) S is irreducible and aperiodic.

(*v*) For each i_0, the number of states in $\mathcal{A}$ which are of the form $(i_0, i_1, \ldots, i_k)^t$ is finite.

Proof: Let F denote the $(k+1) \times (k+1)$ matrix with elements $F_{i,i+1} = 1$, for $i = 0, \ldots, k-1$, $F_{kk} = 1$, and $F_{ij} = 0$ otherwise. Also, let $\mathbf{1}$ denote the vector $(1, 1, \ldots, 1)^t$. We note that $F\mathbf{1} = \mathbf{1}$ and, for any $\mathbf{y} = (y_0, y_1, \ldots, y_k)^t$, $F^r\mathbf{y} = (y_r, \ldots, y_k, y_k, \ldots, y_k)^t$ by induction. Equations (36) and

(37) can then be written in vector form as follows, using σ_n to denote $y_{0n} - (y_{0n} - 1)^+$. Note that $\sigma_n = 1$ if $y_{0n} > 0$, and $\sigma_n = 0$ if $y_{0n} = 0$.

$$\mathbf{y}_{n+1} = F\mathbf{y}_n - \sigma_n \mathbf{1} + \mathbf{v}_n, n = 0, 1, 2, \ldots$$

We can then show that for $n - 1 \geq i \geq 0$

$$\mathbf{y}_n = F^{n-i}\mathbf{y}_i - \sum_{j=i}^{n-1} \sigma_j \mathbf{1} + \sum_{j=i}^{n-1} F^{n-1-j}\mathbf{v}_j \tag{39}$$

Hence, if $\mathbf{v}_j = 0$ for $j = n - 1, \ldots, n - k$, then it follows from (39), with $i = n - k$, that $\mathbf{y}_n$ is a vector of the form $m\,\mathbf{1} = (m, m, \ldots, m)^t$ for some nonnegative integer m. Therefore, since $Pr\,\{\mathbf{v}_j = 0\} > 0$ by assumption, $\mathbf{0}$ is accessible from any state by allowing $\mathbf{v}_j$ to be zero for as many consecutive j's as needed. We assumed earlier that $Pr\,\{v_{rn} > 1\} > 0$ for some r. Hence, since $v_{kn} \geq v_{rn}$, $Pr\,\{v_{kj} = M\} > 0$ for some integer $M > 1$ and all j. Therefore, if $\mathbf{y}_0 = \mathbf{0}$, $v_{kj} = M$ for $j = 0, 1, \ldots, n - 1$ implies $y_{kn} > nM - n$. Hence $Pr\,\{y_{kn} > nM - n,\ \mathbf{y}_0 = 0\} > 0$. For every sequence $\mathbf{v}_j$ such that $\mathbf{v}_j = \mathbf{0}$ for $j = n, n + 1, \ldots, n + k, \ldots$, $\mathbf{y}_{n+k+i}$ remains proportional to $\mathbf{1}$ for all $i \geq 0$. Therefore, for each m, $Pr\,\{\mathbf{y}_{n+k+i} = m\,\mathbf{1},\ \mathbf{y}_0 = 0\}$ is greater than zero for some n,i dependent on m. From (39) we can therefore show that any state of the form $m\,\mathbf{1}$ communicates with $\mathbf{0}$ and hence belongs to $\mathcal{A}$.

If $\mathbf{y}_0 \in \mathcal{A}'$, then we will prove, irrespective of what $\mathbf{v}_j$'s are for $j = 0, 1, \ldots, k - 1$, that $\mathbf{y}_k \in \mathcal{A}$, by showing that $\mathbf{y}_k$ is accessible from a state of the form $m\,\mathbf{1}$ in at most k steps, where

$$m = k + y_{k0} - \sum_{j=1}^{k} \sigma_{k-j} \tag{40}$$

Let $\mathbf{y}_j'$, $j = 0, 1, \ldots, k$ be the sequence of states traced by S' if $\mathbf{y}_0$ is set to zero but $\mathbf{v}_j$, $j = 0, 1 \ldots, k - 1$ are left unchanged. If $\sigma_j' = y_{0j}' - (y_{0j}' - 1)^+$ for $j = 0, 1, 2 \ldots$ then (39) holds with primes on $\mathbf{y}_j$'s and σ_j's, and $\mathbf{y}_0' = 0$. We will first prove that for each k, $\mathbf{y}_k \geq \mathbf{y}_k'$ by showing that for each i:

$$\begin{aligned}&\mathbf{y}_i - \mathbf{y}_i' \geq 0, F(\mathbf{y}_i - \mathbf{y}_i') - (\mathbf{y}_i - \mathbf{y}_i') \geq 0\\ &\qquad \Rightarrow \mathbf{y}_{i+1} - \mathbf{y}_{i+1}' \geq 0, F(\mathbf{y}_{i+1} - \mathbf{y}_{i+1}') - (\mathbf{y}_{i+1} - \mathbf{y}_{i+1}') \geq 0\end{aligned} \tag{41}$$

Suppose the assumptions in (41) hold as they do for $i = 0$. Then $y_{ki} - y_{ki}' \geq y_{k-1,i} - y_{k-1,i}' \geq \cdots \geq y_{0i} - y_{0i}' \geq 0$. Therefore if $y_{0i} > y_{0i}'$, then $(\mathbf{y}_i - \mathbf{y}_i') + (\sigma_i' - \sigma_i)\,\mathbf{1} \geq 0$, which is trivially so if $y_{0i} = y_{0i}'$. Hence, using (39) and corresponding equations for $\mathbf{y}_{i+1}'$,

$$\mathbf{y}_{i+1} - \mathbf{y}_{i+1}' = F(\mathbf{y}_i - \mathbf{y}_i') + (\sigma_i' - \sigma_i)\mathbf{1} \geq (\mathbf{y}_i - \mathbf{y}_i') + (\sigma_i' - \sigma_i)\mathbf{1} \geq 0$$

