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Construction for Group-Balanced Connecting Networks

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We generalize the concept of balanced network to group-balanced network. An s*-stage network is called a group-balanced network if its input switches can be partitioned into groups and its output switches into groups such that the connection pattern (called channel graph) between an input group and an output group is independent of which groups we choose. We show by construction that under a simple divisibility condition, a group-balanced network can be constructed satisfying the following requirements:* (*i*) *the number of stages is specified,* (*ii*) *the size of the switches in each stage is specified,* (*iii*) *the channel graph between an input group and an output group is specified.*

I. INTRODUCTION

An s-stage (connecting) network satisfies the following conditions:

(*i*) The network is composed of switches and links. Switches are arranged in a sequence of s stages.
(*ii*) The switches in a given stage are identical. In particular, they have the same *size,* i.e., the same number of input terminals and output terminals.
(*iii*) Links can exist only between two switches in adjacent stages.

In this paper, we assume that each switch is a rectangular (matrix) switch; i.e., there is a crosspoint connecting every input terminal with every output terminal of that switch. Figure 1 illustrates a three-stage network.

Consider an s-stage network and let S_i denote a switch in the ith stage. Consider the paths in the network which connect an S_1 (input switch), say, the kth, with an S_s (output switch), say the jth. Taking the union of all such paths and replacing each switch on a path by a node, we have

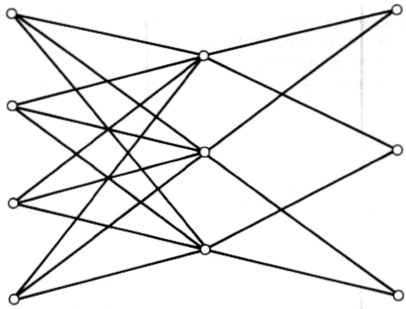

Fig. 1—Three-stage network.

the *channel graph* $G(k, j)$ for that pair of switches. Suppose the collection

$$\{G(k, j): j = 1, 2, \ldots\}$$

is identical for all k, and the collection

$$\{(G(k, j): k = 1, 2, \ldots\}$$

is identical for all j, i.e., the network is symmetric with respect to the switches in the first (last) stage. Then the network is called a *partially balanced network.* If, furthermore, $G(k, j)$ is identical for arbitrary k and j, the network is called a *balanced network.*[4,6]

In this note we generalize the concept of balanced network. An s-stage network is called a *group-balanced network* if its input switches can be partitioned into groups and its output stages into groups such that the connection pattern between an input group and an output group is independent of which groups we choose. This connection pattern will again be referred to as the channel graph between the two groups. When both the input group and the output group contain a single switch, a group-balanced network reduces to a balanced network. Moreover, an s-stage group-balanced network can always be augmented into an $(s + 2)$-stage balanced network by adding a stage before the input stage in such a way that the switches in one input group always connect to the same set of switches in the new stage, and by adding a stage after the output stage with similar connections. We also note that every s-stage network can be viewed as a group-balanced network if all input switches are considered to form one input group and all output switches to form one output group.

The problem of constructing a balanced network with a specified channel graph and given switch sizes has been studied in Refs. 1–8. In this note we give a construction for group-balanced networks under similar conditions.

II. A CONSTRUCTION

For a given node in a channel graph G, we will call the number of links connecting it to a preceding stage its *indegree*, and the number of links connecting it to a succeeding stage its *outdegree*. We assume the specified channel graph is *regular* in the sense that every node in the same stage has the same indegree and outdegree. Let d_i and c_i denote the indegree and outdegree for a node in the ith stage, $i = 1, \ldots, s$. We want to construct a group-balanced network B whose channel graph is the specified one and whose ith stage switches are of given size $n_i \times m_i$ (n_i input terminals and m_i output terminals), $i = 1, \ldots, s$. Note that the number of switches in the ith stage, say l_i, is completely determined from

$$l_i = \prod_{j=0}^{i-1} m_j \prod_{j=i+1}^{s+1} n_j/\lambda, \; i = 1, \ldots, s$$

where $m_0(n_{s+1})$ is defined to be the number of input (output) switches in an input (output) group and λ is the number of paths from the first stage to the last stage in the specified channel graph.

Theorem 1: Suppose d_i divides n_i and c_i divides m_i for every $i = 1, \ldots,$ s. Then the desired B exists.

Proof: The proof is by construction. Without loss of generality, we may assume that the number of stages s is even. For if s is odd, we can always add an $(s + 1)$th stage, which has a single node, to the channel graph by connecting that node with every node in the sth stage. Since $s + 1$ is now even, we can construct an $(s + 1)$-stage group-balanced network (by defining n_{s+1} and m_{s+1} properly) and then delete the $(s + 1)$th stage. Our construction is by induction on s, $(s = 2, 4, 6, \ldots)$.

Let S_i denote a switch of size $n_i \times m_i$. For $s = 2$, take n_2/d_2 groups of switches S_1 and m_1/c_1 groups of switches S_2. Connect every group of S_1 to every group of S_2 according to the specified channel graph G. The resulting network is the desired one.

