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On the Rearrangeability of Some Multistage Connecting Networks

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We generalize the concept of rearrangeability to a finer measure of the connecting power of a network, called the c-rearrangeability function. It can be interpreted as the proportion of calls a network guarantees to connect under a given traffic load. We study the c-rearrangeability function for many well-known rearrangeable networks, including one-sided rearrangeable, two-sided rearrangeable, as well as several other kinds. We also give constructions for some new classes of networks, study their c-rearrangeability functions, and describe conditions under which the networks are rearrangeable. We show that these newly constructed rearrangeable networks compare favorably with the well-known ones with respect to the number of crosspoints.

I. INTRODUCTION

A multistage connecting network can be described by the following (see Fig. 1 for a three-stage example):

(*i*) There are s ordered stages, where $s > 1$ is arbitrary. The ith stage, $i = 1, \cdots, s$, consists of r_i copies of a switch ν_i. The jth copy of ν_i is denoted by ν_{ij}.

(*ii*) Links can exist only between switches of adjacent stages or between $\nu_1(\nu_s)$ and input (output) terminals of the network. The set of links incident to a particular ν_i is partitioned into two subsets. Those which are linked to either ν_{i-1} or input terminals are called input links of ν_i, and those linked to either ν_{i+1} or output terminals are called output links.

(*iii*) The r_1 copies of ν_1 are called input switches of the network. Each ν_1 is connected to n_1 input terminals. The r_s copies of ν_s are called output switches of the network. Each ν_s is connected to n_s output terminals.

The three-stage Clos network is a special case of a multistage connecting network, satisfying the additional restrictions that $s = 3$ and

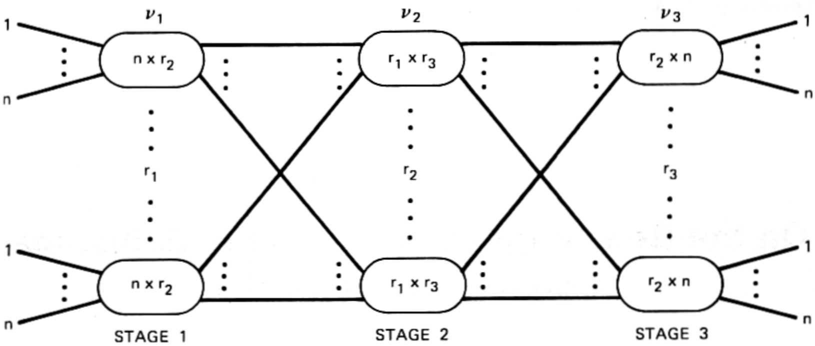


Fig. 1—Generalized three-step Clos network.

that there is exactly one link between every pair (ν_1, ν_2) and every pair (ν_2, ν_3). When ν_1, ν_2, ν_3 themselves are allowed to be multistage connecting networks, then a three-stage Clos network is called a generalized Clos network. For simplicity, we assume $n_1 = n_3 = n$ throughout this paper. Then, a generalized Clos network can be denoted by $C(\nu_1, \nu_2, \nu_3, r_1, r_2, r_3, n)$ (see Fig. 1).

Define a request to be a pair of idle terminals seeking connection. A request becomes a call once the two terminals are connected in the network. An assignment is a set of requests and the size of an assignment is the number of requests in it. An assignment is said to be realizable if every request in it can be simultaneously connected in the network without any link being used more than once. A network is said to be rearrangeable if it can realize every possible assignment.

Consider a multistage connecting network ν. Let $\mathcal{I}$ be the set of input terminals, $\mathcal{O}$ the set of output terminals of ν, and $\mathcal{I} + \mathcal{O} = T$. In many actual cases, not every possible pair in T will generate a request. In general, there could be two subsets $I, \Omega \subseteq T$ such that all requests are generated in the product space $I \times \Omega$. However, the four most important cases are:

(*i*) the one-sided case: $I = \Omega = T$.
(*ii*) the two-sided case: $I = \mathcal{I}, \Omega = \mathcal{O}$.
(*iii*) the input-mixed case: $I = \mathcal{I}, \Omega = T$.
(*iv*) the output-mixed case: $I = T, \Omega = \mathcal{O}$.