Furthermore

$$F(\mathbf{y}_{i+1} - \mathbf{y}_{i+1}') - (\mathbf{y}_{i+1} - \mathbf{y}_{i+1}') = F[F(\mathbf{y}_i - \mathbf{y}_i') - (\mathbf{y}_i - \mathbf{y}_i')] \geq 0$$

since all elements of F are nonnegative. Hence in particular,

$$y_{kk} - y'_{kk} = y_{k0} + \sum_{j=1}^{k} \sigma'_{k-j} - \sum_{j=1}^{k} \sigma_{k-j} \geq 0$$

Therefore from (40), with

$$s = \sum_{j=1}^{k} \sigma'_{k-j},\ s + m \geq k.$$

Now let $\mathbf{y}''_i$ be the sequence of states traced by S' if $\mathbf{y}_0$ were set to $m\mathbf{1}$ while $\mathbf{v}_j$, $j = 0, 1, \ldots, k-1$ were left unchanged. Then, with $\sigma''_i = y''_{0i} - (y''_{0i} - 1)^+$,

$$\mathbf{y}''_i - \mathbf{y}'_i = \left(m + \sum_{j=1}^{i} \sigma'_{i-j} - \sum_{j=1}^{i} \sigma''_{i-j}\right)\mathbf{1} \tag{42}$$

for each i. As before, using (41) we can show $\mathbf{y}''_i - \mathbf{y}'_i \geq 0$. From (42) we have $\mathbf{y}''_{i+1} - \mathbf{y}'_{i+1} = \mathbf{y}''_i - \mathbf{y}'_i + (\sigma'_i - \sigma''_i)\mathbf{1}$, hence $\mathbf{y}''_i = \mathbf{y}'_i \Rightarrow \sigma''_i = \sigma'_i \Rightarrow \mathbf{y}''_{i+1} = \mathbf{y}'_{i+1}$. If $\sigma''_i = 0$ for some $i < k$, then $y''_{0i} = 0$, but $y''_{0i} \geq y'_{0i}$, hence $y''_{0i} = y'_{0i}$ and $\mathbf{y}''_i = \mathbf{y}'_i$ from (42). So $\sigma''_i = 0 \Rightarrow \mathbf{y}''_n = \mathbf{y}'_n$ for $k \geq n \geq i$. So in particular

$$\mathbf{y}''_k - \mathbf{y}'_k = \left(m + \sum_{j=1}^{k} \sigma'_{k-j} - \sum_{j=1}^{k} \sigma''_{k-j}\right)\mathbf{1} = 0 \tag{43}$$

However, we noted earlier that

$$s + m = m + \sum_{j=1}^{k} \sigma'_{k-j} \geq k$$

Hence (43) can only hold if

$$s + m = k = \sum_{j=1}^{k} \sigma''_{k-j}$$

Therefore we have shown that for each $i < k$, $\sigma''_i = 1$. Now we use eqs. (40) and (41) to show

$$\mathbf{y}''_k - \mathbf{y}_k = (m - y_{k0})\mathbf{1} - \sum_{j=1}^{k} \sigma''_{k-j}\mathbf{1} + \sum_{j=1}^{k} \sigma_{k-j}\mathbf{1} = \mathbf{0} \tag{44}$$

Since $\mathbf{y}''_k$ belongs to $\mathcal{A}$, being accessible from $m\mathbf{1}$ belonging to $\mathcal{A}$, we have shown that, starting from any state in $\mathcal{A}'$, S' transitions into a state in $\mathcal{A}$ in at most k steps. Furthermore, every state of $\mathcal{A}$ is accessible from some state of the form $m\mathbf{1}$ in at most k steps. It is clear from the definition of $\mathcal{A}$ that S is irreducible. To show that S is aperiodic, we merely note that $\mathbf{y}_j$ can equal zero for arbitrarily many consecutive j's with positive probability.

We will now prove that the set of states in $\mathcal{A}$ which are of the form $(0, i_1, \ldots, i_k)^t$ is finite. This result is used later to derive conditions for the positive recurrence of S. We just showed that every state in $\mathcal{A}$ is accessible from a state of the form $m\mathbf{1}$ in k steps. In particular, if a state of the form $(0, i_1, \ldots, i_k)^t$ is $\mathbf{y}_k$ with $\mathbf{y}_0 = m\mathbf{1}$ for some m then

$$y_{0k} = m - \sum_{j=1}^{k} \sigma_{k-j} + \sum_{j=1}^{k} v_{k-j,j-1} = 0$$

$$y_{kk} = m - \sum_{j=1}^{k} \sigma_{k-j} + \sum_{j=1}^{k} v_{k,j-1} \tag{45}$$

Hence

$$m - \sum_{j=1}^{k} \sigma_{k-j} \leq 0$$

and

$$\sum_{j=1}^{k} v_{k-j,j-1} = \sum_{i=1}^{\ell} \sum_{j=1}^{k} \mu_{k-j}^{i} x_{j-1}^{i} \leq k$$

Therefore

$$\sum_{i=1}^{\ell} \left(\sum_{j=1}^{k} x_{j-1}^{i} \right) \leq k$$

since, for $i = 1, 2, \ldots, \ell$, $\alpha_0^i = \mu_0^i \geq 1$ and $\mu_j^i \geq \mu_0^i$ for $j = 0, 1, \ldots, k$. Therefore, from eq. (45),

$$\begin{aligned} y_{kk} &\leq \sum_{j=1}^{k} v_{k,j-1} \\ &\leq \sum_{i=1}^{\ell} \mu_k^i \sum_{j=1}^{k} x_{j-1}^{i} \\ &\leq \left(\sum_{i=1}^{\ell} \mu_k^i \right) k \end{aligned} \tag{46}$$

Hence for every state

$$(0, i_1, \ldots, i_k)^t \in \mathcal{A}, i_k \leq k \sum_{i=1}^{\ell} \mu_k^i,$$

hence such states are finite in number from eq. (38). In a similar way we can show for any integer j the states $(i_0, i_1, \ldots, i_k)^t \in \mathcal{A}$ such that $i_0 \leq j$ is a finite set.

