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The Theory of Uniform Cables—Part I: Calculation of Propagation Parameters

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An algorithm for computing the electromagnetic propagation modes and their associated propagation constants (i.e., the loss and phase shift per length) is rigorously developed for uniform cables. The conductors (including the shield, if present) are assumed to have circular cross sections and to be covered by two layers of dielectric insulation. By means of the algorithm, it is now possible to compute the propagation parameters of uniform cables for all frequencies below the microwave range. The algorithm consists of calculating the eigenvectors and eigenvalues of a certain matrix, which are the set of conductor voltages of the modes and the associated propagation constants, respectively. The matrix in question is computed by methods developed in a companion paper on charge densities.

I. INTRODUCTION

Multipair cables consist of a collection of insulated wires surrounded by other dielectric materials, all of which are usually enclosed in a jacketed, metallic shield for mechanical protection and electrical isolation. The wires in a cable are generally helically twisted in pairs and, in practice, other, often undesired, nonuniformities occur along the cable. Nevertheless, the uniform cable, whose wires are straight and parallel, defined by the property that all its longitudinal cross sections are identical, has successfully modeled some aspects of electrical propagation (such as loss per length) over a pair in a cable.[1] Also, some cables are directly modeled as uniform cables; so there is a need for studying uniform cables. Moreover, further experience using the uniform-cable model may lead to a more exact model for nonuniform multipair cables.

A rigorous analysis of electromagnetic propagation over uniform cables, starting from Maxwell's equations, was performed by Carson[2] for cables having homogeneous dielectric material separating the wires.

Subsequently, Mead[3] applied his method to the shielded balanced pair to get a set of formulas for the primary constants (R, L, C, G).

Later, Kuznetsov[4] developed another procedure for calculating the propagation parameters of uniform cables. He too assumed the dielectric separating the wires was homogeneous; also, he did not consider the presence of a shield. His procedure was abstractly stated, but he did derive a concrete formula for the loss and phase-shift per length in the special case of a single pair in free space (see Ref. 4, Chap. 2). Both Kuznetsov and Carson assumed that the frequency of excitation was high enough that the associated skin depth was less than the radius of the wires.

In this paper, the electrical behavior of uniform cables is analyzed from first principles along the lines developed by Kuznetsov. The analysis pertains to all frequencies of excitation below the microwave range, it pertains to cables with or without a shield, and it provides for two layers of dielectric on the conductors* (including the shield). The final result, making use of results from the accompanying paper,[5] is an algorithm, suitable for use on a digital computer, that computes the various propagation parameters of the cable.

The analysis starts with the concept of a modal solution. For a sinusoidal excitation of frequency $\omega/2\pi$, a modal solution for the electric field has the form,

$$\tilde{E} = \exp(i\omega t - \gamma z)\, E,$$

where the Cartesian coordinate z runs along the cable, γ denotes a propagation constant of the cable (giving the loss and phase-shift per length of the mode), and the vector field E is independent of time (t) and z. Thus, the electric field distribution for a modal solution is the same for all transverse cross sections of the cable, apart from a multiplying factor that gives the loss and phase shift per length for the mode, and similarly for the magnetic field. The general solution is a linear combination of modal solutions.

For propagating solutions, in particular, Kuznetsov has shown that there is a potential function V(i.e., a solution of Laplace's equation) defined in the dielectric-region, independent of z, and constant on the surface of the conductors, such that external to the conductors

$$E = -\nabla V + i_z E_z,$$

where i_z denotes the unit vector in the z-direction and the function E_z denotes the z-component of E. (His argument for this electrostatics approximation is enlarged in the next section to account for the shield

* This permits the modeling of a new type of wire insulation recently introduced into cable manufacture: a layer of expanded insulation surrounded by a "skin" of solid insulation. The insulation on the shield is included for formal completeness.

and inhomogeneities in the dielectric). The various propagation modes of the cable, called here the excitation modes, are specified by the set of values for V on the conductors, which are simply the conductor voltages.

