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Vibrations of a Lithium Niobate Fiber

By LYNN O. WILSON

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We discuss wave propagation along a crystalline piezoelectric fiber composed of lithium niobate or some other material in the trigonal 3m crystal class. The crystalline c axis is aligned with the fiber axis. We obtain an analytical description of all the vibrational modes. The method used is to make perturbation expansions about the modes of a hexagonal 6mm piezoelectric fiber, for which exact solutions are known.

I. INTRODUCTION

A single crystal of lithium niobate, grown in the form of a long fiber, has been considered for use as a low-loss acoustic delay line. Lithium niobate is of special interest because it is piezoelectric: it becomes electrically polarized when strained and, conversely, becomes strained when placed in an electric field. This piezoelectricity provides a means for electrically generating and detecting acoustic signals.

In this paper we study mathematically the vibrational properties of a $LiNbO_3$ crystal fiber, with the crystalline c axis aligned along the fiber axis. The problem is by no means simple. We illustrate this by giving a brief history of related problems for which exact solutions have been obtained. The elastic, or acoustic, wave equations for an infinitely long circularly cylindrical isotropic rod were solved exactly by Pochhammer[1] in 1876 and independently by Chree[2] in 1889. Even for an isotropic medium, exact solutions for a rod of finite length have not been obtained. It was not until 1965 that the next full exact solution was found. This was done by Mirsky,[3,4] who determined the vibrational modes of a circularly cylindrical rod consisting of a nonpiezoelectric medium which is transversely isotropic. Such a medium belongs to the hexagonal system of crystals; the crystalline c axis was aligned along the rod or fiber. Certain of the modes obtained by Mirsky, i.e., those which are azimuthally symmetric about the fiber axis, had also been obtained earlier.[5,6] Recently, the author and J. A. Morrison were able to solve the coupled acoustic and electromagnetic wave equations, in the customary quasistatic approximation, for piezoelectric transversely isotropic crystals belonging to the hexagonal 6mm, 622, and 6 crystal classes.[7] Exact solutions were obtained for all the vibrational modes. The author is unaware of any other exact solutions, either for other crystals, or for other orientations of transversely isotropic crystals.

The difficulty lies in the acoustic wave equations which, for a general anisotropic medium, consist of three coupled wave equations for the three vector components of displacement. If piezoelectricity is added via the quasistatic approximation, for which the electric field is represented by the gradient of a potential, there are four coupled equations for four unknown functions. The boundary conditions may also involve all four functions coupled together. For the general anisotropic case, no method has been discovered to decouple the equations. For the specific crystals and orientations discussed above, it was possible to express the elastic displacements (and electric potential) in terms of three (or four) potential functions for which the wave equations decoupled.

Unfortunately, such a serendipitous situation does not exist for the lithium niobate fiber. It belongs to the trigonal 3m crystal class; we cannot expect to find an exact description of the vibrational modes. It will be possible, though, to find an approximate description by means of an infinite series perturbation expansion. We use a technique which is an extension of one used by the author to describe waves travelling along a sapphire fiber.[8] Sapphire is a nonpiezoelectric material belonging to the trigonal $\bar{3}$m crystal class. It is characterized by a stiffness matrix (used in the stress-strain relations) which has almost the same form as that for a transversely isotropic material. There is one additional stiffness coefficient. Since it turns out to be small in magnitude compared to the other stiffness coefficients, it is possible to describe the vibrational modes of a sapphire fiber (with the crystalline c axis aligned with the fiber axis) by means of perturbation expansions about the modes of a transversely isotropic fiber.

The situation for $LiNbO_3$ is similar, albeit somewhat more complicated. We will make an infinite series perturbation expansion about the known solutions for a hexagonal 6mm crystal. The same techniques, incidentally, can be used to describe vibrations of crystals in the trigonal 32 classes. We restrict ourselves to a discussion of trigonal 3m crystals only to keep the analysis from appearing extraordinarily complicated.

For the sapphire fiber, numerical results are available for the lowest-order torsional mode of vibration; they are presented in a paper by the author and M. A. Gatto.[9] A low-frequency asymptotic analysis for that mode was also performed by R. N. Thurston and the author.[10] Excellent numerical agreement between the results of the two independent theories provides a check on the rather complicated analyses involved and encourages us to extend the perturbation technique to a study of $LiNbO_3$.

In Section II we write down the basic equations of motion and boundary conditions. In Section III we apply the perturbation technique and introduce potential functions. In Section IV we solve the differential equations, and in Section V we sketch how to apply the boundary conditions.

Although it would be desirable to present numerical results as well, we shall not do so. Numerical results are not yet available for the unperturbed (hexagonal 6mm) problem. The computational effort required to describe quantitatively the vibrations of a lithium niobate fiber would be even greater than the considerable effort expended to present results for a sapphire fiber.

II. FORMULATION

Consider a single crystal of $LiNbO_3$ (or some other member of the trigonal 3m crystal class), grown in the form of a fiber of circular cross-section, with the crystallographic c axis along the fiber axis. We shall assume that the fiber is infinitely long and has radius R. We adopt a cylindrical coordinate system whose z axis coincides with the fiber axis.

In the quasistatic approximation, where the rotational part of the electric field is neglected, the basic differential equations are[11]

$$\nabla \cdot \mathbf{T} = \rho \frac{\partial^2 \mathbf{u}}{\partial \mathbf{t}^2}, \tag{1}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{2}$$

where $\mathbf{T}$ is the stress, $\mathbf{D}$ is the electric displacement, $\mathbf{u}$ is the elastic displacement, and ρ is the density. The properties of the specific crystal are introduced by means of the constitutive relations

$$\mathrm{T} = -\mathbf{e} \cdot \mathbf{E} + \mathbf{c}{:}\mathbf{S}, \tag{3}$$

$$\mathbf{D} = \epsilon \cdot \mathbf{E} + \mathbf{e}{:}\mathbf{S}, \tag{4}$$

where

$$\mathbf{E} = -\nabla\Phi, \tag{5}$$

$$\mathbf{S} = \nabla_s \mathbf{u}. \tag{6}$$

Here $\mathbf{E}$ denotes the electric field, $\mathbf{S}$ the strain, and Φ the electric potential. The crystal is described by means of the elastic stiffness matrix $\mathbf{c}$, the piezoelectric stress matrix ϵ, and the dielectric permittivity at constant strain matrix ϵ. For a crystal in the trigonal 3m class, these matrices have the following forms in cylindrical coordinates:[12]

$$\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14}C & c_{14}S & 0 \\ c_{12} & c_{11} & c_{13} & -c_{14}C & -c_{14}S & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ c_{14}C & -c_{14}C & 0 & c_{44} & 0 & -c_{14}S \\ c_{14}S & -c_{14}S & 0 & 0 & c_{44} & c_{14}C \\ 0 & 0 & 0 & -c_{14}S & c_{14}C & c_{66} \end{bmatrix}, \tag{7}$$

with

$$c_{66} = \tfrac{1}{2}(c_{11} - c_{12}), \tag{8}$$

$$C = \cos 3\theta, \quad S = \sin 3\theta. \tag{9}$$

$$\mathbf{e} = \begin{bmatrix} -e_{y2}S & e_{y2}S & 0 & 0 & e_{x5} & -e_{y2}C \\ -e_{y2}C & e_{y2}C & 0 & e_{x5} & 0 & e_{y2}S \\ e_{z1} & e_{z1} & e_{z3} & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

$$\epsilon = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{xx} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}. \tag{11}$$

