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Multiple Tone Parameter Estimation From Discrete-Time Observations

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In a previous paper, we discussed estimation of the parameters of a single tone from a finite number of noisy discrete-time observations. In this paper, we extend the discussion to include several tones. The Cramér-Rao bounds are derived and their properties examined. Estimation algorithms are discussed and characterized.

I. INTRODUCTION

In a previous paper,[1] we reported on the estimation of the parameters of tones from a finite number of noisy, discrete-time observations and described the case of a single complex tone. In this report, we discuss the situation when the signal consists of several, say k, tones, either real or complex. By *real signal* we mean

$$s(t) = \sum_{i=1}^{k} b_i \cos(\omega_i t + \theta_i).$$

The corresponding *complex signal* is of the form

$$s(t) + j\check{s}(t) = \sum_{i=1}^{k} b_i \exp[j(\omega_i t + \theta_i)],$$

where $\check{s}(t)$ is the Hilbert transform of $s(t)$.

A computer observes, through the A–D converters, noisy versions of the signal, $X(t)$, and possibly its Hilbert transform $Y(t)$. That is, samples are taken of

$$X(t) = s(t) + W(t), \tag{1}$$

and

$$Y(t) = \check{s}(t) + \check{W}(t), \tag{2}$$

where $W(t)$ and $\check{W}(t)$ are the noise and its Hilbert transform, respectively.

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The observations are made at times denoted t_n. The computer will process one or both sample vectors:

$$\mathbf{X} = [X_0, X_1, \cdots, X_{N-1}]^T \quad \text{and} \quad \mathbf{Y} = [Y_0, Y_1, \cdots, Y_{N-1}]^T,$$

where T denotes matrix transpose,

$$X_n = X(t_n), \tag{3}$$

and

$$Y_n = Y(t_n). \tag{4}$$

We assume the noise samples, W_n and $\check{W}_n$, are independent, zero-mean, gaussian random variables with variance σ^2.

Let $\boldsymbol{\alpha}$ be the p-element vector of unknown signal parameters. We assume all signal parameters are unknown, so that $p = 3k$, and use the convention:

$$\begin{aligned} \alpha_{3i-2} &= \omega_i, \\ \alpha_{3i-1} &= b_i, \end{aligned} \tag{5}$$

and

$$\alpha_{3i} = \theta_i, \qquad i = 1 \text{ to } k.$$

This model describes several situations. The real signal may be received from a data set during a test or it could be a probe signal used to characterize a data-transmission channel. The real and imaginary parts of the complex signal could occur as the result of in-phase and quadrature modulation processes, as described by Palmer.[2] The imaginary part of the complex signal could be the output of a 90-degree phase-shift network (Hilbert transformer) through which the real signal is passed before the sampling is done. This is done in certain types of data sets that use all-digital means to demodulate received signals. Samples of the complex signal are easier to process because of the absence of negative frequency components, as we show below. The model also applies to certain mathematically equivalent, phased-array radar problems, such as the one described in Refs. 3, 4, 5, and 6.

There are two main aspects to the problem of estimating the parameters of the signal: lower bounds to estimation accuracy and algorithms for doing the estimation. In the next section, the properties of the Cramér-Rao (C-R) bounds are explored. There are many other bounds that could be applied but we have only examined the C-R bounds. Section III describes and evaluates some approximations to maximum-likelihood (ML) estimation. In Ref. 1, we found that when the signal consists of a single complex tone ($k = 1$), then ML estimates can be obtained with any desired accuracy. When several tones are present, ML estimation is sufficiently complicated that suboptimum alternatives are attractive.

II. CRAMÉR-RAO BOUNDS

2.1 General theory

Maximum-likelihood estimates of signal parameters are unbiased at high signal-to-noise ratio (s/n).[4,7] We will develop estimation algorithms that have very little bias, so we have only studied the C-R bounds to unbiased estimation accuracy. Even when an estimator has some bias, the unbiased bounds serve as useful goals for estimation accuracy. Since the accuracy of ML estimates approaches the unbiased C-R bounds at high s/n, the unbiased bounds also show what could be done if exact ML estimation algorithms were used.

We found in Ref. 1 that low s/n is that range of s/n where estimation anomalies occur. None of the known bounds seem to be very tight under these conditions.

The first property of the C-R bounds that we consider is one for which we need the following general notation.

Let **V** be a "signal" vector whose typical component is of the form

$$V_n = \sum_{i=1}^{K} b_i g_i(\omega_i, \theta_i, n). \tag{6}$$

Notice that each $g_i(\cdot)$ has an associated level, b_i, and is a function only of n and the ith set of unknown parameters. Time does not necessarily enter into the $g_i(\cdot)$ functions. Let **X** be a noisy observation of **V**. Assume the noise is additive, multivariate normal with zero mean and correlation matrix $\mathbf{R}^{-1}$. If the noise vector is **W**, then

$$\mathbf{X} = \mathbf{V} + \mathbf{W}, \tag{7}$$

and the probability density function of **X** given **V** is

$$f(\mathbf{X}/\mathbf{V}) = \frac{|\mathbf{R}|^{\frac{1}{2}}}{(2\pi)^{N/2}} \exp[-\tfrac{1}{2}(\mathbf{X} - \mathbf{V})^T\mathbf{R}(\mathbf{X} - \mathbf{V})], \tag{8}$$

where N is the dimension of **V** and the T denotes transpose (see Ref. 8, page 207).

The C-R bounds require certain regularity conditions on **V**, which are satisfied by our model.[9] The bounds are the diagonal elements of the inverse of the Fisher information matrix, **J**, whose typical element[10] is:

$$J_{ab} = -E\left\{\frac{\partial^2}{\partial\alpha_b\partial\alpha_a}\log f\right\}, \tag{9}$$

where $E\{\cdot\}$ denotes expected value of $\{\cdot\}$. The bounds are:

$$\mathrm{Var}\,\{\hat{\alpha}_a - \alpha_a\} \geqq J^{aa}, \tag{10}$$

where J^{aa} is the ath diagonal element of $\mathbf{J}^{-1}$ and $\hat{\alpha}_a$ is an unbiased estimate of α_a.

It is easy to show[11] that

$$J_{ab} = \frac{\partial \mathbf{V}^T}{\partial \alpha_b} \mathbf{R} \frac{\partial \mathbf{V}}{\partial \alpha_a}. \tag{11}$$

We now present a few theorems that characterize the C-R bounds. We assume $\mathbf{J}$ is not singular, for reasons discussed below.