The last two are often combined and called the mixed case.

A network is said to be one-sided rearrangeable if it can realize every one-sided assignment. Similarly, we can define two-sided rearrangeable, input-mixed rearrangeable, and output-mixed rearrangeable. Thus, a one-sided rearrangeable network means that every set of pairs of terminals can be simultaneously connected, and a two-sided rearrangeable network means that every set of (input link–output link) terminals can be simultaneously connected. It is clear that one-sided rearrangeability implies mixed rearrangeability, which, in turn, implies two-sided rearrangeability.

Rearrangeability is a strong condition which is manifested in two aspects. First, all the requests in an assignment must be simultaneously connected; i.e., if one request fails, the whole assignment fails. Second, every assignment must be realizable; i.e., if one assignment fails, the whole network fails. Even for a nonrearrangeable network, it is still of interest to know the degree of its nonrearrangeability. We introduce a new concept of rearrangeability in this direction. First, we score an assignment by the largest number of requests it can guarantee to connect simultaneously. Second, we partition the set of all assignments into classes according to the size of an assignment. We score a class by the lowest score achieved by any member in this class. Now, a bad assignment can still bring down the score of its class, but not of the other classes, and not to a score of zero. Thus, we define $R_\nu(c)$, the c-rearrangeability function, as the largest number of requests the network ν can guarantee to connect given any assignment of size c. Thus, $R_\nu(c)/c$ is the proportion of requests ν can guarantee to connect given that the traffic load is approximately $c/(\text{capacity of } \nu)$. When $R_\nu(c) = c$, we say ν is c-rearrangeable. If ν is c-rearrangeable for all c, then c is rearrangeable in the classical sense.

In this paper, we study the c-rearrangeability functions for some well-known rearrangeable networks. We also construct some new classes of networks, study their c-rearrangeability functions, and describe conditions under which the networks are rearrangeable. We show that these newly constructed rearrangeable networks can save a significant number of crosspoints over the well-known networks.

II. ANALYSES OF SOME WELL-KNOWN REARRANGEABLE NETWORKS

As switches are the basic components of a network, to understand the rearrangeable property of a network, we have first to know what the rearrangeable properties of its switches are (the switches mentioned in this paper are all cross-point grid switches). For a switch, the definition of rearrangeability is similar to that for networks, except that input links and output links replace the roles of input terminals and output terminals in a network.

Two links of a switch have direct access to each other if they intersect at a crosspoint. In many networks, the cost of crosspoints still dominates the other costs. Therefore, we would like to minimize the number of crosspoints in a network. A relevant question is, for a given rearrangeable property, which switch has the minimum number of crosspoints? This problem has recently been solved by Chung.[1] However, in our networks, we will stick to the more traditional switches for their engineering feasibility and for ease of comparisons with existing networks. Consider a switch with n input links and m output links. It is called a triangular switch if there is a crosspoint between every pair of links, input or output. Therefore, a triangular switch has $[(n + m)(n + m - 1)]/2$ crosspoints and is clearly one-sided rearrangeable. The switch is called a rectangular switch if there is a crosspoint between every input link and every output link.

A rectangular switch has $n \times m$ crosspoints and is two-sided rearrangeable. The switch is called a trapezoidal switch if there is a crosspoint between every pair of links with at least one of the links belonging to a fixed side. A trapezoidal switch either has $n(n - 1)/2 + nm$ or $m(m - 1)/2 + nm$ crosspoints and is either input-mixed or output-mixed rearrangeable, depending on which side is the fixed side. Note that an $n \times m$ rectangular switch is in fact input-mixed rearrangeable if $m \geqq n - 1$. This is because any pair of input links can be connected through an output link, and there are always enough output links to do it. While the existing networks always use trapezoidal switches when mixed-rearrangeable switches are needed, we will use rectangular switches to save crosspoints when the condition $m \geqq n - 1$ for input-mixed and $n \geqq m - 1$ for output-mixed is met.