Let $\mathbf{n}$ denote a vector normal to the fiber surface, i.e., in the radial direction. For the three mechanical boundary conditions,[13] we shall specify either that the surface tractions vanish:

$$\mathbf{T} \cdot \mathbf{n} = 0 \text{ at } r = R \text{ (free surface)}, \tag{12}$$

or that there is no displacement at the surface:

$$\mathbf{u} = 0 \text{ at } r = R \text{ (clamped surface)}. \tag{13}$$

The free surface condition is the natural one to consider for an acoustic delay line; it is equally simple to show how to solve the problem for the clamped surface condition, so we include it, too.

For the electrical boundary condition,[13] we take either

$$\Phi = 0 \text{ at } r = R \text{ (short-circuit)}, \tag{14}$$

or

$$\mathbf{D} \cdot \mathbf{n} = 0 \text{ at } r = R \text{ (open-circuit)}. \tag{15}$$

The problem is to solve the four differential equations (1) and (2), in conjunction with eqs. (3) to (11), subject to four boundary conditions chosen from (12) to (15). Since we are concerned with waves travelling down the fiber, we assume the solution has an $\exp[i(\omega t - \beta z)]$ dependence, where ω is the angular frequency and β is the propagation constant; β will depend upon ω.

We begin by writing the differential equations and boundary conditions in dimensionless form. Let

$$\hat{c}_{ij} = c_{ij}/c, \quad c = \max_{i,j} |c_{ij}|,$$

$$\hat{e}_{ij} = e_{ij}/e, \quad e = \max_{i,j} |e_{ij}|,$$

$$\hat{\epsilon}_{ij} = \epsilon_{ij}/\epsilon, \quad \epsilon = \max(\epsilon_{xx}, \epsilon_{zz}). \tag{16}$$

Normalize **u** with respect to R, Φ with respct to Re/ϵ, β with respect to R^{-1}, and ω with respect to $(c/\rho)^{1/2}/R$. To simplify notation, we use the same symbols as we used for dimensional quantities, except for the hats on c_{ij}, e_{ij}, and ϵ_{ij}. Upon substituting eqs. (3) to (11) into (1) and (2), we can write the dimensionless differential equations in cylindrical coordinates as

$$\begin{aligned}
&\hat{c}_{11}\left(u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u\right) + \hat{c}_{66}\frac{1}{r^2}u_{\theta\theta} + (\omega^2 - \beta^2\hat{c}_{44})u \\
&\quad - 2i\beta\hat{c}_{14}\cos 3\theta\,\frac{1}{r}u_\theta - 2i\beta\hat{c}_{14}\sin 3\theta\left(u_r - \frac{1}{r}u\right) \\
&\quad + (\hat{c}_{12} + \hat{c}_{66})\frac{1}{r}v_{r\theta} - (\hat{c}_{11} + \hat{c}_{66})\frac{1}{r^2}v_\theta \\
&\quad - 2i\beta\hat{c}_{14}\cos 3\theta\left(v_r - \frac{1}{r}v\right) + 2i\beta\hat{c}_{14}\sin 3\theta\,\frac{1}{r}v_\theta \\
&\quad - i\beta(\hat{c}_{13} + \hat{c}_{44})w_r + 2\hat{c}_{14}\cos 3\theta\left(\frac{1}{r}w_{r\theta} - \frac{1}{r^2}w_\theta\right) \\
&\quad + \hat{c}_{14}\sin 3\theta\left(w_{rr} - \frac{1}{r}w_r - \frac{1}{r^2}w_{\theta\theta}\right) \\
&\quad - i\beta\tau(\hat{e}_{x5} + \hat{e}_{z1})\Phi_r - 2\tau\hat{e}_{y2}\cos 3\theta\left(\frac{1}{r}\Phi_{r\theta} - \frac{1}{r^2}\Phi_\theta\right) \\
&\quad - \tau\hat{e}_{y2}\sin 3\theta\left(\Phi_{rr} - \frac{1}{r}\Phi_r - \frac{1}{r^2}\Phi_{\theta\theta}\right) = 0,
\end{aligned}$$

$$\begin{aligned}
&(\hat{c}_{12} + \hat{c}_{66})\frac{1}{r}u_{r\theta} + (\hat{c}_{11} + \hat{c}_{66})\frac{1}{r^2}u_\theta - 2i\beta\hat{c}_{14}\cos 3\theta\left(u_r - \frac{1}{r}u\right) \\
&\quad + 2i\beta\hat{c}_{14}\sin 3\theta\,\frac{1}{r}u_\theta + \hat{c}_{66}\left(v_{rr} + \frac{1}{r}v_r - \frac{1}{r^2}v\right) + \hat{c}_{11}\frac{1}{r^2}v_{\theta\theta} \\
&\quad + (\omega^2 - \beta^2\hat{c}_{44})v + 2i\beta\hat{c}_{14}\cos 3\theta\,\frac{1}{r}v_\theta + 2i\beta\hat{c}_{14}\sin 3\theta\left(v_r - \frac{1}{r}v\right) \\
&\quad - i\beta(\hat{c}_{13} + \hat{c}_{44})\frac{1}{r}w_\theta + \hat{c}_{14}\cos 3\theta\left(w_{rr} - \frac{1}{r}w_r - \frac{1}{r^2}w_{\theta\theta}\right)
\end{aligned}$$

$$- 2\hat{c}_{14} \sin 3\theta \left(\frac{1}{r} w_{r\theta} - \frac{1}{r^2} w_\theta\right) - i\beta\tau(\hat{e}_{x5} + \hat{e}_{z1}) \frac{1}{r} \Phi_\theta - \tau\hat{e}_{y2}$$

$$\times \cos 3\theta \left(\Phi_{rr} - \frac{1}{r} \Phi_r - \frac{1}{r^2} \Phi_{\theta\theta}\right) + 2\tau\hat{e}_{y2} \sin 3\theta \left(\frac{1}{r} \Phi_{r\theta} - \frac{1}{r^2} \Phi_\theta\right) = 0,$$

$$- i\beta(\hat{c}_{13} + \hat{c}_{44}) \left(u_r + \frac{1}{r} u\right) + 2\hat{c}_{14} \cos 3\theta \left(\frac{1}{r} u_{r\theta} - \frac{2}{r^2} u_\theta\right)$$