Theorem 1: The C-R bounds to unbiased estimation of the parameters ω_i and θ_i of V are functions of b_i but are independent of the other levels, b_j: $j \neq i$. The bound to unbiased estimation of a level, b_i, is independent of all the levels.

Proof: Equation (11) is equivalent to

$$J_{ab} = \sum_n \sum_m R_{nm} \frac{\partial V_n}{\partial \alpha_b} \frac{\partial V_m}{\partial \alpha_a}, \tag{12}$$

where R_{nm} is an element of $\mathbf{R}$.

The elements of $\mathbf{J}$ that are functions of the parameters of g_i and g_j, using the convention given by (5) and the notation $g_i(n) = g_i(\omega_i, \theta_i, n)$, are:

$$J_{3i-2,3j-2} = b_i b_j \sum_n \sum_m R_{nm} \frac{\partial g_i(n)}{\partial \omega_i} \frac{\partial g_j(m)}{\partial \omega_j}. \tag{13a}$$

$$J_{3i-2,3j-1} = b_i \sum_n \sum_m R_{nm} \frac{\partial g_i(n)}{\partial \omega_i} g_j(m). \tag{13b}$$

$$J_{3i-2,3j} = b_i b_j \sum_n \sum_m R_{nm} \frac{\partial g_i(n)}{\partial \omega_i} \frac{\partial g_j(m)}{\partial \theta_j}. \tag{13c}$$

$$J_{3i-1,3j-2} = b_j \sum_n \sum_m R_{nm} g_i(n) \frac{\partial g_j(m)}{\partial \omega_j}. \tag{13d}$$

$$J_{3i-1,3j-1} = \sum_n \sum_m R_{nm} g_i(n) g_j(m). \tag{13e}$$

$$J_{3i-1,3j} = b_j \sum_n \sum_m R_{nm} g_i(n) \frac{\partial g_j(m)}{\partial \theta_j}. \tag{13f}$$

$$J_{3i,3j-2} = b_i b_j \sum_n \sum_m R_{nm} \frac{\partial g_i(n)}{\partial \theta_i} \frac{\partial g_j(m)}{\partial \omega_j}. \tag{13g}$$

$$J_{3i,3j-1} = b_i \sum_n \sum_m R_{nm} \frac{\partial g_i(n)}{\partial \theta_i} g_j(m). \tag{13h}$$

$$J_{3i,3j} = b_i b_j \sum_n \sum_m R_{nm} \frac{\partial g_i(n)}{\partial \theta_i} \frac{\partial g_j(m)}{\partial \theta_j}. \tag{13i}$$

An examination of (13a) through (13i) shows that the submatrix of $\mathbf{J}$ has the form $\mathbf{D}_i\mathbf{Q}_{ij}\mathbf{D}_j$, where

$$\mathbf{D}_i = \begin{bmatrix} b_i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b_i \end{bmatrix} \tag{14}$$

and the matrix $\mathbf{Q}_{ij}$ is not a function of any b_i. It follows that $\mathbf{J}$ has the form

$$\mathbf{J} = \mathbf{DQD}, \tag{15}$$

where

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} & \cdots & \\ \mathbf{0} & \mathbf{D}_2 & & \\ \vdots & & \ddots & \\ & & & \mathbf{D}_k \end{bmatrix} \tag{16}$$

and

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \cdots & \mathbf{Q}_{1k} \\ \vdots & \ddots & \\ \mathbf{Q}_{k1} & & \mathbf{Q}_{kk} \end{bmatrix}. \tag{17}$$

$\mathbf{0}$ is a matrix whose elements are all zeros. From (15),

$$\mathbf{J}^{-1} = \mathbf{D}^{-1}\mathbf{Q}^{-1}\mathbf{D}^{-1}, \tag{18}$$

from which the theorem follows. For example,

$$\text{Var}\,\{\hat{\omega}_1 - \omega_1\} \geqq Q^{11}/b_1^2. \tag{19}$$

This theorem is not entirely new. It is alluded to in Ref. 6. However, this form of the theorem shows that, contrary to Ref. 6 and popular opinion, precisely known sampling times (or antenna element spacing in the equivalent radar problem) are not necessary for the theorem to hold.

The theorem is true whether or not the noise samples are independent and regardless of the sampling times. Of course, if the sampling times are not known, then the C-R bounds cannot be accurately calculated, but that does not obviate the theorem.

It should also be clear that the number of unknown parameters is unimportant to the theorem. Clearly the theorem holds if, for example,

$$g_i(\omega_i, \theta_i, n) = \cos\,(\omega_i t_n + \theta_i),$$

and if

$$t_n = nT.$$

Theorem 2: The bounds associated with the parameters of the first k tones, when there are $k + m$ tones, are not less than the bounds when there are only k tones.

Proof: The matrix $\mathbf{J}$ is always positive semidefinite. Thus, if it is not singular, it is positive definite.

Suppose $\mathbf{J}$ is the Fisher information matrix for $k + m$ tones and is partitioned so that $\mathbf{J}_k$ is the $\mathbf{J}$ matrix associated with k of the tones. This partitioning is always possible. Then write

$$\mathbf{J} = \left[\begin{array}{c|c} \mathbf{J}_k & \mathbf{K} \\ \hline \mathbf{K}^T & \mathbf{J}_m \end{array}\right]. \tag{20}$$

Since $\mathbf{J}$ is positive definite, so are $\mathbf{J}_k$ and $\mathbf{J}_m$.

Write the inverse of $\mathbf{J}$ in the form

$$\mathbf{J}^{-1} = \left[\begin{array}{c|c} \mathbf{U} & \mathbf{W} \\ \hline \mathbf{W}^T & \mathbf{V} \end{array}\right], \tag{21}$$

where $\mathbf{J}_k$ and $\mathbf{U}$ are both $3k$ by $3k$ matrices. Theorem 2 is true if

$$\mathbf{U} \geqq \mathbf{J}_k^{-1}, \tag{22}$$

which means $\mathbf{U} - \mathbf{J}_k^{-1}$ is positive semidefinite and which we now prove.

Using the fact that

$$\mathbf{J}\mathbf{J}^{-1} = \mathbf{I}, \tag{23}$$

one can show that

$$\mathbf{U} = [\mathbf{J}_k - \mathbf{K}\mathbf{J}_m^{-1}\mathbf{K}^T]^{-1}. \tag{24}$$

Observe that $\mathbf{K}\mathbf{J}_m^{-1}\mathbf{K}^T$ is positive semidefinite. That is,

$$\mathbf{K}\mathbf{J}_m^{-1}\mathbf{K}^T \geqq 0. \tag{25}$$

Since $\mathbf{J}_k$ and hence $\mathbf{U}$ are positive definite, (24) and (25) imply that $\mathbf{U}^{-1} \leqq \mathbf{J}_k$, which implies (22).