The set of wire voltages at $z = 0$ (the shield is always assumed to be grounded) is represented by the vector

$$\mathbf{v} = (v_1, \cdots, v_M),$$

where M is the number of wires in the cable. Certain particular voltage-vectors correspond to excitation modes of the cable. Usually, there are M such normalized voltage vectors, each corresponding to a distinct propagation constant. But it can happen that two or more linearly independent voltage vectors correspond to the same propagation constant; in this case, the propagation constant corresponds to the subspace spanned by these vectors, and the dimension of the subspace is called the multiplicity of the propagation constant. For either situation the collection of excitation modes and their associated propagation constants are the specific propagation parameters that are sought. The usual primary constants of a transmission line or mode, if desired, can be determined by standard formulas[6] from the propagation constant in conjunction with the capacitance matrix.

Under the idealized conditions that the conductors are perfectly conducting and the dielectric is homogeneous, there is a single propagation constant,

$$\gamma = ik_e = i\omega(\epsilon\mu)^{1/2}, \tag{1}$$

where ϵ and μ denote the permittivity and permeability of the dielectric, respectively. In such a case, the multiplicity of ik_e is M. This is the so-called TEM mode,[6] where the longitudinal components of the electric and magnetic fields are zero. Other modes are evanescent, since the frequency range considered here lies below their cutoff frequencies. Since the dielectric in the cable may be inhomogeneous and the conductors are good but not perfect, the propagation solutions or modes are perturbations of the TEM mode, the result being in general a set of excitation modes having distinct propagation constants.

In the next section, the algorithm for computing the excitation modes and associated propagation constants is derived. The modes correspond to the eigenvectors of an $M \times M$ matrix, the propagation constants are simple functions of the associated eigenvalues, and the matrix itself is related to the capacitance and admittance matrices of the cable. In the third section, the algorithm is applied to the shielded balanced pair and a high-frequency approximation is indicated. Experimental testing of these results is reported in Ref. 7.

II. PROPAGATION MODES FOR UNIFORM CABLES

In this section, an algorithm for computing the propagation modes and their associated propagation constants is derived for a uniform cable. The analysis pertains to cables having M straight and parallel wires of circular cross section, enclosed by a circular metallic shield.

The geometry, materials, and excitation of the structure to be considered are specified as follows:

(*i*) The radius of the mth conductor is denoted R_m for $m = 0, 1, \cdots, M$, where R_0 refers to the inside radius of the shield. The shield has a uniform thickness th.

(*ii*) The conductors have two layers of insulation with thickness $(R_{m1} - R_m)$ and $(R_{m2} - R_{m1})$ for the wires $(m = 1, \cdots, M)$ and $(R_0 - R_{01})$ and $(R_{01} - R_{02})$ for the shield. The permittivities (possibly complex) are ϵ_m and ϵ_{m1} for the first and second layers, respectively, for $m = 0, 1, \cdots, M$. In general, the subscript "0" will refer to the shield.

(*iii*) The materials are nonmagnetic, and the permeability is μ throughout.

(*iv*) The material in the interstitial space outside the conductors and their insulation has permittivity ϵ (possibly complex).

(*v*) The excitation frequency $\omega/2\pi$ satisfies $(\omega/2\pi)R_0(\epsilon\mu)^{1/2} \ll 1$, that is, the wavelength is much greater than the radius of the cable. Since R_0 generally exceeds 0.1 inch, this means that the frequency is below the microwave range.

(*vi*) The conductivity of a conductor is denoted σ_m and $\omega\epsilon \ll \sigma_m$ for $m = 0, 1, \cdots, M$.

(*vii*) When the cross-sectional plane of the cable is viewed as the complex plane, the centers of the conductors are associated with the complex numbers $b_0 = 0, b_1, \cdots, b_M$. A typical cross section is indicated in Fig. 1.

The electrical behavior of a cable is described by the electric and magnetic fields, $\tilde{E}$ and $\tilde{H}$, which satisfy Maxwell's equations. A modal solution, as discussed in Section I, has the form,

$$\tilde{E} = \exp(i\omega t - \gamma z)E$$
$$\tilde{H} = \exp(i\omega t - \gamma z)H, \tag{2}$$

where the vector fields E and H are independent of z and t. In terms of their longitudinal and transverse parts,

$$E = E^{tr} + i_z E_z$$
$$H = H^{tr} + i_z H_z, \tag{3}$$

where the vector fields E^{tr} and H^{tr} are the projections of E and H, respectively, onto the transverse plane. Maxwell's equations imply that