The transition probabilities for S can be derived from eqs. (36) and (37). Let $P_{\mathbf{i}}^{n} = Pr\,\{y_{0n} = i_0, y_{1n} = i_1, \ldots, y_{kn} = i_k\}$. Then

$$P_{\mathbf{i}}^{n+1} = \sum_{\mathbf{j} \in \mathcal{A}} Pr\,\{\mathbf{y}_{n+1} = \mathbf{i} \,|\, \mathbf{y}_n = \mathbf{j}\}\, P_{\mathbf{j}}^{n} \tag{47}$$

$$P_{\mathbf{i}}^{n+1} = \sum_{\substack{j_0=0\\ \mathbf{j}\in\mathcal{A}}} Pr\,\{v_{0n} = i_0 - j_1, \ldots, v_{k-1,n} = i_{k-1} - j_k, v_{kn}$$

$$= i_k - j_k\}\, P_{\mathbf{j}}^n + \sum_{\substack{j_0>0\\ \mathbf{j}\in\mathcal{A}}} Pr\,\{v_{0n} = i_0 - j_1 + 1, \ldots, v_{k-1,n}$$

$$= i_{k-1} - j_k + 1, v_{kn} = i_k - j_k + 1\}\, P_{\mathbf{j}}^n \tag{48}$$

If the equilibrium probabilities

$$P_{\mathbf{i}} = \lim_{n\uparrow\infty} P_{\mathbf{i}}^n$$

exist, then $P_{\mathbf{i}}^n = P_{\mathbf{i}}$ for every $\mathbf{i} \in \mathcal{A}$ implies $P_{\mathbf{i}}^{n+1} = P_{\mathbf{i}}$, $\mathbf{i} \in \mathcal{A}$. Furthermore $P_{\mathbf{i}}$ is the unique nonnegative solution of

$$\sum_{\mathbf{i}\in\mathcal{A}} P_{\mathbf{i}} = 1$$

$$P_{\mathbf{i}} = \sum_{j_0=0,\mathbf{j}\in\mathcal{A}} p_{i_0-j_1,i_1-j_2,\ldots,i_k-j_k} P_{\mathbf{j}}$$

$$+ \sum_{j_0>0,\mathbf{j}\in\mathcal{A}} p_{i_0-j_1+1,i_1-j_2+1,\ldots,i_k-j_k+1} P_{\mathbf{j}} \tag{49}$$

Here $p_{i_0,i_1,\ldots,i_k} = Pr\,\{v_{0n} = i_0, \ldots, v_{kn} = i_k\}$.

We show next that S is positive recurrent when $Ez_n < 1$.

Theorem 3: S is positive recurrent if

$$E \sum_{i=1}^{\ell} \mu_k^i\, x_n^i < 1$$

Proof: Define a new process c_n as follows:

$$c_{n+1} = (c_n - 1)^+ + \sum_{i=1}^{\ell} \mu_k^i\, x_n^i$$

$$\equiv (c_n - 1)^+ + v_{kn} \tag{50}$$

We know that the Markov chain corresponding to c_n is positive recurrent if $Ev_{kn} < 1$ from Theorem 1. In particular if $Ev_{kn} < 1$ then

$$\lim_{n\uparrow\infty} Pr\,\{c_n = 0\} > 0$$

The processes c_n, b_n as defined by eqs. (50), (34) are related by x_n^i, $i = 1, \ldots, \ell$. Let $b_0 = 0$ and let r be such that $b_{n-i} > 0$ for $i = 0, 1, \ldots, r - 1$ and $b_{n-r} = 0$. Then eq. (34) implies that

$$b_{n+1} = \sum_{m=0}^{r} \sum_{i=1}^{\ell} \sum_{j=0}^{k} \alpha_j^i\, x_{n-j-m}^i - r \tag{51}$$

We also know from eq. (2), assuming $c_0 = 0$, that

$$c_{n+1} \geq \sum_{m=0}^{r+k} \sum_{i=1}^{\ell} \mu_k^i \, x_{n-m}^i - (r+k)$$

From the definition of μ_k^i it can be easily verified that

$$c_{n+1} \geq b_{n+1} - k$$

Hence

$$Pr\,\{b_{n+1} \leq k\} \geq Pr\,\{c_{n+1} = 0\}$$

When $Ev_{kn} < 1$ we know from Theorem 1 that

$$\lim_{n \uparrow \infty} Pr\,\{c_{n+1} = 0\} > 0$$

hence

$$\lim_{n \uparrow \infty} \inf Pr\,\{b_{n+1} \leq k\} > 0$$

Let the set of states $(i_0, i_1, \ldots, i_k)$ in $\mathcal{A}$ with $i_0 \leq k$ be denoted by $\mathcal{A}_k$. Let $P_{\mathbf{ij}}^n$ be the probability of S being in state $\mathbf{i}$ at time n starting from $\mathbf{j}$ at time 0. Then, we have shown that

$$\lim_{n \uparrow \infty} \inf \sum_{\mathbf{i} \epsilon \mathcal{A}} P_{\mathbf{i0}}^n > 0$$

Since cardinality of $\mathcal{A}_k$ is finite

$$\lim_{n \uparrow \infty} \inf P_{\mathbf{i0}}^n > 0$$

for some $\mathbf{i} \,\epsilon\, \mathcal{A}_k$. We also know that $\mathbf{0}$ is accessible from $\mathbf{i}$. So $P_{\mathbf{0i}}^r > 0$ for some r. Therefore

$$\lim_{n \uparrow \infty} \inf P_{\mathbf{00}}^{n+r} \geq \lim_{n \uparrow \infty} \inf P_{\mathbf{0i}}^r \, P_{\mathbf{i0}}^n > 0$$

Hence $\mathbf{0}$ is positive recurrent. Therefore, since S is irreducible and aperiodic, S is positive recurrent.