Another implication of the proof of Theorem 2 is that the bounds for p of $p + m$ unknown parameters are not less than the bounds when only the first p parameters are unknown.

This theorem is also not entirely new, although we have not seen it stated before. A restricted version of the theorem is mentioned in Ref. 12, (page 33), and Problem 2.4.23 in Ref. 10 hints at this kind of result.

The theorem depends upon $\mathbf{J}$ being nonsingular. It is easy, but tedious, to show that if the signal vector is composed of samples of the real or complex signal described in the introduction and only two tones are involved, then $\mathbf{J}$ is singular only if the two tone frequencies are equal, modulo $2\pi/T$, where T is the intersample time. (Remember that a real tone has a component at $+\omega_i$ and another at $-\omega_i$.)

We have not been able to prove this result for an arbitrary number of tones, but all of our calculations of various **J** matrices support the hypothesis that **J** is singular only if two or more of the tone frequencies are equal, modulo $2\pi/T$ (assuming N is large enough).

When two of the tone frequencies are equal, the receiver is receiving one less tone than expected. In this paper, we assume that the correct number of tones, k, is known and that all of the frequencies are distinct.

2.2 Equally spaced samples and independent noise

We now concentrate on the problem described in the introduction. Assume all noise samples are independent with variance σ^2. That is,

$$\mathbf{R} = \frac{1}{\sigma^2}\mathbf{I}, \tag{26}$$

where **I** is an identity matrix.

Define

$$\mu_n = \sum_{i=1}^{K} b_i \cos\,(\omega_i t_n + \theta_i), \tag{27}$$

$$\nu_n = \sum_{i=1}^{K} b_i \sin\,(\omega_i t_n + \theta_i), \tag{28}$$

and

$$t_n = nT; \qquad n = 0, 1, \cdots, N-1. \tag{29}$$

As is mentioned in Ref. 1, the time of the first sample, t_0, has an effect upon bounds and estimation accuracy. We have ignored that problem in this paper and taken t_0 to be zero.

The signal vector is

$$V_n = \begin{cases} \mu_n & n = 0 \text{ to } N-1 \\ \nu_{n-N} & n = N \text{ to } 2N-1 \end{cases} \qquad \text{(complex signal)} \tag{30}$$

or

$$V_n = \mu_n; \qquad n = 0 \text{ to } N-1 \qquad \text{(real signal)}. \tag{31}$$

Then a typical element of **J** is

$$J_{ab} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\left[\frac{\partial \mu_n}{\partial \alpha_a}\frac{\partial \mu_n}{\partial \alpha_b} + \frac{\partial \nu_n}{\partial \alpha_a}\frac{\partial \nu_n}{\partial \alpha_b}\right] \qquad \text{(complex signal)}. \tag{32}$$

The ν_n terms are dropped if the signal is real.

Let $\mathbf{M}(\omega, \theta)$ be the matrix defined by

$$\mathbf{M}(\omega, \theta) = \begin{bmatrix} T^2 \sum n^2 \cos \Delta_n & -T \sum n \sin \Delta_n & T \sum n \cos \Delta_n \\ T \sum n \sin \Delta_n & \sum \cos \Delta_n & \sum \sin \Delta_n \\ T \sum n \cos \Delta_n & -\sum \sin \Delta_n & \sum \cos \Delta_n \end{bmatrix}, \tag{33}$$

where

$$\Delta_n = n\omega T + \theta; \qquad n = 0 \text{ to } N-1. \tag{34}$$

Let $\mathbf{P}_{ij}$ be the matrix defined by

$$\mathbf{P}_{ij} = \mathbf{M}(\omega_i - \omega_j, \theta_i - \theta_j) \tag{35}$$

and let $\mathbf{P}$ be the p-by-p matrix defined by

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \cdots & \mathbf{P}_{1k} \\ \vdots & & \vdots \\ \mathbf{P}_{k1} & & \mathbf{P}_{kk} \end{bmatrix}. \tag{36}$$

Let

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{37}$$

Define a matrix $\mathbf{Q}_{ij}$ by

$$\mathbf{Q}_{ij} = \tfrac{1}{2}[\mathbf{M}(\omega_i - \omega_j, \theta_i - \theta_j) - \mathbf{M}(\omega_i + \omega_j, \theta_i + \theta_j)\mathbf{B}] \tag{38}$$

and a matrix $\mathbf{Q}$ by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \cdots & \mathbf{Q}_{1k} \\ \vdots & & \vdots \\ \mathbf{Q}_{k1} & \cdots & \mathbf{Q}_{kk} \end{bmatrix}. \tag{39}$$

Then it can be shown (13) that $\mathbf{J}$ is given by:

$$\mathbf{J} = \frac{1}{\sigma^2}\mathbf{DPD} \qquad \text{complex tones} \tag{40}$$

or

$$\mathbf{J} = \frac{1}{\sigma^2}\mathbf{DQD} \qquad \text{real tones.} \tag{41}$$

Theorem 3: When the signal consists of two equal-level complex tones, the C-R *bounds for the same parameters (e.g., the two frequencies) are equal. In other words, the mutual interference is reciprocal.*

Proof: The $\mathbf{J}$ matrix is

$$\mathbf{J} = \frac{1}{\sigma^2}\begin{bmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \end{bmatrix}\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12}^T & \mathbf{P}_{11} \end{bmatrix}\begin{bmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \end{bmatrix} \tag{42}$$

because $\mathbf{P}_{11} = \mathbf{P}_{22} = \mathbf{M}(0, 0)$. Observe that $\mathbf{P}_{12}^T = \mathbf{BP}_{12}\mathbf{B}$ because $\mathbf{M}^T(\omega, \theta) = \mathbf{BM}(\omega, \theta)\mathbf{B}$. Thus, $\mathbf{J}^{-1}$ has the form

$$\mathbf{J}^{-1} = \sigma^2\begin{bmatrix} \mathbf{D}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2^{-1} \end{bmatrix}\begin{bmatrix} \mathbf{U} & \mathbf{W} \\ \mathbf{W}^T & \mathbf{V} \end{bmatrix}\begin{bmatrix} \mathbf{D}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2^{-1} \end{bmatrix}, \tag{43}$$

where

$$\mathbf{V} = \mathbf{BUB}. \tag{44}$$

Hence,

$$|U_{ij}| = |V_{ij}| \tag{45}$$

and $U_{ii} = V_{ii}$, which proves the theorem.