Next, consider an s-stage channel graph G for even s. Let f_i be the number of nodes in the ith stage of G. Furthermore, let G^i be the subgraph obtained from G by deleting its first and last $(i - 1)$ stages. Suppose by induction, we have constructed an $(s - 2)$-stage network B' with the specified channel graph G^2. We show show to construct the s-stage network with the specified channel graph G.

Take $(n_s/d_s) \cdot (m_1/c_1)$ copies of B' and label them by $B'(i, j)$ where $i = 1, \ldots, n_s/d_s$ and $j = 1, \ldots, m_1/c_1$. Note that the input (output) switches of B' can be decomposed into $g_2(g_{s-1})$ groups each of which consists of $f_2(f_{s-1})$ switches and the channel graph between every input group and every output group is G^2. Take $g_1 = (n_s/d_s) \cdot (n_2/d_2) \cdot g_2$

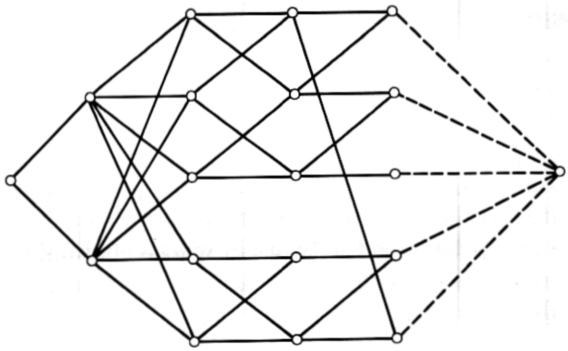

Fig. 2—Specified five-stage channel graph G.

groups (each containing f_1 switches) of S_1 and label the groups by $F(u, v, w)$ where $u = 1, \ldots, n_s/d_s$, $v = 1, \ldots, n_2 d_2$ and $w = 1, \ldots, g_2$. Connect $F(u, v, w)$ to the wth input group of every $B'(u, j)$ according to the connection of nodes in the first two stages of G'. Similarly take $g_s = (m_1/c_1) \cdot (m_{s-1}/c_{s-1}) \cdot g_{s-1}$ groups (each containing f_s switches) of S_s and label the groups by $H(x, y, z)$ where $x = 1, \ldots, m_1/c_1$, $y = 1, \ldots, (m_{s-1}/c_{s-1})$ and $z = 1, \ldots, g_{s-1}$. Connect $H(x, y, z)$ to the zth input group of every $B'(i, x)$ according to the connection of nodes in the last two stages of G'. It is easy to verify that the channel graph between every $F(u, v, w)$ and every $H(x, y, z)$ is the graph G.

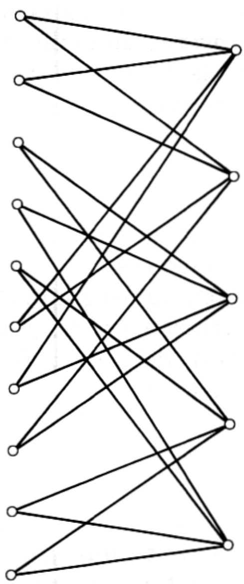

Fig. 3—One of many possible ways of connecting groups, denoted by B^3.

III. EXAMPLES

In this section we illustrate with several examples the scope of applicability of the construction method given in the last section.

Example 1: Let the specified five-stage channel graph G be the one in Fig. 2 (solid lines only), where the specified switch sizes are

$$n_1 = 1, \quad n_2 = 3, \quad n_3 = 2, \quad n_4 = 4, \quad n_5 = 4$$

$$m_1 = 2, \quad m_2 = 5, \quad m_3 = 2, \quad m_4 = 4, \quad m_5 = 1$$

Construction: Since the number of stages is odd, we add an artificial stage (broken lines). We first construct G^3 which has $f_3 = 5$ input nodes and $f_4 = 5$ output nodes. Since

$$\frac{m_3}{c_3} = 1 \quad \text{and} \quad \frac{n_4}{d_4} = 2$$

we take $g_3 = 2$ groups of S_3, $g_4 = 1$ group of S_4 and connect each group of S_3 with the group of S_4 according to G_3. There are many possible ways of connecting, one of which is shown in Fig. 3 and denoted by B^3.

Next we construct G^2 which has two input nodes and five output nodes. Since

$$\frac{m_2}{c_2} = 1, \quad \frac{n_5}{d_5} = 2, \quad \frac{m_4}{c_4} = 2, \quad \frac{n_3}{d_3} = 1$$

we take

$$\frac{n_5}{d_5} \cdot \frac{m_2}{c_2} = 2 \text{ copies of } B^3$$

$$g_2 = \frac{n_5}{d_5} \cdot \frac{n_3}{d_3} \cdot g_3 = 4 \text{ groups of } S_2$$

$$\text{and } g_5 = \frac{m_2}{c_2} \cdot \frac{m_4}{c_4} \cdot g_4 = 2 \text{ groups of } S_5$$

and make connection between groups according to G^2. One possible connection, denoted by B^2, is given in Fig. 4 (solid lines).