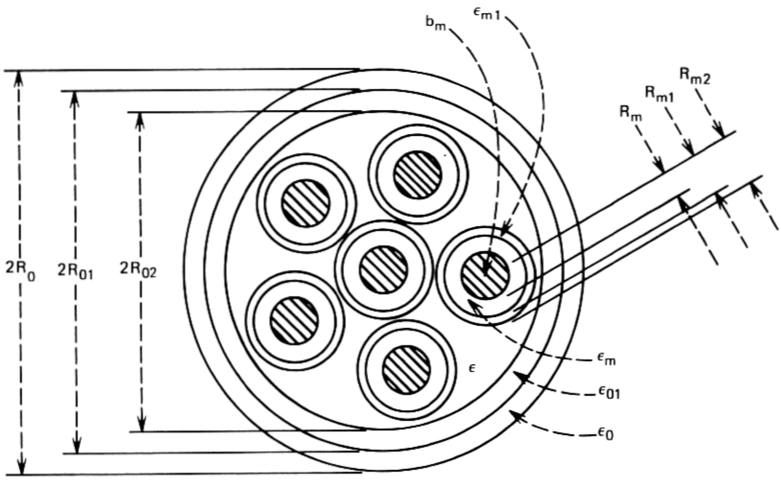

Fig. 1—Typical cable cross section.

(see Ref. 4, page 3)

$$
\begin{aligned}
-X^2E^{tr} &= \gamma\nabla E_z + i\omega\mu\, \nabla H_z \times i_z \\
-X^2H^{tr} &= \gamma\nabla H_z + (k^2/i\omega\mu)\, \nabla E_z \times i_z,
\end{aligned}
\tag{4}
$$

where taking $\tilde{\epsilon}$ as the permittivity as a function of position,

$$
k^2 = \begin{cases} k_m^2 \equiv -i\omega\mu\, \sigma_m & \text{in } m\text{th conductor } m = 0, 1, \cdots, M \\ k_e^2 \equiv \omega^2\mu\tilde{\epsilon} & \text{outside the conductors,} \end{cases}
\tag{5}
$$

where the subscript "e" refers to the region exterior to the conductors, and

$$
X^2 = k^2 + \gamma^2. \tag{6}
$$

The form of the solution in the space separating the conductors is determined by the electrostatics approximation. Kuznetsov[4] has validated this approximation in three steps:

(i) Both E_{ze} and H_{ze} and their derivatives are determined to better than one part in $2.4/k_eR_0$ as solutions of Laplace's equation.

(ii) There is a function V such that $E^{tr} = -\nabla V$ and $\nabla^2 V = 0$.

(iii) On each conductor the tangential part of E^{tr} is zero; so V is constant on the conductors.

For point (i), the original proof must be enlarged to deal with the presence of a shield and inhomogeneities in the dielectric; this is done in Appendix A. Point (ii) is a direct consequence of point (i) because in eq.

(4) both the divergence and curl of $\nabla H_z \times i_z$ are zero, which implies it is the gradient of a solution to Laplace's equation, and point (*iii*) comes from the estimate, ($k_e^2/k_m^2 \ll 1$ for all m) [see Ref. 4, eq. (2.3) and Appendix 1].

The potential which is one on the kth conductor and zero on the others is denoted V_k $k = 1, \cdots, M$. These form a basis set since for any potential V (which is zero on the shield) there are coefficients v_k $k = 1, \cdots, M$ such that

$$V = \sum_{k=1}^{M} v_k V_k. \tag{7}$$

Likewise, when $E_{ze}(k)$ denotes the longitudinal component that corresponds to the potential V_k $k = 1, \cdots, M$, then

$$E_{ze} = \sum_{k=1}^{M} v_k E_{ze}(k) \tag{8}$$

corresponds to the potential V. The coefficients v_k $k = 1, \cdots, M$ form the excitation vector

$$\mathbf{v} = (v_1, \cdots, v_M). \tag{9}$$

To find those excitation vectors that correspond to propagation modes and their associated propagation constants, a set of constraints is developed, one for each conductor, by relating V and E_{ze} at the conductor surfaces. At the outside surface of the mth wire (W_m), eq. (4) implies

$$-X_e^2 E_{\rho m e} = \gamma \frac{\partial E_{ze}}{\partial \rho m} + \frac{i\omega\mu}{R_m} \frac{\partial H_{ze}}{\partial \phi_m}, \quad m = 1, \ldots M, \tag{10}$$

where (ρ_m, ϕ_m) denote polar coordinates based at the center of the mth conductor. But $E_{\rho m e} = -\partial V/\partial \rho_m$, so