V. GENERATING FUNCTIONS FOR JOINT DISTRIBUTIONS

We will now derive expressions for joint distributions of $(y_0, y_1, \ldots, y_k)$ assuming $Ez_n < 1$, so S is positive recurrent. Let

$$E\left(\prod_{r=0}^{k} s_r^{y_{rn}}\right) = \phi_n(s_0, s_1, \ldots, s_k)$$

and

$$E\left(\prod_{r=0}^{k} s_r^{v_{rn}}\right) = \phi_v(s_0, s_1, \ldots, s_k), \; |s_i| \leq 1$$

From eqs. (36) and (37) we have, using independence of $\mathbf{v}_n$ and $\mathbf{y}_n$,

$$\phi_{n+1}(s_0, s_1, \ldots, s_k) = E\left(\left(\prod_{r=0}^{k} s_r\right)^{(y_{0n}-1)^+ - y_{0n}} \prod_{r=0}^{k-1} s_r^{y_{r+1,n}} s_k^{y_{kn}}\right) \times \phi_v(s_0, s_1, \ldots, s_k) \quad (52)$$

Proceeding as in eq. (11) we can show

$$\phi_{n+1}(s_0, s_1, \ldots, s_k) = \left[\phi_n(1, s_0, s_1, \ldots, s_{k-2}, s_{k-1}s_k) \prod_{i=0}^{k} s_i^{-1} + \left(1 - \prod_{i=0}^{k} s_i^{-1}\right) \phi_n(0, s_0, s_1, \ldots, s_{k-1}s_k)\right] \times \phi_v(s_0, s_1, \ldots, s_k) \quad (53)$$

When ϕ_n is the generating function of the equilibrium distribution i.e., when

$$\phi_n(s_0, s_1, \ldots, s_k) = \phi(s_0, s_1, \ldots, s_k) = E \prod_{i=0}^{k} s_i^{y_i} \quad (54)$$

then $\phi_{n+1} = \phi$. Therefore ϕ satisfies

$$\phi(s_0, s_1, \ldots, s_k) = \left[\phi(1, s_0, s_1, \ldots, s_{k-2}, s_{k-1}s_k) \prod_{i=0}^{k} s_i^{-1} + \left(1 - \prod_{i=0}^{k} s_i^{-1}\right) \phi(0, s_0, s_1, \ldots, s_{k-2}, s_{k-1}s_k)\right] \times \phi_v(s_0, s_1, \ldots, s_k) \quad (55)$$

We note that $\phi(0, t_1, \ldots, t_k)$ is a polynomial of finite degree since the set of states $(0, i_1, \ldots, i_k)$ is finite. Knowledge of $\phi(0, t_1, \ldots, t_k)$ determines $\phi(s_0, s_1, \ldots, s_k)$ as follows. If we set $s_0 = s_1 = \cdots = s_{k-1} = 1$ then (55) becomes

$$\phi(1, 1, \ldots, 1, s_k) = [s_k^{-1} \phi(1, 1, \ldots, 1, s_k) + (1 - s_k^{-1})\phi(0, 1, 1, \ldots, 1, s_k)]\phi_v(1, 1, \ldots, 1, s_k)$$

This determines $\phi(1, 1, \ldots, 1, s_k)$ in terms of $\phi(0, 1, 1, \ldots, 1, s_k)$:

$$\phi(1, 1, \ldots, 1, s_k) = \frac{(1 - s_k^{-1})\phi(0,1,1, \ldots, 1, s_k)\phi_v(1, 1, \ldots, 1, s_k)}{1 - s_k^{-1} \phi_v(1, 1, \ldots, 1, s_k)} \quad (56)$$

For $r = 0, 1, \ldots, k$ set

$$\phi^r(s_r, \ldots, s_k) = \phi(1,1, \ldots, 1, s_r, s_{r+1}, \ldots, s_k) \quad (57)$$

Then eq. (56) determines ϕ^k in terms of $\phi(0, 1, \ldots, 1, s_k)$. Using eq. (55) yields:

$$\phi^r(s_r, \ldots, s_k) = \left[\prod_{i=r}^{k} s_i^{-1} \, \phi^{r+1}(s_r, \ldots, s_{k-2}, s_{k-1}\, s_k) + \left(1 - \prod_{i=r}^{k} s_i^{-1}\right) \phi(0, 1, \ldots, s_r, \ldots, s_{k-1} s_k) \right] \times \phi_v(1, 1, \ldots, 1, s_r, \ldots, s_k) \quad (58)$$

So starting with ϕ^k, k applications of (58) yields $\phi^0(s_0, \ldots, s_k) = \phi(s_0, \ldots, s_k)$ in terms of $\phi(0, s_1, \ldots, s_k)$. Equations (15) have a counterpart here. These can be derived in mechanical fashion using formal power series expressions for ϕ and ϕ_v. we will not go into the details here. The derivation is analogous to that given in Sec. VII for the marginals.

In the case $\ell = 1$, an alternate generating function was considered in Ref. 8. The corresponding generating function is obtained by setting

$$u_j = \prod_{i=j}^{k} s_i, \; j = 0, \ldots, k$$

and defining

$$\phi(s_0, s_1, \ldots, s_k) = \Phi(u_0, u_1, \ldots, u_k) = \lim_{n \to \infty} E \left(u_0^{y_{0n}} \prod_{r=1}^{k} u_r^{y_{rn} - y_{r-1,n}} \right)$$

Then corresponding to eq. (55),

$$\Phi(u_0, u_1, \ldots, u_k) = [u_0^{-1} \Phi(u_0, u_0, u_1, \ldots, u_{k-1}) + (1 - u_0^{-1}) \Phi(0, u_0, u_1, \ldots, u_{k-1})] \Phi_v(u_0, u_1, \ldots, u_k) \quad (55')$$

where

$$\Phi_v(u_0, u_1, \ldots, u_k) = E \left(\prod_{r=0}^{k} u_r^{w_{rn}} \right)$$

with

$$w_{rn} = \sum_{i=1}^{\ell} \alpha_r^i \, x_n^i$$

It follows from eq. (55) that, for j = 0, 1, ..., $k - 2$, ($k \geq 2$),

$$\Phi(s, \ldots, s, u_1, \ldots, u_{k-j}) = [s^{-1} \Phi(s, \ldots, s, u_1, \ldots, u_{k-j-1}) + (1 - s^{-1}) \Phi(0, s, \ldots, s, u_1, \ldots, u_{k-j-1})] \times \Phi_v(s, \ldots, s, u_1, \ldots, u_{k-j}) \quad (58')$$

and

$$\Phi(s, \ldots, s, u_1)$$
$$= [s^{-1}\Phi(s, \ldots, s) + (1 - s^{-1})\Phi(0, s, \ldots, s)]\Phi_v(s, \ldots, s, u_1) \quad (58'')$$

These equations are equivalent to eq. (58). If we set $u_1 = s$ in (58″) then we may solve for $\Phi(s, \ldots, s) = \phi(1, \ldots, 1, s)$, as in (56).