Theorem 4: The bounds for two tones, real or complex, are periodic in θ_1 and θ_2 with period π.

Proof: The theorem follows from the easily checked fact that $\mathbf{M}(\omega, \theta + \pi) = -\mathbf{M}(\omega, \theta)$.

Theorem 5: The bounds for real or complex tones are periodic in each frequency with period $2\pi/T$.

Proof: The theorem follows from the fact that $\mathbf{M}(\omega + 2\pi/T, \theta) = \mathbf{M}(\omega, \theta)$.

Theorem 6: The bounds associated with complex tones depend upon the difference frequencies and phases but not upon the absolute values.

Proof: The theorem follows from (35), (36), and (40).

It is, in general, tedious to invert **J** and obtain formulas for the bounds. However, it is a simple matter to have a computer calculate the elements of **J** and its inverse. We have done this to obtain a better understanding of the bounds.

A number of illustrative curves are given in Ref. 13. In the interest of brevity, we will present only two of the figures here.

The main thing we learned from the calculations is that there is a critical frequency separation, $4\pi/NT$, associated with multitone C-R bounds. In Ref. 1, it is shown that when a single complex tone is present, the bounds are independent of the frequency of the tone. When more than one complex tone is present, the bounds approach the single-complex-tone bounds when the minimum frequency separation (modulo $2\pi/T$) exceeds the critical frequency. The multitone bounds increase rapidly as the minimum frequency separation goes below this critical frequency.

This rule applies to a single real tone if it is considered to be two complex tones, one at a frequency, say, of ω_1, and one at $-\omega_1$. Thus, if the frequency of a single real tone is less than $2\pi/NT$, modulo π/T, then its C-R bounds are much larger than the corresponding single-complex-tone bounds.

In all cases, the multitone bounds depend upon the tone phases, as might be expected.

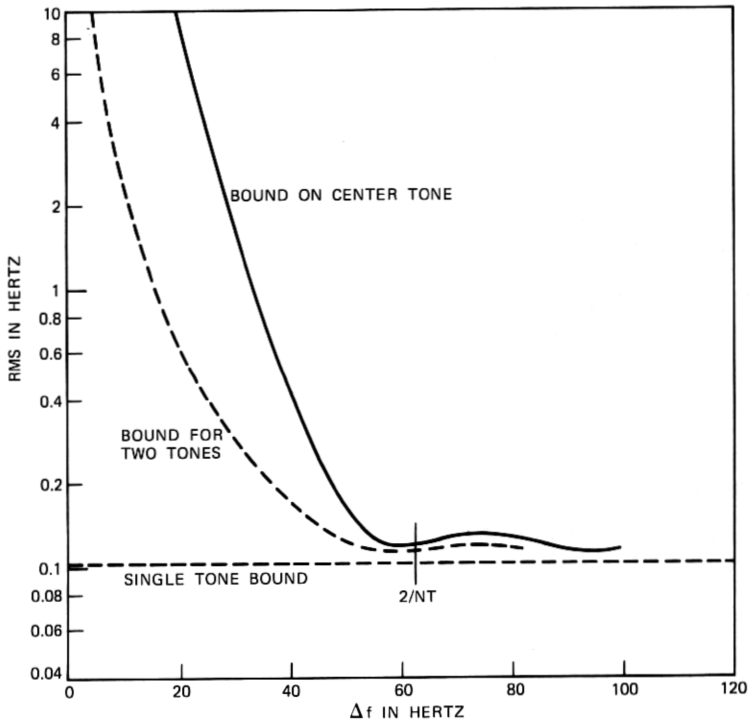

Fig. 1—Frequency estimation bound vs Δf for center of three equally spaced complex tones with worst relative phase and 20 dB s/n. N is 128. $1/T$ is 4000 Hz. Corresponding single and double tone bounds also shown.

Figure 1 illustrates the critical frequency for frequency-estimation bounds. The worst phase, i.e., the phase that gives the largest frequency estimation bound, was used at each difference frequency. Figure 2 shows the critical frequency effect upon the frequency-estimation bound for a single real tone.

To facilitate comparisons, in all figures we used a sampling frequency of 4000 Hz for complex tones and 8000 Hz for real tones. Thus, in both cases, the unknown tone frequencies are assumed to fall in the range of 0 to 4000 Hz.

III. ESTIMATION ALGORITHMS

3.1 General

The ML estimation procedure is conceptually simple. Given that a sample vector, $\mathbf{X}$, is received, the ML estimate of the parameter vector, $\hat{\boldsymbol{\alpha}}$, is the value of $\boldsymbol{\alpha}$ that maximizes the p.d.f. of $\mathbf{X}$. That is, $\hat{\boldsymbol{\alpha}}$ maximizes $f(\mathbf{X}/\mathbf{V})$. $\hat{\boldsymbol{\alpha}}$ may not be unique. Maximum likelihood estimation of the parameters of a single complex tone has been shown to be relatively easy to implement.[1] It was shown in Ref. 1 that single complex-tone ML estimators have variances almost equal to the C-R bounds over a wide range of s/n. No other unbiased estimators could do significantly better over that range of s/n.

Maximum likelihood estimation when several tones are present is much more difficult to implement. However, we show below ways to approximate ML estimation. We start the discussion with complex tones and examine a practical approximation to ML estimation, the resulting bias effects, the use of window functions to reduce bias, and a time-saving interpolation algorithm. Then we briefly discuss how the ideas and results apply to real tones.

Recall from Ref. 1 that we seek to maximize the function

$$L = \frac{2}{N} \sum_n (X_n \mu_n + Y_n \nu_n) - \frac{1}{N} \sum_n (\mu_n^2 + \nu_n^2), \tag{46}$$

where X_n and Y_n are as defined in (3) and (4).