In every network we discuss in this paper, ν_1 and ν_s are always assumed to be two-sided rearrangeable (or stronger). Hence, two terminals from two distinct ν_1 and/or ν_s can be connected if and only if their corresponding switches $\nu_1(\nu_s)$ can be connected. Thus, we can redefine a request as a pair of ν_1 and/or ν_s and an assignment as a collection of requests where each ν_{1i} or ν_{sj} can appear at most n times. If a request is (ν_{1i}, ν_{1i}) or (ν_{sj}, ν_{sj}), then we have to discuss separately how they can be connected.

Consider $\nu = C(\nu_1, \nu_2, \nu_3, r_1, r_2, r_3, n)$ shown in Fig. 2. (In our figures, $\triangle$ represents a one-sided rearrangeable ν_i, $\square$ a two-sided rearrangeable

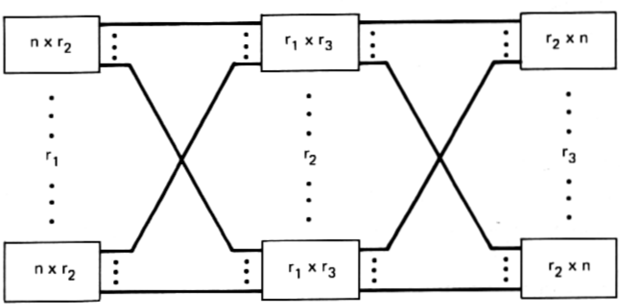
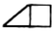
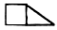


Fig. 2—Ordinary three-stage Clos network.

ν_i,  an input-mixed rearrangeable ν_i, and  an output-mixed rearrangeable ν_i. The number of input links and output links will be shown inside these figures.)

Theorem 1: (See Slepian,[2] Duguid,[3] and Beneš.[4]) *ν is two-sided rearrangeable if and only if $r_2 \geqq n$.*

Theorem 2:

$$R_\nu(c)/c \begin{cases} \cong \min\{r_2/n, 1\}, & \text{for} \quad c > r_2; \\ = 1, & \text{for} \quad c \leqq r_2. \end{cases}$$

Proof: Actually, we will prove an exact expression for $R_\nu(c)/c$. We need only consider the case $n > r_2$ and $c > r_2$ since the other cases are trivial. Consider any assignment of size c. Let $c = pn + q$ where $0 \leqq q < n$. We can assume that the r_2 real ν_2 are embedded in a set of n imaginary ν_2. Then by Theorem 1, all requests can be simultaneously connected by the n ν_2. Rank the n ν_2 according to the number of calls they carry and select the r_2 ν_2 with the highest ranks to be the r_2 real ones. They must carry a total number of calls not less than $\min\{r_2, q\} \times (p+1) + \max\{r_2 - q, 0\} \times p$. On the other hand, when all the requests in an assignment involve only a few ν_1, say as few as possible, then every ν_2 carries essentially the same number of calls, differing at most by one. Hence,

$$R_\nu(c)/c = \begin{cases} (r_2 p + q)/(np + q), & \text{if} \quad r_2 \geqq q; \\ (r_2 p + r_2)/(np + q), & \text{if} \quad r_2 < q. \end{cases}$$

Theorem 2 gives a good approximation to this when c and $r_2 p$ are large relative to q.

To compute the number of crosspoints of a network, we always make the simplifying assumptions that $r_1 = r_3 = n$ and all ν_i are nonblocking switches so that we can easily compare the various networks. Under these assumptions, then, the current network for $r_2 = n$ has $3n^3$ crosspoints.