VI. FINDING $\phi(0, s_1, \ldots, s_k)$

We will show here that a finite system of linear equations can be obtained to solve for the coefficients of the polynomial of finite degree that represents $\phi(0, s_1, \ldots, s_k)$. Let

$$\psi_{\mathbf{j}}(s_1, \ldots, s_k) = \mu \prod_{i=1}^{k} s_i^{j_i}$$

where $\mu = Pr\{y_0 = 0\}$ and let $\theta_{\mathbf{j}}(s_0, s_1, \ldots, s_k)$ be related to $\psi_{\mathbf{j}}$ as $\phi(s_0, s_1, \ldots, s_k)$ is to $\phi(0, s_1, \ldots, s_k)$ in eq. (58). That is, if $\theta_{\mathbf{j}}^r$ is defined as in (57), $\theta_{\mathbf{j}}^r(s_r, \ldots, s_k) = \theta_{\mathbf{j}}(1, 1, \ldots, 1, s_r, s_{r+1}, \ldots, s_k)$, then $\theta_{\mathbf{j}}^r$ satisfies the set of equations equivalent to (58): for $r = 0, 1, \ldots, k - 1$

$$\theta_{\mathbf{j}}^r(s_r, \ldots, s_k) = \left[\prod_{i=r}^{k} s_i^{-1}\, \theta_{\mathbf{j}}^{r+1}(s_r, \ldots, s_{k-2}, s_{k-1}s_k) + \left(1 - \prod_{i=r}^{k} s_i^{-1}\right) \psi_{\mathbf{j}}(1, 1, \ldots, 1, s_r, \ldots, s_{k-2}, s_{k-1}s_k)\right] \times \phi_v(1, 1, \ldots, 1, s_r, \ldots, s_k) \quad (59)$$

and (56) corresponds to

$$\theta_{\mathbf{j}}^k(s_k) = \frac{(1 - s_k^{-1})\psi_{\mathbf{j}}(1, 1, \ldots, 1, s_k)\phi_v(1, 1, \ldots, 1, s_k)}{1 - s_k^{-1}\,\phi_v(1, 1, \ldots, 1, s_k)} \quad (60)$$

From the definition of ϕ_v, $\phi_v(1, 1, \ldots, 1, s_k) = Es_k^{v_{kn}}$ [see above (52)]. Hence from (50) and applying (13), (14) we have

$$\theta_{\mathbf{j}}^k(s_k) = \frac{1}{\mu}\, \psi_{\mathbf{j}}(1, 1, \ldots, s_k)\phi_c(s_k) \quad (61)$$

where $c = \lim_{n\uparrow\infty} c_n$ and $Es_k^c = \phi_c(s_k)$. Hence whenever $\mu > 0$,

$$\frac{1}{\mu}\, \phi_c(\cdot)$$

is a generating function. Now, it can be easily verified that corresponding to each $\mathbf{j}$ the unique solution $\theta_{\mathbf{j}}^0(s_0, \ldots, s_k) = \theta_{\mathbf{j}}(s_0, s_1, \ldots, s_k)$ satisfies an equation similar to (55):

$$\theta_{\mathbf{j}}(s_0, s_1, \ldots, s_k) = \left[\theta_{\mathbf{j}}(1, s_0, s_1, \ldots, s_{k-2}, s_{k-1}s_k) \prod_{i=0}^{k} s_i^{-1} \right.$$
$$\left. + \left(1 - \prod_{i=0}^{k} s_i^{-1}\right) \psi_{\mathbf{j}}(s_0, s_1, \ldots, s_{k-2}, s_{k-1}s_k) \right]$$
$$\times \phi_v(s_0, s_1, \ldots, s_k) \quad (62)$$

The family of such solutions $\theta_{\mathbf{j}}$ are linearly independent. If the generating function of $P_{\mathbf{i}}$ (the equilibrium distribution of S), $\phi(s_0, s_1, \ldots, s_k)$, is such that

$$\phi(0, s_1, \ldots, s_k) = \mu \sum_{\mathbf{j}}{}' c_{\mathbf{j}} \prod_{i=1}^{k} s_i^{j_i} \quad (63)$$

where the sum on the right is over all indices $\mathbf{j} = (0, j_1, \ldots, j_k)$ which are in $\mathcal{A}$, then $\phi(s_0, s_1, \ldots, s_k)$ has the unique representation

$$\phi(s_0, s_1, \ldots, s_k) = \sum_{\mathbf{j}}{}' c_{\mathbf{j}} \theta_{\mathbf{j}}(s_0, s_1, \ldots, s_k) \quad (64)$$

Notice that corresponding to each $\mathbf{j}$ there exists a sequence $P_{\mathbf{i}}(\mathbf{j})$, not necessarily nonnegative, such that

$$\sum P_{\mathbf{i}}(\mathbf{j}) s_0^{i_0} s_1^{i_1} \ldots s_k^{i_k} = \theta_{\mathbf{j}}(s_0, s_1, \ldots, s_k) \quad (65)$$

Hence $P_{\mathbf{i}}$ itself has the representation

$$P_{\mathbf{i}} = \sum_{\mathbf{j}}{}' c_{\mathbf{j}} P_{\mathbf{i}}(\mathbf{j}) \quad (66)$$