$$+ \hat{c}_{14} \sin 3\theta \left(u_{rr} - \frac{1}{r^2} u_{\theta\theta} - \frac{3}{r} u_r + \frac{3}{r^2} u\right)$$

$$- i\beta(\hat{c}_{13} + \hat{c}_{44}) \frac{1}{r} v_\theta + \hat{c}_{14} \cos 3\theta \left(v_{rr} - \frac{1}{r^2} v_{\theta\theta} - \frac{3}{r} v_r + \frac{3}{r^2} v\right)$$

$$- 2\hat{c}_{14} \sin 3\theta \left(\frac{1}{r} v_{r\theta} - \frac{2}{r^2} v_\theta\right) + \hat{c}_{44} \left(w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta}\right)$$

$$+ (\omega^2 - \beta^2\hat{c}_{33})w + \tau\hat{e}_{x5} \left(\Phi_{rr} + \frac{1}{r} \Phi_r + \frac{1}{r^2} \Phi_{\theta\theta}\right) - \tau\beta^2\hat{e}_{z3}\Phi = 0,$$

$$- i\beta(\hat{e}_{x5} + \hat{e}_{z1}) \left(u_r + \frac{1}{r} u\right) - 2\hat{e}_{y2} \cos 3\theta \left(\frac{1}{r} u_{r\theta} - \frac{2}{r^2} u_\theta\right)$$

$$- \hat{e}_{y2} \sin 3\theta \left(u_{rr} - \frac{1}{r^2} u_{\theta\theta} - \frac{3}{r} u_r + \frac{3}{r^2} u\right)$$

$$- i\beta(\hat{e}_{x5} + \hat{e}_{z1}) \frac{1}{r} v_\theta - \hat{e}_{y2} \cos 3\theta \left(v_{rr} - \frac{1}{r^2} v_{\theta\theta} - \frac{3}{r} v_r + \frac{3}{r^2} v\right)$$

$$+ 2\hat{e}_{y2} \sin 3\theta \left(\frac{1}{r} v_{r\theta} - \frac{2}{r^2} v_\theta\right) + \hat{e}_{x5} \left(w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta}\right)$$

$$- \beta^2\hat{e}_{z3}w - \hat{\epsilon}_{xx} \left(\Phi_{rr} + \frac{1}{r} \Phi_r + \frac{1}{r^2} \Phi_{\theta\theta}\right) + \beta^2\hat{\epsilon}_{zz}\Phi = 0, \quad (17)$$

where

$$\tau = \frac{e^2}{\epsilon c}, \quad (18)$$

and u, v, and w are the radial, azimuthal, and longitudinal components of the displacement vector $\mathbf{u}$.

In dimensionless form, the boundary conditions (12) to (15) are

Free surface:

$$\hat{c}_{11}u_r + \hat{c}_{12}(u + v_\theta) - i\beta\hat{c}_{13}w - i\beta\tau\hat{e}_{z1}\Phi + \cos 3\theta[\hat{c}_{14}(-i\beta v + w_\theta) - \tau\hat{e}_{y2}\Phi_\theta] + \sin 3\theta[\hat{c}_{14}(-i\beta u + w_r) - \tau\hat{e}_{y2}\Phi_r] = 0,$$

$$\hat{c}_{66}(u_\theta + v_r - v) + \cos 3\theta[\hat{c}_{14}(-i\beta u + w_r) - \tau\hat{e}_{y2}\Phi_r] - \sin 3\theta[\hat{c}_{14}(-i\beta v + w_\theta) - \tau\hat{e}_{y2}\Phi_\theta] = 0,$$

$$\hat{c}_{44}(-i\beta u + w_r) + \tau \hat{e}_{x5}\Phi_r + \hat{c}_{14} \cos 3\theta(u_\theta + v_r - v) + \hat{c}_{14} \sin 3\theta(u_r - u - v_\theta) = 0 \text{ at } r = 1.$$

Clamped surface:

$$u = v = w = 0 \text{ at } r = 1.$$

Short-circuit:

$$\Phi = 0 \text{ at } r = 1.$$

Open-circuit:

$$-\hat{\epsilon}_{xx}\Phi_r + \hat{e}_{x5}(-i\beta u + w_r) - \hat{e}_{y2} \cos 3\theta(u_\theta + v_r - v) + \hat{e}_{y2} \sin 3\theta(u - u_r + v_\theta) = 0 \text{ at } r = 1. \quad (19)$$

III. PERTURBATIONS AND POTENTIALS

At any given frequency ω, we wish to solve the differential equations (17) and boundary conditions (19) for the elastic displacement components u, v, and w, and for the electric potential Φ; these are functions of r and θ. We also need to determine the propagation constant β. Unfortunately, we have been unable to obtain an exact solution. We shall find an approximate solution by combining two techniques which were applied successfully in earlier papers.[7,8] First, we observe that eqs. (17) and (19) have an exact solution if $\hat{c}_{14} = \hat{e}_{y2} = 0$.[7] In this case, the crystal is a member of the hexagonal 6mm class. We make an infinite series perturbation expansion about any modal solution to that problem. This results in systems of differential equations and boundary conditions for the perturbation contributions to the elastic displacement and electric potential. Second, we write these perturbation contributions in terms of certain potential functions. The differential equations then decouple. With the aid of the boundary conditions, the potential functions can be determined; perturbation contributions to the propagation constant can also be found.

The perturbation technique has been used to describe vibrations of a sapphire fiber.[8] The equations describing that crystal can be obtained from eqs. (17) and (19) by setting Φ and the components of the piezoelectric stress matrix $\mathbf{e}$ to zero.

The potential function technique used here is the same as the one used in obtaining an exact description of the vibrations of a fiber in the hexagonal 6mm class.[7]

For lithium niobate, we find from the definition (16) and the numerical values for the stiffness coefficients[12] that $\hat{c}_{14} \approx 3.6 \times 10^{-2}$. We will use $\hat{c}_{14}$ as a perturbation parameter. This is reasonable since it is small compared to one. Instead of treating $\hat{e}_{y2}$ as a separate perturbation parameter, we write it as a constant multiple of $\hat{c}_{14}$:

$$\hat{e}_{y2} = \xi \hat{c}_{14}. \tag{20}$$

For lithium niobate, it turns out that $\hat{e}_{y2} \approx 6.8 \times 10^{-1}$ and $\xi \approx 18$.[12] The perturbation scheme would work better if $\hat{e}_{y2}$ were smaller than this. It effectively is, in three out of four differential equations and in all but the open-circuit boundary condition, for it is then multiplied by the dimensionless constant $\tau \approx 1.4 \times 10^{-1}$. In the remaining differential equation and boundary condition, however, $\hat{e}_{y2}$ is not multiplied by a small constant in this fashion. How rapidly the perturbation series actually converges will have to be determined numerically.