Next consider $\nu_2 = C(\nu_1, \nu_2, \nu_3, r_1, r_2, r_3, n)$ shown in Fig. 3a.

Theorem 3:[5–7] *ν is one-sided rearrangeable if and only if $r_2 \geqq \lfloor 3n/2 \rfloor$, where $\lfloor x \rfloor$ is, as usual, the integer part of x.*

Theorem 4:

$$R_\nu(c)/c \begin{cases} \cong \min\left\{ r_2 \Big/ \left\lfloor \dfrac{3n}{2} \right\rfloor, 1 \right\}, & \text{for} \quad c > r_2; \\ = 1, & \text{for} \quad c \leqq r_2. \end{cases}$$

Proof: Again, we need only consider the case $\lfloor 3n/2 \rfloor > r_2$ and $c > r_2$. The proof that $R_\nu(c)/c \geqq r_2/\lfloor 3n/2 \rfloor$ uses a similar argument to that

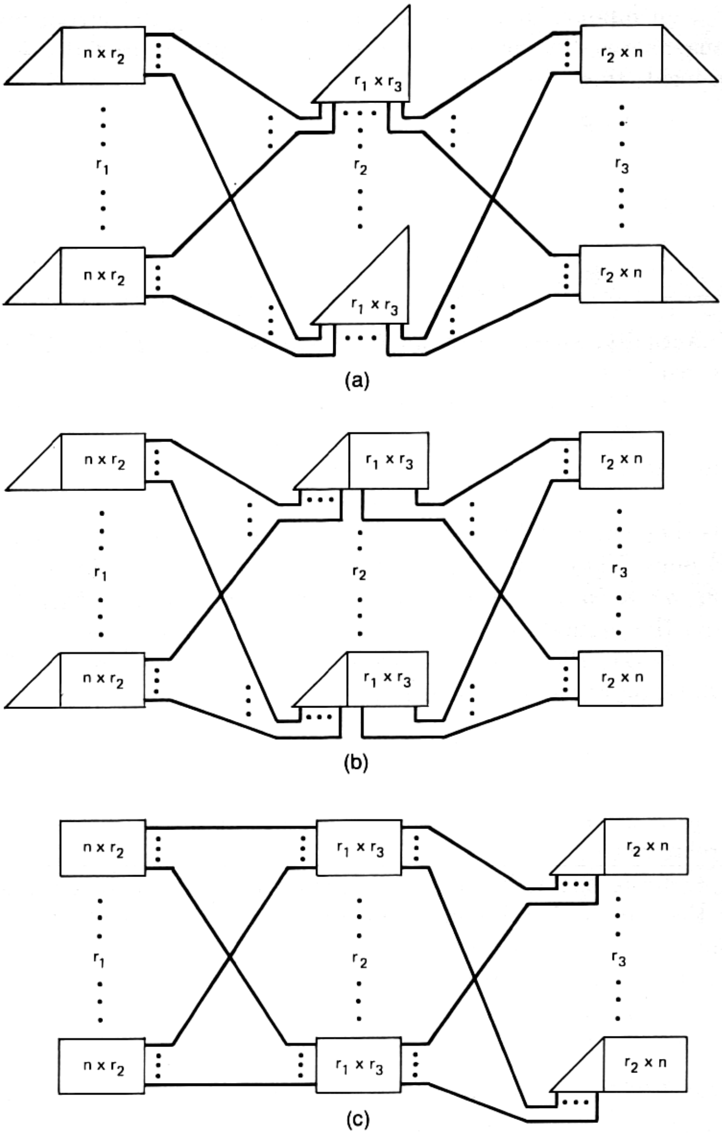


Fig. 3—Mixed three-stage Clos networks.