Furthermore $\theta_{\mathbf{j}}^k(s_k)$ from (61) corresponds to a nonnegative summable sequence. From (59), starting with $r = k - 1$ and going backwards to $r = 0$, we can show that for each $\mathbf{j}$, $P_{\mathbf{i}}(\mathbf{j})$ is the convolution of absolutely summable sequences and hence

$$\sum |P_{\mathbf{i}}(\mathbf{j})| < \infty \quad (67)$$

From eqs. (61), (65), and (66), when $\Sigma_{\mathbf{j}}' c_{\mathbf{j}} = 1$,

$$\sum_{\mathbf{i} \in \mathcal{A}} P_{\mathbf{i}} = \sum_{\mathbf{j}}{}' c_{\mathbf{j}} \theta_{\mathbf{j}}^k(1)$$
$$= \sum_{\mathbf{j}}{}' c_{\mathbf{j}} = 1 \quad (68)$$

We will now show that there is a finite number of linear equations derived by substitution of (66) into (49) which uniquely determine $\{c_{\mathbf{j}}\}$ and hence $P_{\mathbf{i}}$. Let us denote the elements of the transition probability matrix of S, $Pr\,\{\mathbf{y}_{n+1} = \mathbf{i} \,|\, \mathbf{y}_n = \mathbf{j}\}$, by $T_{\mathbf{ij}}$. Then from (40)

$$P_{\mathrm{i}} = \sum_{\mathrm{j}\in\mathcal{A}} T_{\mathrm{ij}} P_{\mathrm{j}} \tag{69}$$

Substitution of (66) yields

$$\sum_{\mathbf{m}}{}' c_{\mathbf{m}} P_{\mathrm{i}}(\mathbf{m}) = \sum_{\mathrm{j}\in\mathcal{A}} T_{\mathrm{ij}} \sum_{\mathbf{m}}{}' c_{\mathbf{m}} P_{\mathrm{j}}(\mathbf{m}) \tag{70}$$

which is a set of linear equations for $c_{\mathbf{m}}$. Hence any solution $\{d_{\mathbf{m}}\}$ of (70) has the property that $Q_{\mathrm{i}} = \Sigma'_{\mathbf{m}}\, d_{\mathbf{m}} P_{\mathrm{i}}(\mathbf{m})$ satisfies (69), hence

$$Q_{\mathrm{i}} = \sum_{\mathrm{j}\in\mathcal{A}} T_{\mathrm{ij}} Q_{\mathrm{j}} \tag{71}$$

Now let T^{n}_{ij} be the n-step transition matrix of S. Then using (71) we have

$$Q_{\mathrm{i}} = \sum_{\mathrm{j}\in\mathcal{A}} T^{n}_{\mathrm{ij}}\, Q_{\mathrm{j}} \tag{72}$$

Since S is positive recurrent

$$\lim_{n\uparrow\infty} T^{n}_{\mathrm{ij}} = P_{\mathrm{i}} \tag{73}$$

Furthermore since $\Sigma_{\mathrm{i}} |P_{\mathrm{i}}(j)| < \infty$ for each $\mathbf{j}$, $\Sigma_{\mathrm{i}} |Q_{\mathrm{i}}| < \infty$. Hence taking limits of both sides of (72) and interchanging limit and sum on the right hand side of (72) we have

$$\begin{aligned} Q_{\mathrm{i}} &= \lim_{n\uparrow\infty} \sum_{\mathrm{j}} T^{n}_{\mathrm{ij}}\, Q_{\mathrm{j}} \\ &= P_{\mathrm{i}} \sum_{\mathrm{j}} Q_{\mathrm{j}} \end{aligned} \tag{74}$$

However, if $\Sigma'_{\mathrm{j}}\, d_{\mathrm{j}} = 1$ then, from (68), $\Sigma_{\mathrm{i}}\, Q_{\mathrm{i}} = 1$. Therefore

$$Q_{\mathrm{i}} = P_{\mathrm{i}} \tag{75}$$

Since $P_{\mathrm{i}}(\mathbf{j})$ are linearly independent, $c_{\mathrm{j}} = d_{\mathrm{j}}$ for each $\mathbf{j}$, and $\{c_{\mathrm{j}}\}$ are the unique solution of (70).

Remark 1: A similar set of equations can be obtained by substituting (64) into (55) and equating the coefficients of like powers on both sides of (55).

Remark 2: Note that $\phi(0, s_1, \ldots, s_k) = \mu$ in the case when, for each i, j, $\alpha^i_j > 0$. Hence (58) may be used repeatedly to obtain an expression for $\phi(s_0, s_1, \ldots, s_k)$. Herbert[6] considered this model when $\ell = 1$.

In the alternate formulation $\Phi(0, u_1, \ldots, u_k)$ is a multinomial. Moreover, from (34) and (35), since $\alpha^i_0 > 0$, $i = 1, \ldots, \ell$, $y_{0n} = 0 \Rightarrow x^i_{n-1} = 0$, $i = 1, \ldots, \ell$, which implies $y_{kn} = y_{k-1,n}$. Hence $\Phi(0, u_1, \ldots, u_k)$ is independent of u_k. From (58′) and (58″), $\Phi(s, u_1, \ldots, u_k)$ may be expressed in terms of $\Phi(0, s, \ldots, s, u_1, \ldots, u_{k-j-1}), j = 0, \ldots, k-2$, and $\Phi(0, s, \ldots, s)$. If we let $s \to 0$ in this expression, and equate $\Phi(0, u_1, \ldots, u_k)$ with the finite part, we obtain a system of homogeneous linear equations for the coefficients in the multinomial. In general, we also obtain a (consistent) set of homogeneous linear equations from finiteness conditions.