We first make a perturbation expansion for the propagation constant:

$$\beta = \sum_{m=0}^{\infty} (\hat{c}_{14})^m \beta_m. \tag{21}$$

When we make a perturbation expansion for the elastic displacements and electric potential, it is convenient also to make a Fourier expansion in θ. Because of the three-fold symmetry of the crystal about the z axis, the Fourier expansion only needs to include multiples of 3θ, rather than θ. With Z used to represent u, v, w, or Φ, we assume that

$$Z(r, \theta) = \sum_{m=0}^{\infty} (\hat{c}_{14})^m \sum_{n=-\infty}^{\infty} e^{iN\theta} e^{i3n\theta} Z^{m,n}(r). \tag{22}$$

To begin the perturbation scheme, we choose (for $m = n = 0$) $u^{0,0}(r) e^{iN\theta}, \ldots, \Phi^{0,0}(r)e^{iN\theta}$, and β_0 to be a modal solution to the unperturbed problem, i.e., that for a hexagonal 6mm crystal. N can be any integer. It determines which type of modal solution is being considered. $N = 0$ corresponds to an azimuthally symmetric mode, $|N| = 1$ to a flexural mode, and $|N| > 1$ to a higher-order flexural mode. For $m = 0$ and $n \neq 0$, set $u^{0,n}, \ldots, \Phi^{0,n}$ to zero. The problem then is to determine $u^{m,n}(r), \ldots, \Phi^{m,n}(r)$, and β_m for $m > 0$. We will see that the displacement and electric potential contributions vanish when $|n| > m$. The functions thus need only be determined in the "triangular" region $m = 0, 1, 2, 3, \ldots$ and $|n| \leq m$.

We next write the perturbation contributions to the elastic displacements and electric potential in terms of certain potential functions:

$$u^{m,n}(r) = \frac{d}{dr} \sum_{\ell=1}^{3} \psi_\ell^{m,n}(r) - \frac{s}{r} \psi_4^{m,n}(r),$$

$$v^{m,n}(r) = i \left[\frac{s}{r} \sum_{\ell=1}^{3} \psi_\ell^{m,n}(r) - \frac{d\psi_4^{m,n}(r)}{dr} \right],$$

$$w^{m,n}(r) = i \sum_{\ell=1}^{3} \mu_\ell \psi_\ell^{m,n}(r),$$

$$\Phi^{m,n}(r) = i \sum_{\ell=1}^{3} \eta_\ell \psi_\ell^{m,n}(r), \tag{23}$$

with

$$s = 3n + N. \tag{24}$$

The μ_ℓ and η_ℓ, $\ell = 1, 2, 3$, are constants (independent of m and n) which must be determined.

Substitute the potential functions defined in (23) into the perturbation expansions. Substitute these, in turn, into the differential equations (17). After considerable algebra, we find that the terms multiplied by $(\hat{c}_{14})^m e^{i(3n+N)\theta}$ yield the following system of differential equations for $\psi_1^{m,n}, \ldots, \psi_4^{m,n}$.

$$\frac{d}{dr} \sum_{\ell=1}^{3} \{\hat{c}_{11} \nabla_s^2 \psi_\ell^{m,n} + [(\omega^2 - \beta_0^2 \hat{c}_{44}) + \beta_0 \mu_\ell (\hat{c}_{13} + \hat{c}_{44}) + \beta_0 \eta_\ell \tau (\hat{e}_{x5} + \hat{e}_{z1})] \psi_\ell^{m,n}\} - \frac{s}{r} [\hat{c}_{66} \nabla_s^2 \psi_4^{m,n} + (\omega^2 - \beta_0^2 \hat{c}_{44}) \psi_4^{m,n}] = F_1^{m,n}(r), \tag{25}$$

$$\frac{s}{r} \sum_{\ell=1}^{3} \{\hat{c}_{11} \nabla_s^2 \psi_\ell^{m,n} + [(\omega^2 - \beta_0^2 \hat{c}_{44}) + \beta_0 \mu_\ell (\hat{c}_{13} + \hat{c}_{44}) + \beta_0 \eta_\ell \tau (\hat{e}_{x5} + \hat{e}_{z1})] \psi_\ell^{m,n}\} - \frac{d}{dr} [\hat{c}_{66} \nabla_s^2 \psi_4^{m,n} + (\omega^2 - \beta_0^2 \hat{c}_{44}) \psi_4^{m,n}] = F_2^{m,n}(r), \tag{26}$$

$$\sum_{\ell=1}^{3} [\mu_\ell \hat{c}_{44} + \eta_\ell \tau \hat{e}_{x5} - \beta_0 (\hat{c}_{13} + \hat{c}_{44})] \nabla_s^2 \psi_\ell^{m,n} + \sum_{\ell=1}^{3} [(\omega^2 - \beta_0^2 \hat{c}_{33}) \mu_\ell - \beta_0^2 \hat{e}_{z3} \tau \eta_\ell] \psi_\ell^{m,n} = F_3^{m,n}(r), \tag{27}$$

$$\sum_{\ell=1}^{3} [\hat{\epsilon}_{xx} \eta_\ell + \beta_0 (\hat{e}_{x5} + \hat{e}_{z1}) - \hat{e}_{x5} \mu_\ell] \nabla_s^2 \psi_\ell^{m,n} + \sum_{\ell=1}^{3} [-\beta_0^2 \hat{\epsilon}_{zz} \eta_\ell + \beta_0^2 \hat{e}_{z3} \mu_\ell] \psi_\ell^{m,n} = F_4^{m,n}(r), \tag{28}$$

with

$$\nabla_s^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{s^2}{r^2}. \tag{29}$$

The functions $F_j^{m,n}(r)$, $j = 1, \ldots, 4$ are written in the Appendix. They are written in terms of functions which have been determined in earlier stages of the iterative procedure. When $n = 0$, they also involve the constant β_m, which must be found. The functions vanish when $m = 0$ or when $m > 0$ and $|n| > m$.

In a similar fashion, the perturbation procedure yields a system of boundary conditions.