After carefully arranging the terms, we obtain the likelihood func-

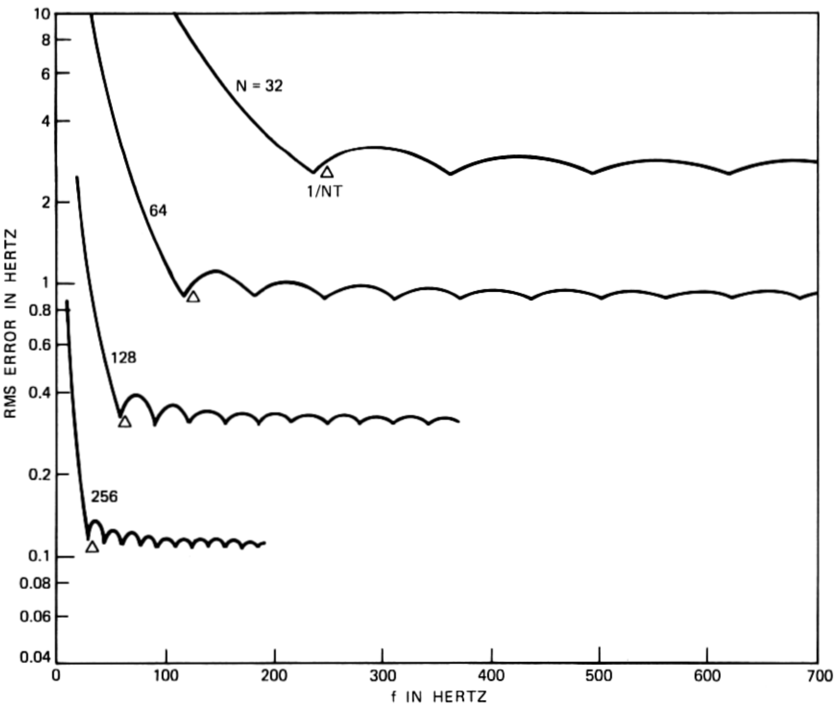

Fig. 2—Frequency estimation bounds vs frequency for single real tone at 20 dB s/n and worst phase.

tion in a form similar to that used in Ref. 1:

$$L = \sum_{i=1}^{k} \{2b_i \operatorname{Re}[e^{-j\theta_i} A(\omega_i)] - b_i^2\} - \frac{1}{N} \sum_{i \neq m} \sum_{m} b_i b_m \sum_{n} \cos(n\omega_i T - n\omega_m T + \theta_i - \theta_m), \quad (47)$$

where

$$A(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} (X_n + jY_n) e^{-jn\omega T}. \quad (48)$$

L as given by (47) has two main terms and would be difficult to maximize by a simple program. It can be done, but a lot of work is involved. We notice, however, that when there is only one tone ($k = 1$), the second term of (47) vanishes. Also, when N is large and $k > 1$, the magnitude of the second term is still relatively small and does not involve the data. Thus, we are led to drop the second term in L and maximize the remainder. This, of course, will only give ML estimates when $k = 1$ and will give "almost ML" estimates otherwise.

3.2 An almost ML algorithm

Suppose the cross-product terms in (47) are dropped. Then to make estimates, we need to maximize

$$L_1 = \sum_{i=1}^{k} 2b_i \operatorname{Re}[e^{-j\theta_i} A(\omega_i)] - b_i^2. \quad (49)$$

From Ref. 1, each frequency estimate, $\hat{\omega}_i$ maximizes $|A(\omega)|$. Then the corresponding level and phase estimates are

$$\hat{b}_i = |A(\hat{\omega}_i)| \quad (50)$$

and

$$\hat{\theta}_i = \arg[A(\omega_i)]. \quad (51)$$

The function $|A(\omega)|$ has many maxima and large peaks near the frequency of each tone. Thus, the frequencies of these large peaks, as illustrated in Fig. 3, are taken to be the frequency estimates, $\hat{\omega}_i$. Due to the periodicity of $|A(\omega)|$, all the ω_i should be confined to a range no wider than $\omega_s = 2\pi/T$ to avoid ambiguous frequency estimates. Normally the range $(0, 2\pi/T)$ is used. When real tones are involved, the range should not exceed π/T.

3.3 Bias

Consider the case of only two tones. An example of $|A(\omega)|$ when the noise power is zero is shown on Fig. 3.

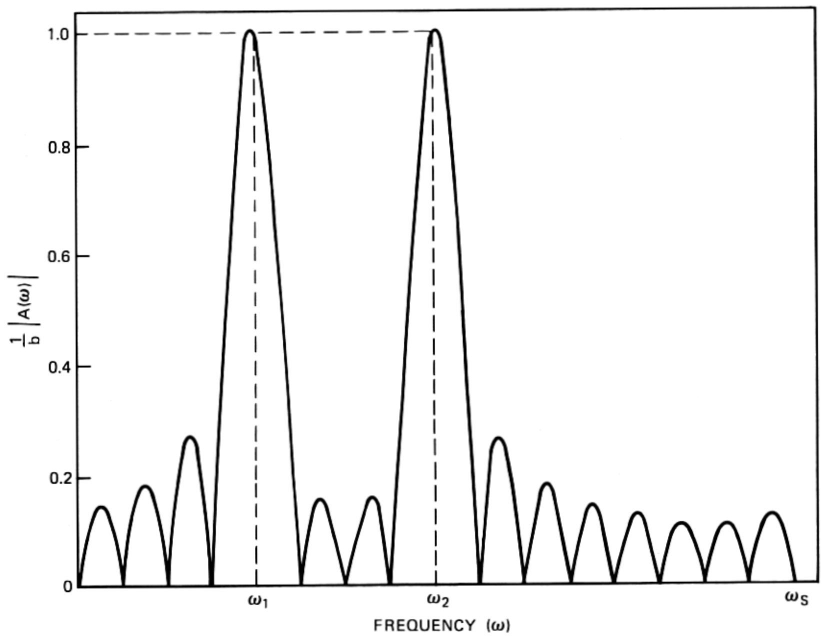

Fig. 3—Shape of $|A(\omega)|$ from two complex tones of equal phase and level, without noise. N is 16.

The figure has large peaks near ω_1 and ω_2. The peaks in the example are actually both displaced away from the average of the two frequencies. Thus, the penalty for neglecting the cross-product term in (47) is a bias in estimates of frequencies and levels.

The frequency and level bias in the zero-noise case is easily calculated. An example of such calculations is shown on Fig. 4. The figure shows the dependence of frequency estimation bias on the difference frequency (Δf) of the two tones. When two tones have almost the same frequency, the two large peaks merge into one at a frequency equal to the average of the two tone frequencies. This accounts for the negative slope of $-\frac{1}{2}$ at low Δf on Fig. 4. There is also a dependence upon the difference phase ($\Delta\theta$).

Figure 4 shows the bias for one of the two complex tones. The bias for the other has the same magnitude but opposite sign. In general, the magnitudes of the biases for two tones are not equal. However, they are equal when the two tones are equal-level complex tones.