$$X_e^2 \int_0^{2\pi} \frac{\partial V}{\partial \rho_m} d\phi_m = \gamma \int_0^{2\pi} \frac{\partial E_{ze}}{\partial \rho_m} d\phi_m \quad m = 1, \cdots, M, \tag{11}$$

where the derivatives are evaluated on W_m. By eqs. (7) and (8), this is expressed as

$$X_e^2 \sum_{k=1}^{M} v_k \int_0^{2\pi} \frac{\partial V_k}{\partial \rho_m} d\phi_m = \gamma \sum_{k=1}^{M} v_k \int_0^{2\pi} \frac{\partial E_{ze}(k)}{\partial \rho_m} d\phi_m \quad m = 1, \cdots, M. \tag{12}$$

The mk-element of the $M \times M$ capacitance matrix C is

$$C_{mk} = \epsilon_m R_m \int_0^{2\pi} \frac{\partial V_k}{\partial \rho_m} d\phi_m, \tag{13}$$

and by eq. (4) for the transverse magnetic field, the mk-element of the $M \times M$ admittance matrix Y is

$$Y_{mk} = \oint_{W_m} H_{\phi m}(k)\, ds = \frac{i\omega\epsilon_m R_m}{X_e^2} \int_0^{2\pi} \frac{\partial E_{ze}(k)}{\partial \rho_m}\, d\phi_m. \tag{14}$$

Thus, the set of constraints in eq. (12) is jointly represented by the matrix equation,

$$i\omega\, C\mathbf{v} = \gamma Y \mathbf{v}, \tag{15}$$

where the excitation vector $\mathbf{v}$ is viewed here as a column vector.

But, as will be shown, the matrix Y has the form,

$$Y = \frac{i\omega\gamma}{X_e^2}\, \tilde{Y}, \tag{16}$$

where the matrix $\tilde{Y}$ is independent of γ. Consequently, eq. (15) can be expressed

$$C\mathbf{v} = (\gamma^2/X_e^2)\, \tilde{Y}\mathbf{v} \text{ or } C^{-1}\, \tilde{Y}\mathbf{v} = (X_e^2/\gamma^2)\, \mathbf{v}. \tag{17}$$

This means that to correspond to an excitation mode, the excitation vector $\mathbf{v}$ must be an eigenvector of the matrix $C^{-1}\tilde{Y}$, and conversely. If λ is the associated eigenvalue, then the corresponding propagation constant γ is

$$\gamma = ik_e(1 - \lambda)^{-1/2}. \tag{18}$$

The remaining problem is to calculate the matrices C and $\tilde{Y}$.

The elements of the capacitance matrix C can be computed by the method developed in Ref. 5 (see in particular Section III). This consists of inverting a matrix and is a standard operation on a digital computer. The elements of the $\tilde{Y}$-matrix are evaluated by deriving a boundary-value problem for the function E_{ze} (in the region separating the conductors), then again applying methods from Ref. 5 to solve it.

As the first step in determining $\tilde{Y}$, in Appendix B it is shown that E_{ze} satisfies the following conditions:

(*i*) $\nabla^2 E_{ze} = 0$, where ∇^2 denotes the Laplacian, $(\partial^2/\partial x^2) + (\partial^2/\partial V^2)$.

(*ii*) Across dielectric interfaces E_{ze} is continuous and $\delta(\partial E_{ze}/\partial \rho) = \gamma\delta(\partial V/\partial \rho)$, where $\partial/\partial\rho$ denotes the normal derivative and δ denotes the difference or jump.

(*iii*) $(\partial E_{ze}/\partial \rho_m) - D_\omega E_{ze} = \gamma(\partial V/\partial \rho_m)$ at $\rho_m = R_m$ for $m = 0, 1, \cdots, M$, where D_ω is a linear operator such that for $m = 0, 1, \cdots, M$

$$D_\omega\colon e^{in\phi_m} \to (X_{cm}/\alpha_{nm})\, e^{in\phi_m} \quad n = 0, \pm 1, \pm 2, \ldots, \tag{19}$$

and the coefficients α_{nm} (which depend on ω) are defined in eq. (58) for $m = 0$ and in eq. (52) for $m \neq 0$, both in Appendix B. In other words, in condition (iii), D_ω alters the Fourier series of E_{ze} on each conductor by multiplying the respective Fourier components.