in the proof of Theorem 2. To prove the reverse inequality, label all the ν_1 and ν_3 by the numbers 1 to $r_1 + r_3$. Consider the $\lfloor 3n/2 \rfloor$ requests,

$$\begin{aligned} &(1, 2), \cdots, (1, 2), \quad (\lfloor n/2 \rfloor \text{ of them}), \\ &(1, 3), \cdots, (1, 3), \quad (\lfloor n/2 \rfloor \text{ of them}), \\ &(2, 3), \cdots, (2, 3), \quad (\lfloor (n+1)/2 \rfloor \text{ of them}). \end{aligned}$$

No two of them can be carried by the same ν_2, since otherwise the two will share a link. Consider an assignment whose requests can be partitioned into sets of $\lfloor 3n/2 \rfloor$ requests of the above type and a remainder set which is a subset of the above type of $\lfloor 3n/2 \rfloor$ requests. Then the number of calls carried by each ν_2 can differ at most by one. Hence, $R_\nu(c)/c \leqq r_2/\lfloor 3n/2 \rfloor +$ (a constant)$/c \cong r_2/\lfloor 3n/2 \rfloor$. The proof is completed.

Under the simplifying assumptions previously stated, the number of crosspoints for this network for $r_2 = \lfloor 3n/2 \rfloor$ is $7n^3$. Note that this compares favorably with the one-sided network[8] obtained from a two-sided rearrangeable network where both sides are $\mathcal{I} + \mathcal{O}$. Such a network needs $12n^3$ crosspoints. It is also better than the one obtained by joining three two-sided networks together, the first having $\mathcal{I}$ in both sides, the second having $\mathcal{O}$ in both sides, and the third having $\mathcal{I}$ in one side and $\mathcal{O}$ the other. Such a network needs $9n^3$ crosspoints.

We can easily obtain an input-mixed rearrangeable network from the above one-sided rearrangeable network by changing ν_2 from one-sided rearrangeable to input-mixed rearrangeable and ν_3 from output-mixed rearrangeable to two-sided rearrangeable (Fig. 3b). Let ν be the network shown in Fig. 3b.

Theorem 5:

$$R_\nu(c)/c \begin{cases} \cong \min \left\{ r_2 \Big/ \left\lfloor \dfrac{3n}{2} \right\rfloor, 1 \right\}, & \text{for} \quad c > r_2, \\ = 1, & \text{for} \quad c \leqq r_2. \end{cases}$$

Proof: The proof is similar to the proofs for Theorems 3 and 4.

For $r_2 = 3n/2$, this network has $(23/4)n^3$ crosspoints. However, we can also obtain an input-mixed rearrangeable network by joining two two-sided rearrangeable networks together; one is $(\mathcal{I}, \mathcal{O})$-two-sided and the other has $\mathcal{I}$ in both sides. Such a network needs $6n^3$ crosspoints.

III. A NEW INPUT-MIXED REARRANGEABLE NETWORK

Consider $\nu = C(\nu_1, \nu_2, \nu_3, r_1, r_2, r_3, n)$ shown in Fig. 3c.

Theorem 6: *ν is input-mixed rearrangeable if*

(*i*) $r_2 \geqq n$,
(*ii*) $(r_2 - 1)r_3 \geqq nr_1$.

Proof: We first explain how a request is connected in this network. A (ν_{1i}, ν_{3j}) request is still connected through some ν_2 which has an idle link to ν_{1i} and an idle link to ν_{3j} just as is done in the networks of Section II. But a (ν_{1i}, ν_{1j}) request cannot be connected in this manner since ν_2 cannot connect two input links. Instead, we will connect both ν_{1i} and ν_{1j} to some ν_{3k} and then use the input-mixed rearrangeable property of ν_3 to complete the connection. One question is whether there are enough input links of ν_3 to accommodate all (ν_1, ν_1) requests. Now each (ν_1, ν_3) request takes up one input link of ν_1 and one of ν_3, and each (ν_1, ν_1) request takes up two input links of ν_1 and two of ν_3. Hence, regardless of the distribution of (ν_1, ν_1) requests relative to the (ν_1, ν_3) requests, the maximum number of ν_3 input links needed is, except for a minor correction, the maximum number of ν_1 input links available, which is nr_1. The minor correction is because each (ν_1, ν_1) request takes up a pair of ν_3 input links from the same ν_3. Hence, occasionally, a ν_3 input link may be wasted since it has no partner. Discounting one input link from each ν_3, we obtain condition *(ii)*.