3.4 Window functions

In discrete Fourier transform (DFT) work, window functions (also called weighting functions) are often used to minimize the effects of one tone upon another. The modification of the DFT of samples of one

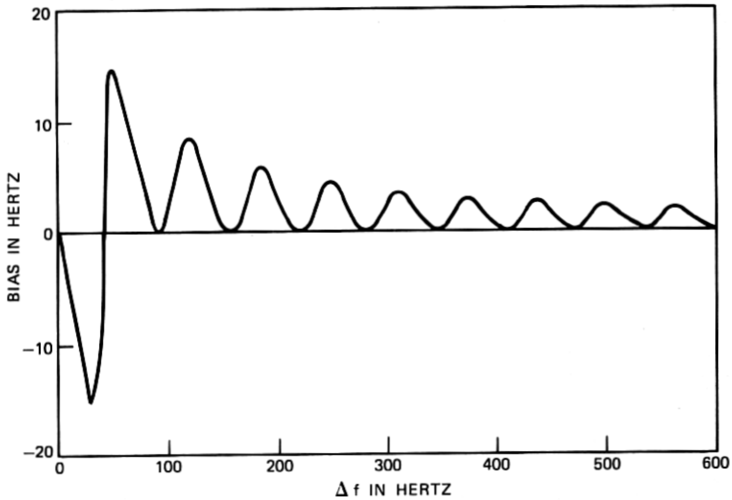

Fig. 4—Bias in peak frequency of $|A(\omega)|$ from two equal-level, equal-phase complex tones vs difference frequency, without noise. N is 64.

tone by the presence of samples of another tone is called *leakage.* See Rife and Vincent[14] for a discussion of leakage and how window function will reduce it.

In the time domain, a window function (or time window), say $h(t)$, is characterized by its samples, $h(nT)$. In use, each data sample, $X_n + jY_n = Z_n$, is multiplied by $h_n = h(nT)$ before $A(\omega)$ is computed. Thus, $A(\omega)$ becomes

$$A(\omega) = \frac{1}{N} \sum_n h_n Z_n e^{-jn\omega T}. \tag{52}$$

When a window function is used, the bias in the frequencies of the peaks of $|A(\omega)|$ is modified. If a good window is used, the bias can be greatly reduced. The penalty, as we see below, is an increase in the variance of $\hat{\omega}_i$ and $\hat{b}_i$. Palmer also reported this penalty in Ref. 2.

In the context of the DFT, window functions can be written in the form

$$h_n = 1 + \sum_{i=1}^{M} d_i \cos\,(2\pi in/N). \tag{53}$$

The number M, which can be assumed to be less than $N/2$, and the d_i define particular windows. With h_n in this form,

$$\frac{1}{N} \sum_{n=0}^{N-1} h_n = 1.$$

Table I—Values of d_i in ascending order of i for various windows

Hanning	Standard	Taylor
−1	−1.43596 0.497536 −0.061576	−1.03538 0.0824936 −0.00116197 −0.00188862 −0.00123387 −0.000671595 −0.000275885

A window that is better than many at reducing bias is the one identified by Rife and Vincent[14] as $g_3(t)$. We call this the *standard* window. Another useful window is one of the Taylor windows.[14] These windows are defined in Table I.

Figure 5 is an attempt to summarize the way window functions affect bias. The curves on the figure compare upper bounds to the bias associated with each of the previously defined window functions. The curves were obtained by computing at each frequency the bias at the worst phase (the phase that gave the largest bias). The resulting curves were flattened as indicated for the Taylor curve.

Figure 5 shows the Taylor window does the best job when the tone frequency separation is small. At large separations, however, the standard window does much better. The figure also shows how bad the bias is if no window is used.

Windowing increases the variance of frequency and level estimates. It can be shown[13] that the increase in variance is related to the function.

$$\eta = \frac{1}{N} \sum_{n=0}^{N-1} h_n^2. \tag{54}$$

It is easy to show that

$$\eta = 1 + \tfrac{1}{2} \sum_{i=1}^{M} d_i^2, \qquad \text{if} \qquad N > 2M. \tag{55}$$

Thus, η is not a function of N. Simulations verify that larger RMS errors are associated with larger values of η. Some values of η are tabulated below.

Window	η
None	1.00
Hanning	1.50
Taylor	1.54
Standard	2.16

Durrani et al. call the sum η a *dispersion factor*.[15] They have compared many windows and have tabulated their parameters, including dispersion factors. Other windows are mentioned in Blackman and Tukey.[16]

The data on Fig. 6 shows the general effects of windows on RMS errors when a single complex tone is present. The Hanning window produces almost the same RMS error as the Taylor window and is not shown on the figure.

Bias contributes to RMS errors more than variance does at high s/n,

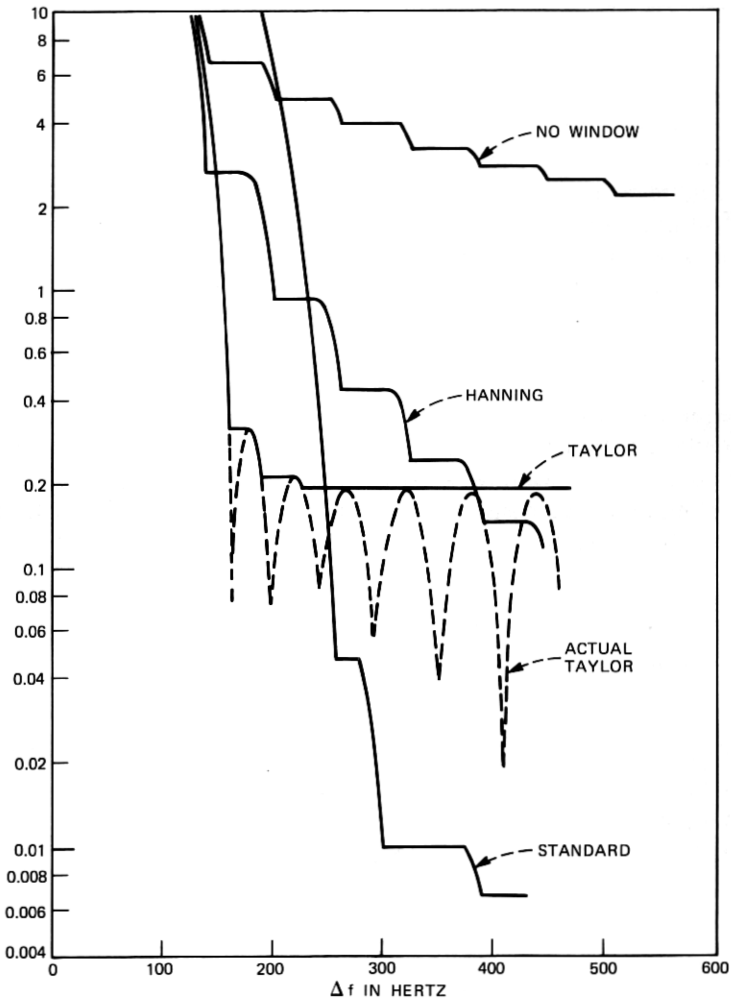

Fig. 5—Magnitude of frequency estimation bias for two equal-level complex tones using window functions. Curves are leveled as described in the text. Worst-phase was used at each frequency. N is 64.