VII. GENERATING FUNCTIONS FOR MARGINALS AND FINITE SOLVABILITY

The joint distributions of $(y_0, y_1, \ldots, y_k)$ have $(k+1)$ arguments. We will see that we can reduce the problem to "$k+1$ one-dimensional problems" when we are only interested in the marginal distributions of $y_0, y_1, \ldots, y_k$. Let us denote the generating functions of y_i by $\Phi_i(s)$ and those of v_{rn} by $\phi_{rv}(s)$. Then

$$\begin{aligned}\phi_i(s) &= \phi\,(1, 1, \ldots, \overset{i}{s}, 1, \ldots, 1)\\ &= \Phi\,(\overset{i+1}{s, \ldots, s}, \overset{k-i}{1, \ldots, 1}),\ i = 0, \ldots, k\end{aligned} \tag{76}$$

From (55) we then obtain for $r = 0, \ldots, k-1$

$$\phi_r(s) = \left[s^{-1}\phi_{r+1}(s) + (1 - s^{-1})\phi\,(0, 1, \ldots, \overset{r+1}{s}, \ldots, 1)\right]\phi_{rv}(s)$$

and

$$\phi_k(s) = [s^{-1}\phi_k(s) + (1 - s^{-1})\phi(0, 1, \ldots, 1, s)]\phi_{kv}(s) \tag{77}$$

Note that

$$\begin{aligned}&\phi\,(0, 1, \ldots, \overset{r}{s}, \ldots, 1)\\ &\qquad = \Phi\,(0, \overset{r}{s, \ldots, s}, \overset{k-r}{1, \ldots, 1}),\ r = 1, \ldots, k\end{aligned}$$

Therefore once the c_j have been determined from the method presented above, Eq. (77) gives the marginal distributions. Once again we can translate (77) into linear equations for the distributions themselves as in (15). The marginals are finitely solvable in the sense that a finite number of components of the marginal distributions can be solved for from a finite number of linear equations.

For each $r = 1, 2, \ldots, k$ let γ_{rj} be the coefficient of s^j in the polynomial

$$\phi(0, 1, \ldots, \overset{r}{s}, \ldots, 1)$$

denoted by $c_r(s)$. Equating coefficients of like powers of s_r^j on both sides of (63) after setting $s_i = 1$ for $i \neq r$ yields

$$\gamma_{rj} = \mu \sum_{j_r=j}{}' c_{\mathbf{j}} \tag{78}$$

Therefore since the $c_{\mathbf{j}}$'s can be determined as the solutions to a finite system of linear equations, so can the γ_{rj}'s.

Let

$$\phi_r(s) = \sum_{j=0}^{\infty} \pi_{rj} s^j, \quad \Pi_{rj} = \sum_{i=0}^{j} \pi_{rj}$$

and $F_r(s) = \Sigma_{j=0}^{\infty} \Pi_{rj} s^j$ for $|s| < 1$ and r = 0, 1, . . . , k. Then $F_r(s) = \phi_r(s)/(1-s)$, and eqs. (77) become

$$F_r(s) = s^{-1}[F_{r+1}(s) - c_{r+1}(s)]\phi_{rv}(s),\ r = 0, 1, \ldots, k-1 \tag{79}$$

$$F_k(s) = \frac{\phi_{kv}(s)c_k(s)}{\phi_{kv}(s) - s} \tag{80}$$

From (79) we can show that, for each $r = 0, 1, \ldots, k-1$, $\{\Pi_{rj}\}_{j=0}^{N+r}$ are determined from $\{\Pi_{r+1,j}\}_{j=0}^{N+r+1}$ by a finite set of linear equations, for any N. Let the sequence $\{\delta_{rj}\}$ correspond to $s^{-1}[F_{r+1}(s) - c_{r+1}(s)]$. From the definition of $F_{r+1}(s)$ and $c_{r+1}(s)$ it follows that $\Pi_{r+1,0} = \gamma_{r+1,0}$. Therefore $\delta_{rj} = 0$ for $j < 0$ and

$$\begin{aligned}\delta_{rj} &= \Pi_{r+1,j+1} - \gamma_{r+1,j+1} \text{ for } 1 \leq j+1 \leq \text{degree of } c_{r+1}(s)\\ &= \Pi_{r+1,j+1} \text{ for } j+1 > \text{degree of } c_{r+1}(s)\end{aligned} \tag{81}$$

From (79), the sequence $\{\Pi_{rj}\}_{j=0}^{N+r}$ is the convolution of $\{\delta_{rj}\}$ with $\{p_{rj}\}$—the sequence of probabilities corresponding to the characteristic function $\phi_{rv}(s)$. Therefore, since the sequence $\{p_{rj}\}$ is known a priori, we can find Π_{rj} as:

$$\Pi_{rj} = \sum_{i=0}^{j} \delta_{r,j-i} p_{ri},\ j = 0, 1, \ldots, N+r \tag{82}$$

Hence, we observe that $\{\Pi_{0j}\}_{j=0}^{N}$ can be determined as solutions to a finite system of linear equations using $\{\Pi_{kj}\}_{j=0}^{N+k}$.

In order to find $\{\Pi_{kj}\}_{j=0}^{N+k}$ we proceed as in (15). Equating the coefficients of like powers of s^j in

$$\sum_{j=0}^{\infty} \Pi'_{kj} s^j = \frac{\phi_{kv}(s)}{\phi_{kv}(s) - s} \tag{83}$$

yields:

$$\Pi'_{k0} = 1$$

$$\Pi'_{kj} = \left(p_{kj} + \Pi'_{k,j-1} - \sum_{i=1}^{j} p_{ki}\Pi'_{k,j-i}\right) \Big/ p_{k0}$$

$$j = 1, 2, \ldots\ldots, N+k \tag{84}$$

Therefore $\{\Pi'_{kj}\}_{j=0}^{N+k}$ can be determined uniquely as solutions of (84). From (80) and (83), $\{\Pi_{kj}\}_{j=0}^{N+k}$ is the convolution of $\{\Pi'_{kj}\}$ with $\{\gamma_{kj}\}$,

$$\Pi_{kj} = \sum_{i=0}^{j} \Pi'_{k,j-i}\gamma_{ki}, \, j = 0, 1 \ldots . N + k \tag{85}$$

Therefore we have shown that each of the Π_{rj}, and hence the marginal distributions π_{rj}, $r = 0, 1, \ldots, k, j = 0, 1, \ldots, N + r$, can be found, for any finite N, as solutions to a finite system of linear equations.