If condition *(ii)* is satisfied, then each (ν_{1i}, ν_{1j}) request can be replaced by two requests (ν_{1i}, ν_{3k}) and (ν_{1j}, ν_{3k}). Hence, an input assignment is turned into a two-sided assignment. The requirement that each ν_{3k} must appear no more than n times is irrelevant here because the connection of (ν_{1i}, ν_{3k}) does not involve any output links of ν_{3k}. By Theorem 1, the derived two-sided assignment is rearrangeable if $r_2 \geqq n$. Theorem 6 is proved.

If $r_1 = r_3$, then $r_2 = n + 1$ satisfies both conditions of Theorem 6. Furthermore, since the size is right, we can use rectangular switches for ν_3 for mixed-rearrangeable property. This network has $3n^3 + 3n^2$ crosspoints (under the simplifying assumptions) as compared to $(23/4)n^3$ for the input-mixed rearrangeable network in Section II.

Since a (ν_{1i}, ν_{1j}) request takes twice as many links to connect as a (ν_{1i}, ν_{3j}) request, one might suspect that the blocking probability for the former request is much larger. This is not necessarily true, however, since there are only r_2 distinct connecting paths of two links for a (ν_{1i}, ν_{3j}) request but $\binom{r_2}{2} r_3$ paths of four links for a (ν_{1i}, ν_{1j}) request.

Theorem 7: Let ν be the network in Theorem 6. Then

$$R_\nu(c)/c \cong \min\left\{1, \frac{r_2}{n}\right\} \times \min\left\{1, \frac{\max\{2c - nr_1, 0\} + (r_2 - 1)r_3}{2c}\right\}.$$

Proof: For the time being, suppose $r_2 = n$. Consider any assignment of size c and let u be the number of (ν_1, ν_3)-type requests in it. Then $u \geqq \max\{2c - nr_1, 0\}$ since $u + 2(c - u) = 2c - u$ input links of ν_1 are required while only nr_1 are available. If there are not enough input links of ν_3 to take care of all c requests, then priority should be given to (ν_1, ν_3) type requests to maximize the number of requests connected. The priority is due to the fact that a (ν_1, ν_3) request needs only one input link of ν_3 while a (ν_1, ν_1) request needs two. For $u \geqq r_2 r_3$, the maximum number of requests connectable is r_2r_3; for $u < r_2r_3$, the maximum is approximately

$$u + \frac{(r_2 - 1)r_3 - u}{2} = \frac{u + (r_2 - 1)r_3}{2} < r_2r_3.$$

The worst case occurs when u is at its minimum, i.e., $u = \max \times \{2c - nr_1, 0\}$. But still the network guarantees to connect at least $\frac{1}{2}$ [max $(2c - nr_1, 0\} + (r_2 - 1)r_3]$ requests. Now look at the distribution of these calls in the ν_2. If $r_2 < n$, select the $r_2\nu_2$ that carry the most calls. In this way, we obtain Theorem 7.