This boundary problem is simplified by putting

$$E_{ze} = \gamma V + E'_{ze}. \tag{20}$$

Then the corresponding conditions on the function E'_{ze} are

(i) $\nabla^2 E'_{ze} = 0$.

(ii) Both E'_{ze} and its derivatives are continuous across dielectric interfaces.

(iii) $(\partial E'_{ze}/\partial \rho_m) - D_\omega E'_{ze} = \gamma D_\omega V$ at $\rho_m = R_m$ for $m = 0, 1, \cdots, M$.

The function E'_{ze} restricted to a conductor surface is represented by a Fourier series expansion and this by an infinite vector whose components are the Fourier coefficients. If, consistent with the notation in Ref. 5, $\mathbf{u}$ denotes the concatenation of these vectors for each conductor and $\mathbf{p}$ the concatenation of the vectors corresponding to $\epsilon_m(\partial E'_{ze}/\partial \rho_m)$ $m = 0, 1, \cdots, M$, then condition (iii) can be expressed by the matrix equation,

$$\mathbf{p} - \tilde{D}_\omega \mathbf{u} = \tilde{D}_\omega \, \mathbf{V}, \tag{21}$$

where $\tilde{D}_\omega$ denotes the diagonal matrix with elements $\epsilon_m X_m/\alpha_{nm}$ for $0 \leqq m \leqq M$ and $n = 0, 1, 2, \cdots$. Also, there is a vector β and matrices T_0, H_0, and G_0 (defined in Ref. 5) such that

$$\mathbf{u} = T_0\beta \text{ and } \mathbf{p} = G_0\beta - H_0\mathbf{u}, \tag{22}$$

where the subscript "0" emphasizes the context of homogeneous dielectric, in accordance with condition (ii) for E'_{ze}. It follows that

$$(G_0 - H_0T_0 - \tilde{D}_\omega \, T_0)\,\beta = \gamma \tilde{D}_\omega \mathbf{V} \tag{23}$$

or

$$\beta = -\gamma[(H_0 + \tilde{D}_\omega)T_0 - G_0]^{-1}\,\tilde{D}_\omega \mathbf{V}. \tag{24}$$

But it follows from eq. (14) and eq. (16) that for $1 \leqq m, k \leqq M$,

$$\tilde{Y}_{mk} - C_{mk} = \frac{\epsilon_m R_m}{\gamma}\int_0^{2\pi} \frac{\partial E'_{ze}(k)}{\partial \rho_m}\, d\phi_m = 2\pi R_m P_{0k}/\gamma = 2\pi\epsilon\beta_{0k}/\gamma, \tag{25}$$

where the latter equality is shown in Ref. 5, eq. (9). Therefore, the mk element of the $\tilde{Y}$-matrix is computed by carrying out the matrix inversion in eq. (24) for $V = V_m$ to give β_{0k}. Of course, in practice, this inversion is performed on a truncated approximation of the matrix $[(H_0 + \tilde{D}_\omega)T_0 - G_0]$.

III. APPLICATION AND EXTENSIONS

In this section, the general results of the last section are applied to the so-called shielded balanced pair. Also, extensions of the results to cables without a shield and cables with circular holes in the interstitial dielectric are discussed.

The shielded balanced pair (SBP) consists of a pair of identical wires placed symmetrically inside a circular shield (the center of the shield lies on the line joining the centers of the wires and midway between them). Since the structure is symmetric about the perpendicular bisector of the line joining the wire centers, there will be a balanced mode: the potential and fields will be antisymmetric about this line and, in particular, the wire-voltages will be equal and opposite in sign. The problem is to determine the propagation constant (γ) of the SBP. For convenience, the dielectric will be assumed to be homogeneous with permittivity ϵ.

The excitation vector $\mathbf{v} = (1, -1)$ is an eigenvector for the matrix $C^{-1}\tilde{Y}$ and the corresponding eigenvalue is

$$\lambda = (\tilde{Y}_{11} - \tilde{Y}_{12})/(C_{11} - C_{12}). \tag{26}$$

Equation (18) gives γ in terms of λ. When $V_{12} \equiv V_1 - V_2$,

$$C_{11} - C_{12} = \epsilon \oint_{W_1} \frac{\partial V_{12}}{\partial \rho_1} ds \equiv Q_1, \tag{27}$$

which is the total charge on wire no. 1 (W_1), and

$$\tilde{Y}_{11} - \tilde{Y}_{12} = (\epsilon/\gamma) \oint_{W_1} \frac{\partial E_{ze}}{\partial \rho_1} ds. \tag{28}$$

These two quantities are evaluated as in section II (relative to the potential function V_{12}): Q_1 by methods from Ref. 5, and $\tilde{Y}_{11} - \tilde{Y}_{12}$ by eqs. (24) and (25).