VIII. A LIMITING CASE

For each m let d_{jm} be a nondecreasing sequence of nonnegative integers such that

$$(i) \quad d_{j0} = 0, \, d_{j1} = j; \, j = 0, 1, \ldots, k$$

$$(ii) \quad \lim_{m \uparrow \infty} d_{jm} - d_{j-1,m} = \infty, \, j = 1, \ldots, k \tag{86}$$

We define a sequence of processes $\{z_n^m\}$ which will be time-scaled versions of z_n. Let

$$z_n^m = \sum_{i=1}^{\ell} \sum_{j=0}^{k} \alpha_j^i \, x_{n-d_{jm}}^i$$

We observe that z_n^1 is the same as z_n, and z_n^0 is the "fastest" version of z_n, in the sense that all the packets triggered by x_n^i are bunched together and arrive at the same time. As m increases the different delayed contributions of x_n^i are spread farther and farther apart in time. The limiting case can then be interpreted as the "slowest"; see Ref. 7.

Let $\{\eta_n\}$, $n = 0, 1, 2, \ldots$, be a sequence of independent identically distributed random variables such that for each n the distribution of η_n is the same as that of z_n. We will show that η_n then corresponds to the slowest case: the finite dimensional distributions of the processes $\{z_n^m\}$, $m = 0, 1, \ldots$ converge to the corresponding distributions of $\{\eta_n\}$ as $m \uparrow \infty$. Indeed, let $n_1 < n_2 < \cdots < n_s$ be nonnegative integers. Then

$$Pr\,\{z_{n_1}^m = i_1, z_{n_2}^m = i_2, \ldots, z_{n_s}^m = i_s\} = \prod_{j=1}^{s} Pr\,\{z_{n_j}^m = i_j\} \tag{87}$$

for large enough m, in particular for every m such that $d_{jm} - d_{j-1,m} > n_s - n_1$, $j = 1, \ldots, k$. However from the definition of η_n, $Pr\,\{z_{n_j}^m = i_j\} = Pr\,\{\eta_{n_j} = i_j\}$. Therefore from the independence of η_n and (87)

$$\begin{aligned} &Pr\,\{z_{n_1}^m = i_1, z_{n_2}^m = i_2, \ldots, z_{n_s}^m = i_s\} \\ &= P\,\{\eta_{n_1} = i_1, \eta_{n_2} = i_2, \ldots, \eta_{n_s} = i_s\} \end{aligned} \tag{88}$$

for large enough m.

We now define a sequence of processes b_n^m, b_n^∞ corresponding to z_n^m, η_n respectively. Formally let

$$b_{n+1}^m = (b_n^m - 1)^+ + z_n^m$$
$$b_{n+1}^\infty = (b_n^\infty - 1)^+ + \eta_n \tag{89}$$

Since $Ez_n^m = E\eta_n$, if $Ez_n < 1$ then for each m, $\lim_{n\uparrow\infty} b_n^m = b^m$ is a well-defined random variable, and so is $b^\infty = \lim_{n\uparrow\infty} b_n^\infty$. We can then show that

$$\lim_{m\uparrow\infty} b^m = b^\infty \tag{90}$$

from Theorem 22 in Ref. 9, since z_n^m, η_n are nonnegative. Hence the distribution of b^∞ approximates the distribution of b^m for sufficiently large m. Therefore for each j

$$\lim_{m\uparrow\infty} Pr\,\{b^m \le j\} = Pr\,\{b^\infty \le j\} \tag{91}$$

Therefore b^∞ is the steady-state queue size corresponding to the "slowest" version of z_n. Let

$$\phi_x(s_1, \ldots, s_\ell) = E\left(\prod_{i=1}^{\ell} s_i^{x_n^i}\right)$$

Then it is easy to verify that $Es^{z_n^0}\, and\, Es^{\eta_n}$ are given by

$$\phi_x^0 = \phi_x(s^{\mu_k^1}, s^{\mu_k^2}, \ldots, s^{\mu_k^\ell}) \text{ and } \phi_x^\infty = \prod_{i=0}^{k} \phi_x\,(s^{\alpha_i^1}, s^{\alpha_i^2}, \ldots, s^{\alpha_i^\ell})$$

respectively. If $\phi^0 = Es^{b^0}$ and $\phi^\infty = Es^{b^\infty}$ then

$$\phi^0 = \frac{(1 - s^{-1})\phi_x^0(s)\mu}{1 - s^{-1}\phi_x^0(s)}$$
$$\phi^\infty = \frac{(1 - s^{-1})\phi_x^\infty(s)\mu}{1 - s^{-1}\phi_x^\infty(s)} \tag{92}$$

In the special case when $\ell = 1$, and (omitting the superscript) $\alpha_j = 0$ or 1 for each j, we have an interesting special relationship between ϕ^0 and ϕ^∞. Let $f_n^0 = Pr\,\{b^0 \le n\}$, $f_n^\infty = Pr\,\{b^\infty \le n\}$ and $F^0 = \Sigma f_n^0 s^n$, $F^\infty = \Sigma f_n^\infty s^n$. Then F^0 and F^∞ are $\phi^0/1 - s$ and $\phi^\infty/1 - s$ respectively for $|s| < 1$. We will show that

$$f_n^\infty = f_{n\mu_k}^0 \tag{93}$$

equivalently

$$Pr\,\{b^\infty \le n\} = Pr\,\{b^0 \le n\mu_k\} \tag{94}$$

Let ω be a primitive μ_kth root of unity. Then for $|s| < 1$

$$\frac{1}{\mu_k} \sum_{i=0}^{\mu_k - 1} F^0(\omega^i s) = \frac{\phi_x(s^{\mu_k})}{\mu_k} \sum_{i=0}^{\mu_k - 1} \frac{\mu}{\phi_x(s^{\mu_k}) - \omega^i s}$$

$$= \frac{\mu[\phi_x(s^{\mu_k})]^{\mu_k}}{[\phi_x(s^{\mu_k})]^{\mu_k} - s^{\mu_k}} = F^{\infty}(s^{\mu_k}) \tag{95}$$

Therefore

$$\sum_{n=0}^{\infty} f_n^{\infty} s^{n\mu_k} = \frac{1}{\mu_k} \sum_{i=0}^{\mu_k - 1} \sum_{m=0}^{\infty} f_m^0 (\omega^i s)^m$$

$$= \sum_{n=0}^{\infty} f_{n\mu_k}^0 s^{n\mu_k} \tag{96}$$

Since f_n^0 and f_n^{∞} are both increasing and bounded by 1, (96) shows that (93) holds.

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