Free surface:

$$\sum_{\ell=1}^{3}\left[\hat{c}_{11}\frac{d^2}{dr^2}+\hat{c}_{12}\left(\frac{d}{dr}-s^2\right)+\beta_0\hat{c}_{13}\mu_\ell+\beta_0\tau\hat{e}_{z1}\eta_\ell\right]\psi_\ell^{m,n} - 2\hat{c}_{66}s\left(\frac{d}{dr}-1\right)\psi_4^{m,n} = K_1^{m,n} \text{ at } r = 1,$$

$$\hat{c}_{66}\left[\sum_{\ell=1}^{3} 2s\left(\frac{d}{dr}-1\right)\psi_\ell^{m,n} - \left(\frac{d^2}{dr^2}-\frac{d}{dr}+s^2\right)\psi_4^{m,n}\right] = K_2^{m,n} \text{ at } r = 1,$$

$$\sum_{\ell=1}^{3}\left[\hat{c}_{44}(\mu_\ell-\beta_0)+\eta_\ell\tau\hat{e}_{x5}\right]\frac{d}{dr}\psi_\ell^{m,n}+\beta_0\hat{c}_{44}s\psi_4^{m,n} = K_3^{m,n} \text{ at } r = 1; \tag{30}$$

Clamped surface:

$$\sum_{\ell=1}^{3}\frac{d}{dr}\psi_\ell^{m,n} - s\psi_4^{m,n} = 0 \text{ at } r = 1,$$

$$s\sum_{\ell=1}^{3}\psi_\ell^{m,n} - \frac{d\psi_4^{m,n}}{dr} = 0 \text{ at } r = 1,$$

$$\sum_{\ell=1}^{3}\mu_\ell\psi_\ell^{m,n} = 0 \text{ at } r = 1. \tag{31}$$

Short-circuit:

$$\sum_{\ell=1}^{3}\eta_\ell\psi_\ell^{m,n} = 0 \text{ at } r = 1, \tag{32}$$

Open-circuit:

$$\sum_{\ell=1}^{3}\left[-\hat{\epsilon}_{xx}\eta_\ell+\hat{e}_{x5}(\mu_\ell-\beta_0)\right]\frac{d\psi_\ell^{m,n}}{dr}+\beta_0\hat{e}_{x5}s\psi_4^{m,n} = K_4^{m,n} \text{ at } r = 1. \tag{33}$$

The constants $K_j^{m,n}$, $j = 1, \ldots, 4$ are written in the Appendix. Like the $F_j^{m,n}(r)$, they vanish when $m = 0$ and are known when $m > 0$; when $n = 0$, they also involve β_m.

IV. SOLUTION OF THE DIFFERENTIAL EQUATIONS

We now show how to decouple the differential equations (25) to (28) and solve them. First, let

$$H_1^{m,n}(r) = \sum_{\ell=1}^{3} \{\hat{c}_{11}\nabla_s^2\psi_\ell^{m,n} + [(\omega^2 - \beta_0^2\hat{c}_{44}) + \beta_0\mu_\ell(\hat{c}_{13} + \hat{c}_{44}) + \beta_0\eta_\ell\tau(\hat{e}_{x5} + \hat{e}_{z1})]\psi_\ell^{m,n}\}, \quad (34)$$

$$H_2^{m,n}(r) = \hat{c}_{66}\nabla_s^2\psi_4^{m,n} + (\omega^2 - \beta_0^2\hat{c}_{44})\psi_4^{m,n}. \quad (35)$$

Then by using (25), (26), and procedures similar to those exhibited in Ref. 8, we can show that, except in a certain special case to be discussed later,

$$H_1^{m,n}(r) = \tfrac{1}{2}r^s \int_0^r x^{-s}[F_1^{m,n}(x) + F_2^{m,n}(x)]\,dx + \tfrac{1}{2}r^{-s} \int_0^r x^s[F_1^{m,n}(x) - F_2^{m,n}(x)]\,dx, \quad (36)$$

$$H_2^{m,n}(r) = \tfrac{1}{2}r^s \int_0^r x^{-s}[F_1^{m,n}(x) + F_2^{m,n}(x)]\,dx - \tfrac{1}{2}r^{-s} \int_0^r x^s[F_1^{m,n}(x) - F_2^{m,n}(x)]\,dx, \quad (37)$$

Now consider eqs. (27), (28), and (34). They are equivalent to the three decoupled equations

$$\nabla_s^2\psi_\ell^{m,n} + p_\ell^2\psi_\ell^{m,n} = f_\ell^{m,n}, \; \ell = 1, 2, 3, \quad (38)$$

provided that

$$p_\ell^2 = [(\omega^2 - \beta_0^2\hat{c}_{44}) + \beta_0\mu_\ell(\hat{c}_{13} + \hat{c}_{44}) + \beta_0\eta_\ell\tau(\hat{e}_{x5} + \hat{e}_{z1})]/\hat{c}_{11}, \quad (39)$$

from (34),

$$-p_\ell^2\,[\mu_\ell\hat{c}_{44} + \eta_\ell\tau\hat{e}_{x5} - \beta_0(\hat{c}_{13} + \hat{c}_{44})] + [(\omega^2 - \beta_0^2\hat{c}_{33})\mu_\ell - \beta_0^2\hat{e}_{z3}\tau\eta_\ell] = 0, \quad (40)$$

from (27), and, from (28),

$$-p_\ell^2\,[\hat{\epsilon}_{xx}\eta_\ell + \beta_0(\hat{e}_{x5} + \hat{e}_{z1}) - \hat{e}_{x5}\mu_\ell] + [-\beta_0^2\hat{\epsilon}_{zz}\eta_\ell + \beta_0^2\hat{e}_{z3}\mu_\ell] = 0. \quad (41)$$

These imply that the p_ℓ^2 satisfy the cubic equation

$$(\hat{\epsilon}_{xx}p_\ell^2 + \beta_0^2\hat{\epsilon}_{zz})[(\hat{c}_{11}p_\ell^2 + \beta_0^2\hat{c}_{44} - \omega^2)(\hat{c}_{44}p_\ell^2 + \beta_0^2\hat{c}_{33} - \omega^2) - p_\ell^2\beta_0^2(\hat{c}_{13} + \hat{c}_{44})^2] + \tau(\hat{e}_{x5}p_\ell^2 + \beta_0^2\hat{e}_{z3})[(\hat{c}_{11}p_\ell^2 + \beta_0^2\hat{c}_{44} - \omega^2) \times (\hat{e}_{x5}p_\ell^2 + \beta_0^2\hat{e}_{z3}) - 2p_\ell^2\beta_0^2(\hat{c}_{13} + \hat{c}_{44})(\hat{e}_{x5} + \hat{e}_{z1})] + \tau p_\ell^2\beta_0^2(\hat{e}_{x5} + \hat{e}_{z1})^2(\hat{c}_{44}p_\ell^2 + \beta_0^2\hat{c}_{33} - \omega^2) = 0, \quad (42)$$

and that

$$\mu_\ell = \beta_0 \{(\hat{c}_{11}p_\ell^2 + \beta_0^2\hat{c}_{44} - \omega^2)(\hat{e}_{z3}\hat{\epsilon}_{xx} - \hat{e}_{x5}\hat{\epsilon}_{zz}) - p_\ell^2(\hat{e}_{x5} + \hat{e}_{z1})[\tau\hat{e}_{x5}(\hat{e}_{x5} + \hat{e}_{z1}) + \hat{\epsilon}_{xx}(\hat{c}_{13} + \hat{c}_{44})]\} \times \{\beta_0^2(\hat{c}_{13} + \hat{c}_{44})(\hat{e}_{z3}\hat{\epsilon}_{xx} - \hat{e}_{x5}\hat{\epsilon}_{zz}) - \tau\hat{e}_{x5}(\hat{e}_{x5} + \hat{e}_{z1})(\hat{e}_{x5}p_\ell^2 + \beta_0^2\hat{e}_{z3}) - \hat{\epsilon}_{xx}(\hat{e}_{x5} + \hat{e}_{z1})(\hat{c}_{44}p_\ell^2 + \beta_0^2 c_{33} - \omega^2)\}^{-1} \quad (43)$$

and

$$\eta_\ell = [\hat{c}_{11}p_\ell^2 + \beta_0^2\hat{c}_{44} - \omega^2 - \mu_\ell\beta_0(\hat{c}_{13} + \hat{c}_{44})]/[\beta_0\tau(\hat{e}_{x5} + \hat{e}_{z1})]. \quad (44)$$