A suggestive formula for γ is obtained by applying Green's identity to eq. (28). Since both V_{12} and E_{ze} are antisymmetric functions, and since $V_{12} = 0$ on the surface of the shield (Sh),

$$\oint_{W_1} \frac{\partial E_{ze}}{\partial \rho_1} ds = \frac{1}{2} \oint_{\Gamma} \frac{\partial E_{ze}}{\partial \eta} V_{12}\, ds, \tag{29}$$

where Γ denotes the entire boundary of the dielectric (W_1, W_2, Sh) and $\partial/\partial\eta$ denotes the normal derivative on Γ (into the dielectric). Since both E_{ze} and V_{12} satisfy Laplace's equation, Green's identity (see Ref. 8, Vol. 2, page 252) implies that the derivative in the integrand can be switched to give

$$\epsilon \oint_{W_1} \frac{\partial E_{ze}}{\partial \rho_1} ds = \frac{\epsilon}{2} \oint_{\Gamma} \frac{\partial V_{12}}{\partial \eta} E_{ze}\, ds = \oint_{W_1} q_1 E_{ze}\, ds + \frac{1}{2} \oint_{Sh} q_0 E_{ze}\, ds, \tag{30}$$

where q_1 and q_0 denote the charge densities on W_1 and Sh, respectively. Thus,

$$\gamma = ik_e\left[1 - (Q_1)^{-1} \oint_{W_1} q_1(E_{ze}/\gamma)ds - (2Q_1)^{-1} \oint_{Sh} q_0(E_{ze}/\gamma)ds\right]^{-1/2}; \quad (31)$$

the Fourier components of $(E_{ze}/\gamma) - V_{12}$ on W, and Sh are $T_0\beta$ with β given by eq. (24). This formula for γ indicates the contribution from each conductor. A comparison between measurement and theory for the SBP based on eq. (31) is presented in Ref. 7.

For high frequencies (i.e., ω for which $\omega\mu\sigma R_1 \gg 1$), the formula for E_{ze} can be simplified by neglecting the first term of condition (*iii*) (see Ref. 4, eq 5.8, where X_e should be X_i). This gives

$$E_{ze} = -(\gamma/\epsilon)D_\omega^{-1}\, q_m \quad (m = 0, 1, 2). \quad (32)$$

This approximation together with the approximation

$$(1 - \delta)^{-1/2} \cong 1 + \tfrac{1}{2}\delta$$

gives

$$\gamma = ik_e\left\{1 + (2Q_1)^{-1}\left[\oint_{W_1} q_1(D_\omega^{-1}\, q_1)ds + \tfrac{1}{2} \oint_{Sh} q_0(D_\omega^{-1}\, q_0)ds\right]\right\}. \quad (33)$$

The integrals in (31) and (33) can be expressed in terms of the Fourier coefficients of the charge density for convenience in calculating these expressions.

The development in Section II can be extended to cables without a shield and to cables with circular holes in the interstitial dielectric. The calculation of the charge components, and hence, the capacitance and admittance matrices, in both of these situations was treated in Ref. 5, Section III. Again, eq. (17) must be solved for eigenvectors and eigenvalues; but in the absence of a shield, the capacitance matrix is not invertible and the problem is what Kato[9] calls an eigen-problem in the generalized sense. In principle, these can be solved on a digital computer as before.

The results presented here in conjunction with the computation techniques of the companion paper on charge densities allow computation of the excitation modes and their associated propagation constants for uniform cables under a broad range of conditions. These include capacitive and resistive unbalance, caged shields that consist of a collection of unconnected wires, and many decades of excitation frequencies. As the simplest, nontrivial example, the formulation was applied to the shielded balanced pair. As indicated in Ref. 7, the calculations agree reasonably well with measurements.