Corollary: *ν is input-mixed c-rearrangeable if $r_2 \geqq n$ and either*

$$c \leqq \frac{(r_2 - 1)r_3}{2} \quad \text{or} \quad c \geqq \frac{(r_2 - 1)r_3}{2} \geqq \frac{nr_1}{2}.$$

IV. A NEW ONE-SIDED REARRANGEABLE NETWORK

Consider $\nu = C(\nu_1, \nu_2, \nu_3, r_1, r_2, r_3, n)$, where ν_1 and ν_3 are input- and output-mixed rearrangeable and ν_2 is one-sided rearrangeable. Also assume n and r_2 are even. We construct a ν' from ν by inserting something between the pair $(\nu_{2,2i-1}, \nu_{2,2i})$ for each $i = 1, \cdots, r_2/2$ (see Fig. 4) to provide some limited access between links of $\nu_{2,2i-1}$ and links of $\nu_{2,2i}$. One way to do this is to insert two two-sided rearrangeable networks μ_{i1} and μ_{i2} between $\nu_{2,2i-1}$ and $\nu_{2,2i}$. The input links of μ_{i1} are the extensions of the n links of $\nu_{2,2i-1}$ and the output links of μ_{i2} are the extensions of the n links of $\nu_{2,2i}$. μ_{i1} has $\frac{1}{3}(r_1 + r_3)$ output links that become the input links of μ_{i2}. Thus any link of $\nu_{2,2i-1}$ can seize an output link of μ_{i1} and then connect to any link of $\nu_{2,2i}$ in μ_{i2}. Of course, $\frac{1}{3}(r_1 + r_3)$ such connections can be made simultaneously.

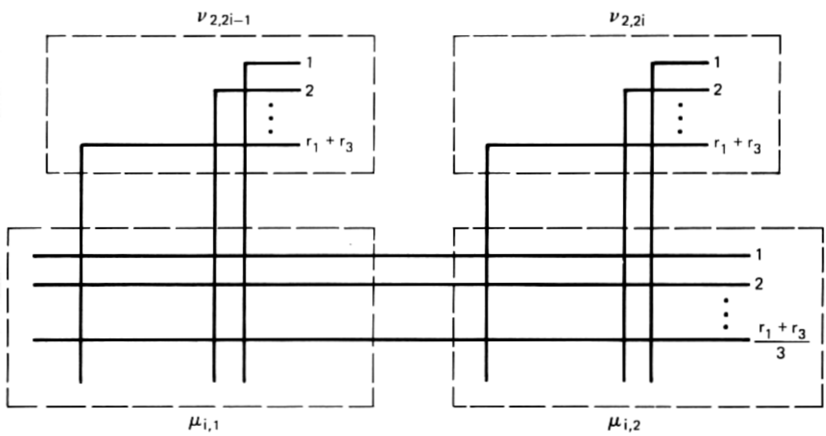


Fig. 4—One-sided rearrangeable networks.

Theorem 8: *ν is one-sided rearrangeable if and only if $r_2 \geqq n$.*

Proof: For a given assignment, define an assignment graph by taking all ν_1 and ν_3 as vertices and every request as an edge. We can augment the assignment graph to become a regular graph of degree n by adding suitable edges to it. By a theorem of Petersen (see Refs. 9 or 10), a regular graph of even degree is 2-factorable; i.e., the assignment graph can be decomposed into $(n/2)$ 2-factors, where a 2-factor is a subgraph in which every vertex is of degree 2. Hence, a 2-factor consists of a set of disjoint circuits. Now any circuit of length 1 represents a request from two terminals of the same switch. We can connect them within that switch because of its mixed-rearrangeable property. Aside from that, we can partition all edges in an odd circuit into three sets such that edges in the same set are all disjoint; and we can partition all edges in an even circuit into two such sets. Since all circuits in a 2-factor are disjoint, we can combine those sets into three large sets A_1, A_2, A_3, such that the edges in each large set are all disjoint. As each edge represents a request, all the requests in A_1 can be connected through, say, ν_{21}, since they are all disjoint (we can ignore those edges which are augmented to the assignment graph). Similarly, all the requests in A_2 can be connected through ν_{22}. For a request in A_3, say (x, y), if (x, y) is disjoint with every request in $A_1(A_2)$, then we can connect it through $\nu_{21}(\nu_{22})$. Otherwise, suppose x has appeared in A_1 and y in A_2. Then we connect x to ν_{22} and y to ν_{21} and then connect them through μ_{11} and μ_{12}. We do this for every request in A_3. Therefore, all requests in a 2-factor can be connected by a pair of ν_2. There are $n/2$ 2-factors; hence, $n/2$ pairs of ν_2 will suffice. If we have less than $n/2$ pairs of ν_2, then there is no way to handle the $3n/2$ requests given in the proof of Theorem 4. Hence, Theorem 8 is proved.