Finally, we construct G^1 whose output stage can be ignored (since it is artificial) as long as we define $m_5 = n_6 = 1$. Take

$$\frac{n_6}{d_6} \cdot \frac{m_1}{c_1} = 1 \text{ copy of } B^2$$

$$\text{and } g_1 = \frac{n_6}{d_6} \cdot \frac{n_2}{d_2} \cdot g_2 = 12 \text{ groups of } S_1$$

The final product is given in Fig. 4 with broken lines added.

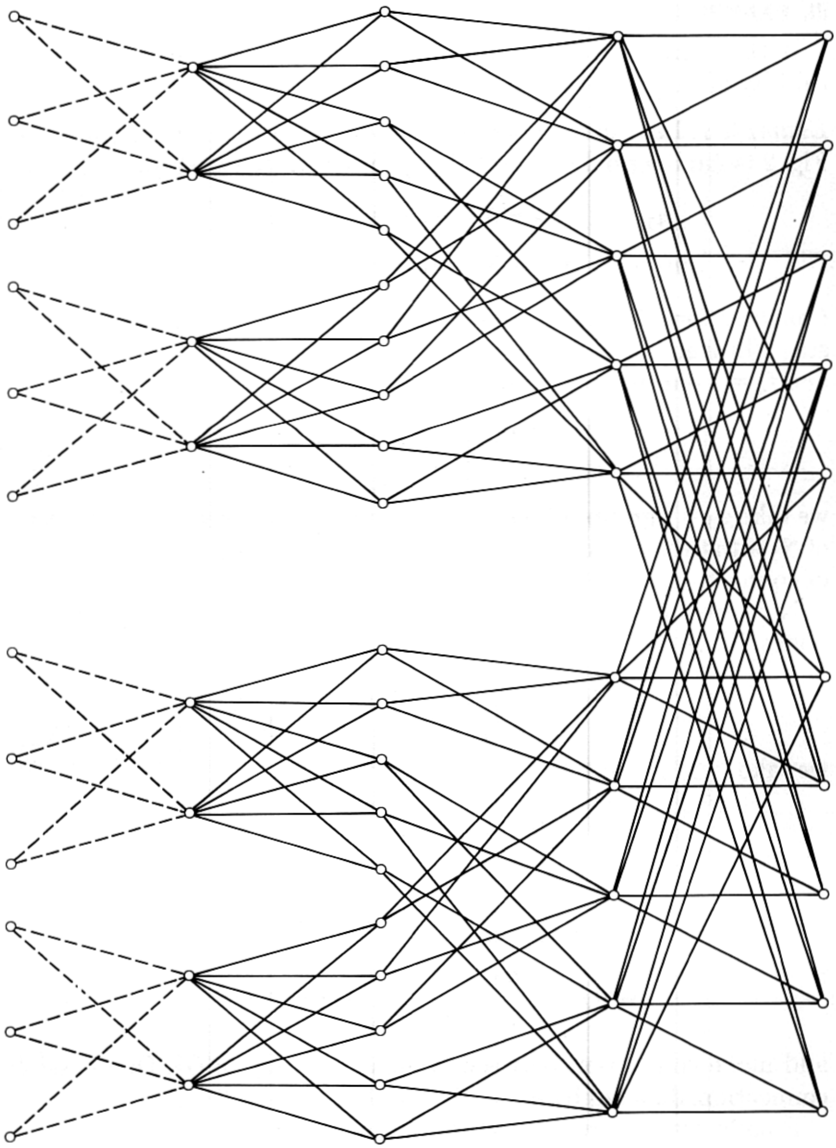

Fig. 4—A possible connection between groups according to G^2.

Example 2. Consider the channel graph in Fig. 5. Suppose we specify $n_1 = 1, n_2 = 1, n_3 = 2, m_1 = 3, m_2 = 2, m_3 = 1$. Then our construction fails since m_1 is not divisible by c_1. However, a balanced network having these parameters and the specified channel graph does exist as shown in Fig. 6.

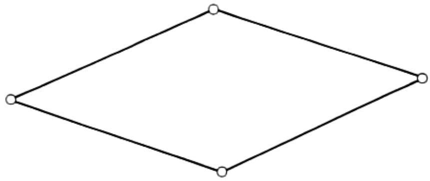

Fig. 5—Channel graph.

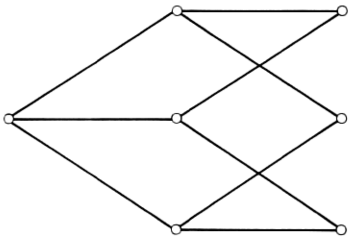

Fig. 6—Specified channel graph.

It turns out that the balanced network in Fig. 6 can be constructed by a method in Ref. 6. However, that method is not able to construct the network in Example 1.

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