APPENDIX A

The Electrostatics Approximation

The proof of the electrostatics approximation is broken into three parts in Section III. In this appendix the first part is verified; for the other two parts the reader is referred to Kuznetsov's work.[4] It is shown in particular that within an error for E_{ze} of one part in $(2.4/k_eR_0)$ (typically 250 for 10 MHz),

$$\nabla^2 E_{ze} = 0, \tag{34}$$

and the same argument will hold for H_{ze}.

As indicated in eq. (1.5) of Ref. 5,

$$(\nabla^2 + X_e^2)E_{ze} = 0. \tag{35}$$

If E_{ze}^0 is a function such that $E_{ze}^0 = E_{ze}$ on Γ, E_{ze}^0 is continuous across dielectric interfaces and $\delta\,(\partial E_{ze}^0/\partial\rho) = \delta(\partial E_{ze}/\partial\rho)$ there, and

$$\nabla^2 E_{ze}^0 = 0, \tag{36}$$

then $E_{ze} = E_{ze}^0 + E_{ze}^1$, where $E_{ze}^1 = 0$ on Γ, E_{ze}^1 and its derivatives are continuous across dielectric interfaces, and

$$(\nabla^2 + X_e^2)E_{ze}^1 = -X_e^2 E_{ze}^0. \tag{37}$$

When ψ_1 denotes the first eigenfunction of $-\nabla^2$ that is zero on Γ, then the L^2-norm of E_{ze}^1 is bounded by

$$\|E_{ze}^1\| \leqq \left|\frac{X_e^2}{X_e^2 - \lambda_1^2}\right| \|E_{ze}^0\|, \tag{38}$$

where λ_1 is the eigenvalue associated with ψ_1. The smallest eigenvalue of $-\nabla^2$ in the circle of radius R_0 relative to a boundary condition of zero on the circumference is $2.4048/R_0$ [2.4048 is the first zero of $J_0(x)$] and (see Ref. 8, Vol. 1, page 409, Theorem 3)

$$\lambda_1^2 \geqq (2.4048/R_0)^2. \tag{39}$$

Hence,

$$\|E_{ze}^1\| \leqq (X_eR_0/2.4048)^2 \|E_{ze}^0\|, \tag{40}$$

and since $X_e \sim k_e$,

$$E_{ze} \cong E_{ze}^0, \tag{41}$$

which was to be proved. Clearly, the same argument holds for the x and y derivatives of E_{ze}.

APPENDIX B

Boundary Problem for E_{ze}

From the various conditions on E_z, a boundary problem for E_{ze} can be deduced. The result is stated as three conditions in Section II. In this appendix these conditions are verified.

(i) From the electrostatics approximation, $\nabla^2 E_{ze} = 0$.

(ii) Since E_z is a tangential component of E, it is continuous across interfaces. The difference in normal-derivative across the dielectric interfaces is deduced from eq. (4). Since H_{ze} is continuous across interfaces,

$$-\delta(X_e^2 E_{\rho me}) = \gamma \delta \left(\frac{\partial E_{ze}}{\partial \rho_m}\right) \quad m = 0, 1, \cdots, M \tag{42}$$

at the interfaces. But by the usual boundary condition for the normal E field, $\delta(k_e^2 E_{\rho e}) = 0$; so for $m = 0, 1, \cdots, M$,

$$\delta \left(\frac{\partial E_{ze}}{\partial \rho_m}\right) = -\gamma\, \delta(E_{\rho e}) = \gamma \delta \left(\frac{\partial V}{\partial \rho_m}\right) \tag{43}$$

at the dielectric interfaces.

(iii) By the same reasoning as in (ii)

$$\frac{\partial E_{ze}}{\partial \rho_m} - \frac{\partial E_{zm}}{\partial \rho_m} = -\gamma (E_{\rho e} - E_{\rho m}) \text{ at } \rho_m = R_m, \tag{44}$$

where E_{zm} and $E_{\rho m}$ refer to components of E in the mth conductor for $m = 0, 1, \cdots, M$. Since

$$E_{\rho m} = (k_e^2/k_m^2) E_{\rho e} \ll E_{\rho e} \tag{45}$$

for all m, eq. (44) to a good approximation becomes

$$\frac{\partial E_{ze}}{\partial \rho_m} - \frac{\partial E_{zm}}{\partial \rho_m} = \gamma \frac{\partial V}{\partial \rho_m} \quad \text{for } m = 0, 1, \cdots, m. \tag{46}$$

Finally, it is proved that

$$\frac{\partial E_{zm}}{\partial \rho_m} = D_{\omega m} E_{zm} \quad \text{for } m = 0, 1, \cdots, M, \tag{47}$$

and since $E_{zm} = E_{ze}$ at $\rho_m = R_m$, this will verify the third condition. This is proved first for the wires, then for the shield.