Furthermore, by eqs. (34), (27), and (28), the $f_\ell^{m,n}(r)$, $\ell = 1, 2, 3$, must satisfy

$$\hat{c}_{11} \sum_{\ell=1}^{3} f_\ell^{m,n} = H_1^{m,n}, \quad (45)$$

$$\sum_{\ell=1}^{3} [\mu_\ell\hat{c}_{44} + \eta_\ell\tau\hat{e}_{x5} - \beta_0(\hat{c}_{13} + \hat{c}_{44})]f_\ell^{m,n} = F_3^{m,n}, \quad (46)$$

$$\sum_{\ell=1}^{3} [\hat{\epsilon}_{xx}\eta_\ell + \beta_0(\hat{e}_{x5} + \hat{e}_{z1}) - \hat{e}_{x5}\mu_\ell]f_\ell^{m,n} = F_4^{m,n}. \quad (47)$$

These equations can be solved for the $f_\ell^{m,n}(r)$. By (35), eq. (38) also holds when $\ell = 4$, provided that

$$p_4^2 = (\omega^2 - \beta_0^2\hat{c}_{44})/\hat{c}_{66}, \quad (48)$$

$$f_4^{m,n} = H_2^{m,n}/\hat{c}_{66}. \quad (49)$$

The next step is to solve the uncoupled differential equations (38). The functions $f_j^{m,n}(r)$, $j = 1, \ldots, 4$, are either determined completely ($n \neq 0$) or else involve β_m in a known way ($n = 0$). Using the fact that $\psi_j^{m,n}$ is bounded at $r = 0$ to evaluate an integration constant, we have as a solution to (38)

$$\psi_j^{m,n}(r) = \left[A_j^{m,n} + \frac{\pi}{2}\int_r^1 xY_s(p_jx)f_j^{m,n}(x)\,dx\right]J_s(p_jr) + \frac{\pi}{2}\int_0^r xJ_s(p_jx)f_j^{m,n}(x)\,dx\;Y_s(p_jr) \text{ if } p_j^2 > 0,$$

$$\psi_j^{m,n}(r) = \left[A_j^{m,n} - \int_r^1 xK_s(q_jx)f_j^{m,n}(x)\,dx\right]I_s(q_jr) - \int_0^r xI_s(q_jx)f_j^{m,n}(x)\,dx\;K_s(q_jr) \text{ if } p_j^2 \equiv -q_j^2 < 0. \quad (50)$$

For any values of m and n, there are four constants $A_j^{m,n}$, $j = 1, \ldots, 4$, which remain to be evaluated. When $n = 0$, β_m must also be found. In the next section, we will show how to apply the boundary conditions to evaluate these constants.

It appears from (36), (37), (50), and (61) that a double integration must be performed computationally to obtain $H_1^{m,n}(r)$ and $H_2^{m,n}(r)$. Use of (38) and integration by parts, however, can reduce this to a single integration.

There is one special case for which the above analysis is not quite correct. By using arguments similar to those in Ref. 8, we can show that if $p_\ell^2 = 0$ for some ℓ, then

$$H_1^{m,n}(r) = C^{m,n}r^s + \tfrac{1}{2}r^s \int_1^r x^{-s}[F_1^{m,n}(x) + F_2^{m,n}(x)]\,dx + \tfrac{1}{2}r^{-s} \int_0^r x^s[F_1^{m,n}(x) - F_2^{m,n}(x)]\,dx,$$

$$H_2^{m,n}(r) = C^{m,n}r^s + \tfrac{1}{2}r^s \int_1^r x^{-s}[F_1^{m,n}(x) + F_2^{m,n}(x)]\,dx - \tfrac{1}{2}r^{-s} \int_0^r x^s[F_1^{m,n}(x) - F_2^{m,n}(x)]\,dx, \quad (51)$$

if $n \neq 0$. Here $C^{m,n}$ is a constant which remains to be determined. (When every p_ℓ^2 is nonzero, $C^{m,n}$ is arbitrary in the sense that changing it merely changes the constant by which the entire solution is multiplied.) Also, in this special case we have when $n = 0$,

$$H_1^{m,0}(r) = C_1^{m,n} - \int_r^1 F_1^{m,0}(x)\,dx,$$

$$H_2^{m,0}(r) = C_2^{m,n} - \int_r^1 F_2^{m,0}(x)\,dx, \quad (52)$$

where $C_1^{m,n}$ and $C_2^{m,n}$ must be determined. Now it can be shown that if the fiber is vibrating in the lowest-order torsional mode (with $v^{0,0}$ proportional to r and $u^{0,0} = w^{0,0} = \Phi^{0,0} = N = 0$), then $p_1^2 = p_4^2 = 0$. For this case, the solutions of (38) for $j = 1$ and 4 are

$$\psi_j^{m,n}(r) = \left[A_j^{m,n} - \frac{1}{2s}\int_r^1 x^{-s+1} f_j^{m,n}(x)\,dx\right] r^s - \frac{1}{2s}\int_0^r x^{s+1} f_j^{m,n}(x)\,dx\,r^{-s} \text{ if } n \neq 0, \quad (53)$$

$$\psi_j^{m,0}(r) = \int_r^1 x \ln x f_j^{m,0}(x)\,dx + \int_0^r x f_j^{m,0}(x)\,dx \ln r \text{ if } n = 0. \quad (54)$$

In eq. (54), an integration constant has been set to zero because it does not affect the final solution. It follows from eqs. (23), (43), (44), and (48), that when $p_1^2 = p_4^2 = 0$ and $n \neq 0$, the constants $A_1^{m,n}$ and $A_4^{m,n}$ appear in the displacements and electric potential only in the combination $A_1^{m,n} - A_4^{m,n}$. Thus for $n \neq 0$, the constants to be evaluated are $A_1^{m,n} - A_4^{m,n}$,

$A_2^{m,n}$, $A_3^{m,n}$, and $C^{m,n}$. When $n = 0$, we must find $A_2^{m,0}$, $A_3^{m,0}$, $C_1^{m,0}$, $C_2^{m,0}$, and β_m. In the next section, we show how to use the boundary conditions to determine these constants.

V. EVALUATION OF THE BOUNDARY CONDITIONS

For any pair (m, n), there are four boundary conditions, three of which are either eqs. (30) or (31), and the fourth of which is either (32) or (33); there are also four unknown constants to be found. When $n = 0$, β_m must be determined, too.