Theorem 9:

$$R_\nu(c)/c \begin{cases} \cong \min\left\{\dfrac{r_2}{n}, 1\right\}, & \text{for} \quad c > \dfrac{3r_2}{2}, \\ = 1, & \text{for} \quad c \leqq \dfrac{3r_2}{2}. \end{cases}$$

Proof: Omitted.

For $r_2 = n$, this network has $(16/3)n^3$ crosspoints versus the $7n^3$ for the standard one-sided rearrangeable network.

V. A *c*-REARRANGEABILITY THEOREM

We have seen that the c-rearrangeability functions of many networks discussed in previous sections are such that $R_\nu(c)/c \cong \alpha$, a constant, over most of the range of c. Consider $\nu = C(\nu_1, \nu_2, \nu_3, r_1,$

$r_2, r_3, n)$ such that $R_{\nu_i}(c)/c \cong \alpha_i$ for two-sided assignment. What can we say about $R_\nu(c)$ for two-sided assignment? If the blocking in ν_1, ν_2, ν_3 and the blocking due to the structure of ν all act independently, then we should have

$$R_\nu(c) \cong \alpha_1\alpha_2\alpha_3 \min\left\{1, \frac{r_2}{\max\{\alpha_1, \alpha_3\} \times n}\right\}.$$

The reason for the last term is because at most $\max\{\alpha_1, \alpha_3\} \times n$ requests from the same switch can get through to the second stage. However, we show that the blocking in ν_1 and the blocking in ν_3 can be coordinated so that the requests blocked in one stage form a subset of the requests blocked in the other stage. Without loss of generality, suppose $\alpha_1 \leqq \alpha_3$. Then

Theorem 10:

$$R_\nu(c) \geqq \alpha_1\alpha_1 \min\left\{1, \frac{r_2}{\alpha_1 n}\right\}.$$

Proof: For any given assignment, consider its bipartite assignment graph G. Let d_i be the degree of vertex i. We want to find a subgraph G' such that the degree of vertex ν in G' is $d_i' = \alpha_1 d_i$ (treating it as an integer). Then G' is the set of requests that will get through both ν_1 and ν_3.

Let V_1 be the set of vertices corresponding to ν_1 and V_3 the set corresponding to ν_3. Let $d(X)$ denote the sum of degrees over all vertices of X in G, and define $d'(X)$ similarly for X in G'. Finally, let $d_Y(X)$ denote the degree sum of X in G when the set Y is deleted from G. Then a theorem of Gale[11] on network flows has the following interpretation.[12]

Gale's Theorem: *G' exists if and only if there do not exist two sets $S \subseteq V_1$, $T \subseteq V_3$ such that either*

$$d'(S) > d'(T) + d_T(S),$$

or

$$d'(T) > d'(S) + d_S(T).$$

In our case, $d'(S) = \alpha_1 d(S)$ and $d'(T) = \alpha_1 d(T)$. Without loss of generality, suppose $d(S) \geqq d(T)$. Then the second inequality in Gale's theorem certainly cannot hold. To check the first, note that

$$d_T(S) \geqq d(S) - d(T) \geqq \alpha_1 d(S) - \alpha_1 d(T) = d'(S) - d'(T).$$

Hence, the first inequality also does not hold. We conclude that G' exists and Theorem 10 is proved.

When the involved numbers are large, the discrepancy caused by assuming $\alpha_i d_i$ an integer is certainly negligible.

VI. ACKNOWLEDGMENT

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