Inside a wire,

$$\nabla^2 E_{zm} + X_m^2 E_{zm} = 0 \quad m = 1, \cdots, M, \tag{48}$$

and since $\gamma^2 \ll X_m^2$, the quantity X_m^2 can be replaced by k_m^2. The solution for E_{zm} has the form,

$$E_{zm} = \sum_n a_n \frac{J_n(k_m \rho_m)}{J_n(k_m R_m)} \exp(in\phi), \tag{49}$$

where the a_n are the Fourier coefficients of E_z at $\rho_m = R_m$ and J_n denotes the nth order Bessel function of the first kind. The radial derivative at $\rho_m = R_m$ is

$$\frac{\partial E_{zm}}{\partial \rho_m} = k_m \sum_n a_n \frac{J_n'(k_m R_m)}{J_n(k_m R_m)} \exp(in\phi_m) = k_m D_{\omega m} E_{zm}, \tag{50}$$

where $D_{\omega m}$ is a linear operator, which for $m = 1, \cdots, M$,

$$D_{\omega m}: \exp(in\phi_m) \rightarrow (k_m/\alpha_{nm}) \exp(in\phi_m) \quad n = 0, \pm 1, \pm 2, \cdots, \tag{51}$$

and

$$\alpha_{nm} = \frac{J_n(k_m R_m)}{J_n'(k_m R_m)}. \tag{52}$$

Inside the shield,

$$\nabla^2 E_{z0} + k_0^2 E_{z0} = 0, \tag{53}$$

where X_0^2 has been replaced by k_0^2. The solution has the form,

$$E_{z0} = \sum_n a_n \frac{b_n J_n(k_0 \rho_0) + c_n Y_n(k_0 \rho_0)}{b_n J_n(k_0 R_0) + c_n Y_n(k_0 R_0)} \exp(in\phi_0), \tag{54}$$

where the a_n are the Fourier coefficients of E_z at $\rho_0 = R_0$, the b_n and c_n are determined by reference to the boundary condition on the outside surface of the shield ($\rho_0 = R_0 + th \equiv R_0'$), and Y_n denotes the nth order Bessel function of the second kind. Assuming the cable is not in the vicinity of sources or other conductors, the potential outside will be zero; hence, there is no charge on the outer surface of the shield, and by eq. (4) $\partial E_z / \partial \rho_0$ is continuous at $\rho_0 = R_0'$. Since outside the cable, E_z has the form

$$E_z = \sum_{n \neq 0} f_n \left(\frac{R_0'}{\rho_0}\right)^{|n|} \exp(in\phi_0), \tag{55}$$

for some f_n, it follows from the continuity of $\partial E_z / \partial \rho_0$ that in eq. (54),

$$b_n = Y_{n-1}(k_0 R_0') \quad \text{and} \quad c_n = -J_{n-1}(k_0 R_0') \quad n = 0, \pm 1, \pm 2, \cdots. \tag{56}$$

The radial derivative of E_z at $\rho_0 = R_0$ is

$$\frac{\partial E_{z0}}{\partial \rho_0} = k_0 \sum_n (a_n / \alpha_{n0}) \exp(in\phi_0), \tag{57}$$

where

$$\alpha_{n0} = \frac{Y_{n-1}(k_0R_0')J_n(k_0R_0) - J_{n-1}(k_0R_0')Y_n(k_0R_0)}{Y_{n-1}(k_0R_0')J_n'(k_0R_0) - J_{n-1}(k_0R_0')Y_n'(k_0R_0)} \tag{58}$$

$$n = 0, \pm 1, \pm 2, \cdots.$$

Hence, at $\rho_0 = R_0$

$$\frac{\partial E_{z0}}{\partial \rho_0} = k_0 D_{\omega 0} E_{z0}, \tag{59}$$

where $D_{\omega 0}$ is a linear operator as in eq. (51) with $m = 0$. The direct sum[10] of the $D_{\omega m}$ for $m = 0, 1, \cdots, M$ is D_ω, as given in eq. (19).

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