When the solutions $\beta_j^{m,n}$ to the differential equations (38) are substituted into the appropriate boundary conditions from eqs. (30) to (33), a system of equations results which can be written in matrix form as

$$\mathbf{J}^n \mathbf{A}^{m,n} = \mathbf{V}^{m,n}. \tag{55}$$

We will not write down here the specific components of these matrices and vectors, although it is straightforward to do so. The important things to know are the following: The 4×4 matrix $\mathbf{J}^n$ involves Bessel functions. It depends upon β_0, but is known once this is determined. The vector $\mathbf{A}^{m,n}$ consists of the four unknown constants to be determined. The vector $\mathbf{V}^{m,n}$ contains known constants: $K_j^{m,n}$, Bessel functions, integrals involving $f_j^{m,n}$. When $n = 0$, it also contains β_m linearly.

Incidentally, from a computational viewpoint, it is never necessary to differentiate the functions $\psi_j^{m,n}$ numerically, either for substitution into the boundary conditions or into the functions listed in the Appendix. Equation (38) can be used to eliminate all second derivatives of $\psi_j^{m,n}$ with respect to r. Differentiation with respect to r of the solutions (50) and the use of standard relations between Bessel functions and their derivatives result in analytical expressions for $d\psi_j^{m,n}/dr$.

The procedure for solving the differential equations and applying the boundary conditions is an iterative one. We start with $m = 0$. We choose a modal solution when $n = 0$ and set all $\psi_j^{0,n}$ to zero when $n \neq 0$. The $\psi_j^{0,0}$ satisfy eq. (38) with $f_j^{0,0} = 0$. The boundary conditions for this case are

$$\mathbf{J}^0 \mathbf{A}^{0,0} = 0. \tag{56}$$

From this we obtain for a nontrivial solution the frequency equation

$$\det \mathbf{J}^0 = 0, \tag{57}$$

which determines β_0 as a function of ω. This, of course, is the same as the dispersion relation for the hexagonal 6mm case about which we are perturbing.

As we iterate on m, we can see from the equations in the Appendix that $F_j^{m,n} = K_j^{m,n} = 0$ whenever $|n| > m$. It follows that $\mathbf{V}^{m,n} = 0$ in this case and, since $\det \mathbf{J}^n \neq 0$, that $\mathbf{A}^{m,n} = 0$. Thus all $\psi_j^{m,n} = 0$ whenever $|n| > m$.

If $0 < |n| \leq m$, $\mathbf{V}^{m,n}$ is in general nonzero. Since $\det \mathbf{J}^n \neq 0$, we can immediately obtain $\mathbf{A}^{m,n}$ from eq. (55).

If $n = 0$, the analysis is slightly more complicated. It was explained in detail in Ref. 8, so we merely give the results here. To obtain β_m, replace any column of $\mathbf{J}^0$ by $\mathbf{V}^{m,n}$ and set the determinant of the resulting matrix to zero. The unknown vector $\mathbf{A}^{m,0}$ can be written as

$$\mathbf{A}^{m,0} = C_m \mathbf{A}^{0,0} + \mathbf{D}^{m,0}, \tag{58}$$

where $\mathbf{D}^{m,0}$ has *three* unknown components and $D_j^{m,0} = 0$ for some j for which $A_j^{0,0} \neq 0$. Then the equation

$$\mathbf{J}^0 \mathbf{D}^{m,0} = 0 \tag{59}$$

can be solved for $\mathbf{D}^{m,0}$. Furthermore, C_m is arbitrary in the sense that varying it varies the constant by which the full solution is multiplied. We set $C_m = 0$.

In this manner, the functions $\psi_j^{m,n}$ can be determined iteratively, starting with $m = 0$. For any given value of m, nontrivial results are obtained only when $|n| \leq m$. The perturbation contributions to the elastic displacements and electric potential are then found from eq. (23). The full solution is given by eqs. (21) and (22).

APPENDIX

Let

$$\begin{aligned} s &= 3n + N, \\ s_+ &= 3(n+1) + N, \\ s_- &= 3(n-1) + N, \end{aligned} \tag{60}$$

where n and N are integers. Then

$$\begin{aligned} F_1^{m,n}(r) + F_2^{m,n}(r) = \left(\frac{d}{dr} - \frac{s}{r}\right) \sum_{j=0}^{m-1} \Bigg\{ & \gamma_{m-j} \hat{c}_{44} \left[\sum_{\ell=1}^{3} \psi_\ell^{j,n} + \psi_4^{j,n} \right] \\ & - \beta_{m-j} \sum_{\ell=1}^{3} \left[(\hat{c}_{13} + \hat{c}_{44})\mu_\ell + \tau(\hat{e}_{x5} + \hat{e}_{z1})\eta_\ell \right] \psi_\ell^{j} \\ - \left[\nabla_{s+}^2 - \frac{2(1-s_+)}{r} \left(\frac{d}{dr} + \frac{s_+}{r}\right) \right] \Bigg\{ & \sum_{j=0}^{m-1} 2\beta_{m-j-1} \left[\sum_{\ell=1}^{3} \psi_\ell^{j,n+1} - \psi_4^{j,n+1} \right] \\ & + \sum_{\ell=1}^{3} (\tau \xi \eta_\ell - \mu_\ell) \psi_\ell^{m-1,n+1} \Bigg\}, \end{aligned}$$

$$F_1^{m,n}(r) - F_2^{m,n}(r) = \left(\frac{d}{dr} + \frac{s}{r}\right) \sum_{j=0}^{m-1} \left\{ \gamma_{m-j} \hat{c}_{44} \left[\sum_{\ell=1}^{3} \psi_\ell^{j,n} - \psi_4^{j,n} \right] \right.$$

$$- \beta_{m-j} \sum_{\ell=1}^{3} \left[(\hat{c}_{13} + \hat{c}_{44}) \mu_\ell + \tau (\hat{e}_{x5} + \hat{e}_{z1}) \eta_\ell \right] \psi_\ell^j$$

$$+ \left[\nabla_{s-}^2 - \frac{2(1+s_-)}{r} \left(\frac{d}{dr} - \frac{s_-}{r} \right) \right] \left\{ \sum_{j=0}^{m-1} 2\beta_{m-j-1} \left[\sum_{\ell=1}^{3} \psi_\ell^{j,n-1} + \psi_4^{j,n-1} \right] \right.$$

$$\left. + \sum_{\ell=1}^{3} (\tau \xi \eta_\ell - \mu_\ell) \psi_\ell^{m-1,n-1} \right\},$$

$$F_3^{m,n}(r) = \frac{1}{2} \left\{ \left(\frac{d}{dr} - \frac{s_-}{r} \right) \nabla_{s-}^2 - \frac{2(2+s_-)}{r} \right.$$