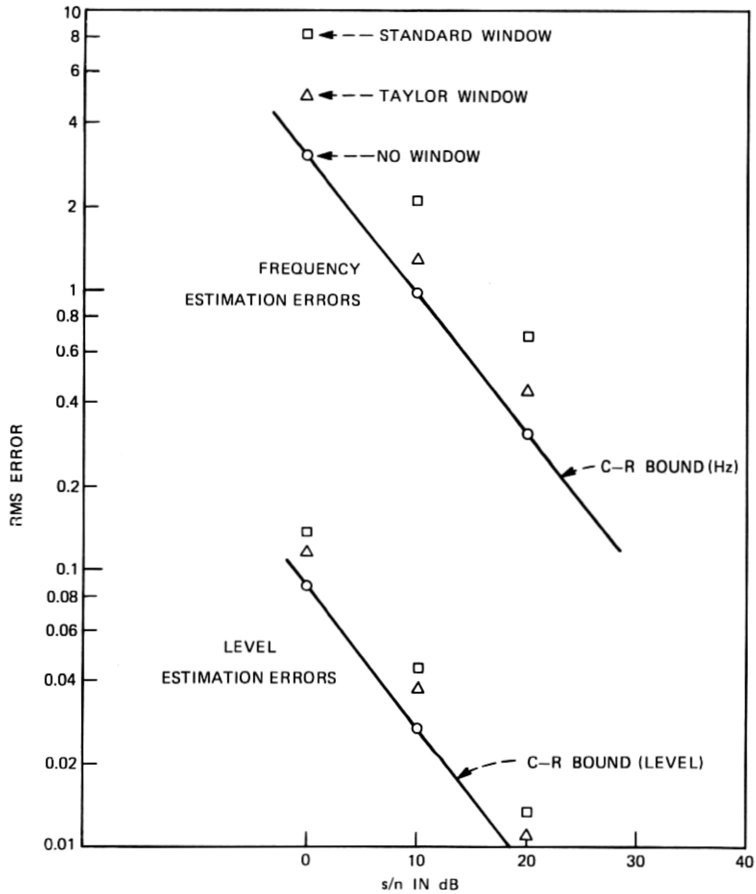

Fig. 6—Simulation results showing the effect of window functions on estimation variance with a single complex tone. N is 64.

while estimation variance controls RMS errors at low s/n. Thus, while a given window may produce lower RMS errors than another at high s/n, the roles may be reversed at low s/n. The "best" window for a given application will, therefore, depend upon the tone frequency spacings, the expected s/n, and possibly other factors. Figure 7 illustrates this point. On the figure, the Taylor window is best at 10 dB s/n, but the standard window is best at 40 dB s/n, where the bias associated with the Taylor window causes the RMS error curve to level off.

3.5 Interpolation

Maximization of $|A(\omega)|$ involves a search routine. A two-step algorithm that has a coarse search and a fine search was described in

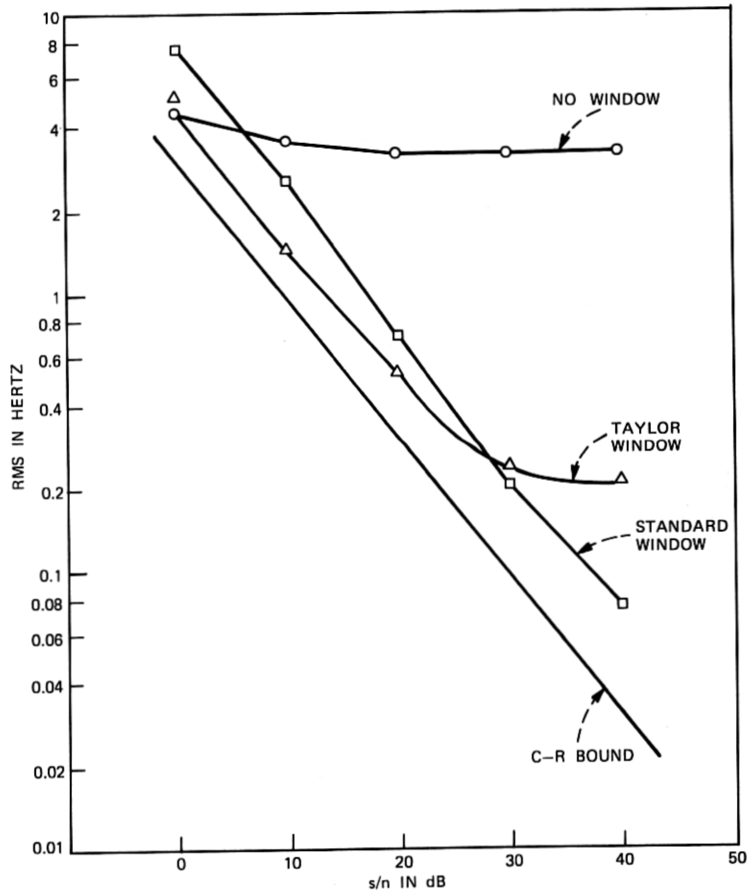

Fig. 7—Simulation results showing combined effects of bias and variance on frequency estimation for one of two complex tones. Frequency difference is 380 Hz; worst-phase was used for each window. N is 64.

Ref. 1. Fine searches are time consuming. This can be serious if computer time is important. One way to trade accuracy for speed is to use an interpolation algorithm on the DFT of the input data to arrive at frequency and level estimates.

Rife and Vincent developed several interpolation algorithms.[14] The one we investigate here is the following.

Assume the output of the FFT is the set:

$$A_k = \frac{1}{N} \sum_{n=0}^{N-1} h_n Z_n e^{-j2\pi nk/N}, \qquad k = 0 \text{ to } N-1. \tag{56}$$

Suppose a coarse search is conducted over $0 < k < N$. This results in locating $|A_l|$ which is the largest $|A_k|$ in the interval. Choose $\alpha = \pm 1$ such that $|A_{l+\alpha}| \geqq |A_{l-\alpha}|$

Let

$$a_1 = |A_l| \tag{57}$$

and

$$a_2 = |A_{l+\alpha}|. \tag{58}$$

Assume the sampling frequency is $\omega_s = 2\pi/T$.