$$\left. \times \left[\nabla_{s-}^2 - \frac{2(1+s_-)}{r} \left(\frac{d}{dr} - \frac{s_-}{r} \right) \right] \right\} \left[\sum_{\ell=1}^{3} \psi_\ell^{m-1,n-1} + \psi_4^{m-1,n-1} \right]$$

$$- \frac{1}{2} \left\{ \left(\frac{d}{dr} + \frac{s_+}{r} \right) \nabla_{s+}^2 - \frac{2(2-s_+)}{r} \left[\nabla_{s+}^2 - \frac{2(1-s_+)}{r} \left(\frac{d}{dr} + \frac{s_+}{r} \right) \right] \right\}$$

$$\times \left[\sum_{\ell=1}^{3} \psi_\ell^{m-1,n+1} - \psi_4^{m-1,n+1} \right] + \sum_{j=0}^{m-1} \left[\beta_{m-j} (\hat{c}_{13} + \hat{c}_{44}) \nabla_s^2 \sum_{\ell=1}^{3} \psi_\ell^{j,n} \right.$$

$$\left. + \gamma_{m-j} \sum_{\ell=1}^{3} (\hat{c}_{33} \mu_\ell + \tau \hat{e}_{z3} \eta_\ell) \psi_\ell^{j,n} \right],$$

$$F_4^{m,n}(r) = -\tfrac{1}{2}\xi \left\{ \left(\frac{d}{dr} - \frac{s_-}{r} \right) \nabla_{s-}^2 \frac{2(2+s_-)}{r} \right.$$

$$\left. \times \left[\nabla_{s-}^2 - \frac{2(1+s_-)}{r} \left(\frac{d}{dr} - \frac{s_-}{r} \right) \right] \right\} \left[\sum_{\ell=1}^{3} \psi_\ell^{m-1,n-1} + \psi_4^{m-1,n-1} \right]$$

$$+ \tfrac{1}{2}\xi \left\{ \left(\frac{d}{dr} + \frac{s_+}{r} \right) \nabla_{s+}^2 - \frac{2(2-s_+)}{r} \left[\nabla_{s+}^2 - \frac{2(1-s_+)}{r} \left(\frac{d}{dr} + \frac{s_+}{r} \right) \right] \right\}$$

$$\times \left[\sum_{\ell=1}^{3} \psi_\ell^{m-1,n+1} - \psi_4^{m-1,n+1} \right] + \sum_{j=0}^{m-1} \left[\beta_{m-j} (\hat{e}_{x5} + \hat{e}_{z1}) \nabla_s^2 \sum_{\ell=1}^{3} \psi_\ell^{j,n} \right.$$

$$\left. + \gamma_{m-j} \sum_{\ell=1}^{3} (\hat{e}_{z3} \mu_\ell - \hat{\epsilon}_{zz} \eta_\ell) \psi_\ell^{j,n} \right] \qquad (61)$$

where

$$\gamma_m = \sum_{j=0}^{m} \beta_j \beta_{m-j}. \qquad (62)$$

$$K_1^{m,n} = \frac{1}{2} \sum_{j=0}^{m-1} \beta_{m-1-j} \left\{ \left(\frac{d}{dr} - s_- \right) \left[\sum_{\ell=1}^{3} \psi_\ell^{j,n-1} + \psi_4^{j,n-1} \right] \right.$$

$$\left. - \left(\frac{d}{dr} + s_+ \right) \left[\sum_{\ell=1}^{3} \psi_\ell^{j,n+1} - \psi_4^{j,n+1} \right] \right\}$$

$$-\frac{1}{2}\sum_{\ell=1}^{3}(\mu_\ell-\tau\xi\eta_\ell)\left[\left(\frac{d}{dr}-s_-\right)\psi_\ell^{m-1,n-1}-\left(\frac{d}{dr}+s_+\right)\psi_\ell^{m-1,n+1}\right]$$

$$-\sum_{j=0}^{m-1}\beta_{m-j}\sum_{\ell=1}^{3}(\hat{c}_{13}\mu_\ell+\hat{e}_{z1}\tau\eta_\ell)\psi_\ell^{j,n}\text{ at }r=1,$$

$$K_2^{m,n}=\frac{1}{2}\sum_{j=0}^{m-1}\beta_{m-1-j}\left\{\left(\frac{d}{dr}-s_-\right)\left[\sum_{\ell=1}^{3}\psi_\ell^{j,n-1}+\psi_4^{j,n-1}\right]\right.$$

$$\left.+\left(\frac{d}{dr}+s_+\right)\left[\sum_{\ell=1}^{3}\psi_\ell^{j,n+1}-\psi_4^{j,n+1}\right]\right\}$$

$$-\frac{1}{2}\sum_{\ell=1}^{3}(\mu_\ell-\tau\xi\eta_\ell)\left[\left(\frac{d}{dr}-s_-\right)\psi_\ell^{m-1,n-1}\right.$$

$$\left.+\left(\frac{d}{dr}+s_+\right)\psi_\ell^{m-1,n+1}\right]\text{ at }r=1,$$

$$K_3^{m,n}=\hat{c}_{44}\sum_{j=0}^{m-1}\beta_{m-j}\left\{\sum_{\ell=1}^{3}\left[s\psi_\ell^{j,n}+\left(\frac{d}{dr}-s\right)\psi_\ell^{j,n}\right]-s\psi_4^{j,n}\right\}$$

$$+\frac{1}{2}\left[\nabla_{s-}^2-2(1+s_-)\left(\frac{d}{dr}-s_-\right)\right]\left[\sum_{\ell=1}^{3}\psi_\ell^{m-1,n-1}+\psi_4^{m-1,n-1}\right]$$

$$-\frac{1}{2}\left[\nabla_{s+}^2-2(1-s_+)\left(\frac{d}{dr}+s_+\right)\right]\left[\sum_{\ell=1}^{3}\psi_\ell^{m-1,n+1}-\psi_4^{m-1,n+1}\right]\text{ at }r=1,$$

$$K_4^{m,n}=\hat{e}_{x5}\sum_{j=0}^{m-1}\beta_{m-j}\left\{\sum_{\ell=1}^{3}\left[s\psi_\ell^{j,n}+\left(\frac{d}{dr}-s\right)\psi_\ell^{j,n}\right]-s\psi_4^{j,n}\right\}$$

$$+\tfrac{1}{2}\xi\left[\nabla_{s-}^2-2(1+s_-)\left(\frac{d}{dr}-s_-\right)\right]\left[\sum_{\ell=1}^{3}\psi_\ell^{m-1,n-1}+\psi_4^{m-1,n-1}\right]$$

$$-\tfrac{1}{2}\xi\left[\nabla_{s+}^2-2(1-s_+)\left(\frac{d}{dr}+s_+\right)\right]\left[\sum_{\ell=1}^{3}\psi_\ell^{m-1,n+1}-\psi_4^{m-1,n+1}\right]$$

$$\text{at }r=1.\qquad(63)$$

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