The formulas from Rife and Vincent are:

$$\hat{\omega} = \frac{\omega_s}{N}(l + \alpha\delta) \tag{59}$$

and

$$\hat{b} = \frac{2\pi a_1 X}{\sin(\pi X)\left[1 + \sum_{n=1}^{M} d_n \delta^2/(\delta^2 - n^2)\right]}, \tag{60}$$

where the d_n define a window and

$$\delta = \frac{C_1 a_2 - C_2 a_1}{C_3 a_2 + a_1}. \tag{61}$$

The numbers C_1, C_2, and C_3 are given by Rife and Vincent in Table II for several windows.

The interpolation formulas give estimates that are only a little worse than the fine search gives. RMS frequency errors are typically increased by about 30 percent when interpolation is used. RMS level errors increase less.

When many tones are present, window functions can provide a satisfactory reduction of leakage as long as the minimum frequency separation is no less than about $8\pi/NT$. The data on Fig. 8 illustrate this point. The tone phases were all made random for these simulations. Thus, the points indicate the RMS errors one might encounter in a working system. The bound shown on the figure is the (unbiased) C-R bound maximized over the possible phases of the center tone.

We consider a real-tone estimation system to be equivalent to a complex-tone system if the two systems have the same useful bandwidth and the same frequency resolution. This means (*i*) the real sampling frequency is twice the complex sampling frequency and (*ii*) the total sampling time, NT, is the same for the real tones as for the

Table II — Constants for computing delta in eq. (61)

Window	C_1	C_2	C_3
None	1	0	1
Hanning	2	1	1
Taylor	1.96339	1.01643	0.893534
Standard	3.6020	2.5862	1.0317

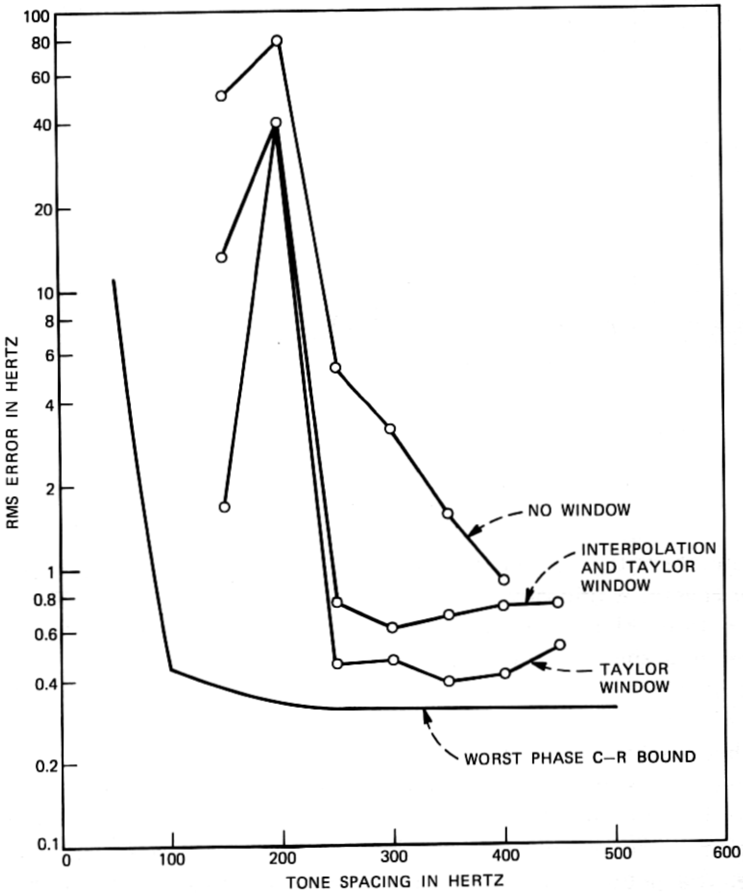

Fig. 8—Simulation results showing effects of Taylor window and interpolation algorithm upon frequency estimates of center tone of three equally spaced real tones. Center tone frequency is 2000 Hz. All have random phase. N is 128 and s/n is 20 dB.

complex. For example, a real-tone system using $1/T = 8000$ Hz and $N = 32$ is equivalent to a complex-tone system using $1/T = 4000$ Hz and $N = 16$.

The estimation algorithms described above for complex tones can be applied to real tones whose frequencies, in Hz, are in the range $(1/NT, 1/2T - 1/NT)$. The resulting accuracies are about the same as in the equivalent complex case.

IV. CONCLUSIONS

We have studied the problem of estimating the parameters, such as level and frequency, of several sinusoidal signals from a number of noisy observations, taken at discrete-time instants. Gaussian noise and ideal analog-to-digital conversion were assumed. The nature of the problem led us to study the generalized Cramér-Rao lower bounds to estimation accuracy and maximum likelihood estimation. The complexity of maximum likelihood estimation algorithms led us to examine several algorithms that yield estimates that are almost, but not exactly, maximum likelihood estimates of the signal parameters.

We were able to obtain estimators that have negligible bias, at least at high s/n. Thus, we considered in detail only the generalized C-R bound for unbiased estimators. Even when the resulting numbers are not, strictly speaking, lower bounds (e.g., when an estimator is biased), the unbiased estimation bounds can be considered to be desirable objectives for estimators.

Several properties of the bounds were derived from the properties of the **J** matrix. Other properties, such as the existence of critical frequencies, were revealed from computations.

The **J** matrix in the real tone cases is more complicated than in the complex cases and does not have quite the same structural properties. Thus, for example, the lower bounds for a single complex tone are not also lower bounds for the equivalent single real tone. On the other hand, the bounds for the case of many real tones approach the bounds for the equivalent complex cases when none of the real tones have frequency differences less than $2/NT$, modulo $1/2T$ (in Hz).

The cases of many complex tones and of real tones present some difficulties. Maximum likelihood estimation is difficult to implement because of the presence of cross-product terms. To properly implement ML estimation, multidimensional search procedures over a nonconvex function would be necessary. We found that when the tone frequencies are separated far enough, the cross-product terms could be neglected, thereby permitting the use of a simple algorithm whose estimates are almost equal to ML estimates.

The penalty for dropping the cross-product terms is a bias in frequency and level estimates. We found that the use of a suitable window function will reduce the bias to the point where it can be neglected when the minimum frequency separation of the tones is $4/NT$. Three window functions were discussed and compared.

We found that the use of a window to reduce bias increased the variance of the frequency and level estimates. The RMS error of frequency estimates is increased by about 35 percent with Taylor window and by over 100 percent with standard window. The use of the interpolation formulas increases RMS frequency errors by another 30 percent or so. Level estimates are affected less by windows and interpolation. All of these figures apply when the s/n is above threshold.

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