THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 55, No. 8, October 1976
Printed in U.S.A.

Alarm Statistics of the Violation Monitor and Remover

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(Manuscript received March 1, 1976)

Four of the Bell System digital transmission systems, T1 Outstate, T2, 3A-RDS (radio system at T3 rate), and T4M, have violation monitor and removers (VMRs) located at the receiving-end maintenance offices. Among other things, they monitor the lines, remove violations in the pulse transmission code, and generate alarms to initiate maintenance actions. This paper investigates the alarm statistics of the four types of VMR under the assumption that the information bits are statistically independent. It is found that all the VMRs have very sharp alarm thresholds. The results of the T4M VMR are presented in detail. Curves are given to show the various statistics obtained.

I. INTRODUCTION

Digital transmission systems serving large numbers of message channels should be continuously monitored to check the quality of service. This can be achieved by putting monitors at maintenance offices along the digital transmission route. An ideal monitor should provide the exact number of errors made in transmission. Since line errors cannot be directly measured in service, alternative criteria have to be used for performance monitoring. For instance, bipolar coding[1] can be employed so that the monitor can detect line errors from the violations of the coding sequence, and parity bits can be inserted into the transmitted digital stream so that the monitor can detect line errors if the received parity bits differ from those calculated from the received signal. The monitor generates alarms to initiate maintenance actions when the detected violation rates are greater than a predetermined threshold.

In some cases the digital stream has a periodic, identifiable pulse sequence called "frame format" to which the monitor must synchronize before it can detect violations. The monitor is said to be in-frame when it recognizes the location of the frame-pulse sequence. High line-error rates may alter the frame pulses such that they are unrecognizable by the monitor, which is then said to be out-of-frame. The in-frame condition is necessary to identify the various signal components that are multiplexed to form the digital stream. The monitor removes all violations detected so that violations do not propagate beyond the maintenance office; hence, the name "violation monitor and remover" (VMR) was designated. Removal of a violation is not an attempt to correct the line error. It is perfomed to guarantee that the VMR output is violation-free so that if an alarm condition exists, it will not propagate to the next maintenance office. The VMR performs other functions as well. If it is out-of-frame, a pseudorandom signal with proper frame format will replace the received digital stream at the VMR output in order to prevent alarm propagation.

Four of the Bell System digital transmission systems, T1 Outstate, T2, 3A-RDS (radio system at T3 rate), and T4M have VMRs located at the receiving-end maintenance offices. The T1 Outstate system uses bipolar coding. The T2 system utilizes B6ZS[1] (bipolar with six zeros extraction) coding. Both the 3A-RDS and the T4M systems employ added parity bits for performance monitoring. The VMR for each system has its own alarm rules. The durations of time for alarm generation and alarm release at various error rates are important system parameters. This paper investigates the alarm statistics of the four types of VMR under the assumption that the information bits are statistically independent; i.e., each bit is a Bernoulli trial. The derivations for the T4M VMR[2] are presented in detail in Section II. Those related to the other VMRs are discussed in Appendix A. Section III discusses some of the results obtained and their significance in digital transmission systems.

II. THE T4M VMR

2.1 Alarm strategy

The T4M digital transmission line[2] has a transmission rate of 274 megabits per second (Mb/s) with the information transmitted in a binary format. Its frame format[3] contains 196 bits of which 192 are information bits and 4 are housekeeping bits. One of the latter is a parity bit used to check the 192 information bits. The alarm strategy of the VMR at low-parity violation rates is implemented in the following manner. The first single parity violation that is observed triggers a 100-ms timer and a counter. If the counter accumulates more than 31 parity violations before the 100-ms measuring timer times out, a 3-ms waiting timer is immediately triggered. At the end of 3 ms, another 100-ms timer is triggered and the counter starts counting again. During this second 100-ms period, if the counter overflows; i.e., it accumulates more than 31 violations, a VMR alarm is generated immediately. The 3-ms waiting timer is employed so that a short burst of errors will not cause an alarm. Since the transmission rate is 274 Mb/s and violation is checked once every 196 bits, the alarm threshold violation rate is set at

$$\frac{31 \times 196}{2.74 \times 10^{8} \times 0.1} = 0.222 \times 10^{-3}.$$

It will be shown in the next section that this violation rate corresponds to an error rate of approximately 1.1×10^{-6}.

To avoid oscillatory alarms near the threshold violation rate, hysteresis is designed into the VMR alarm system. A 1-second release timer is used to measure the violation rate when the VMR is in the alarm state. The release timer is free-running and is not synchronized to the VMR alarm. The alarm is released only after a full duration of the release timer is passed and the 31-violation counter does not overflow. Thus, whenever an alarm is generated, it will last at least 1 second. This produces a release-error-rate threshold of about 1.1×10^{-7}.

When the VMR is out-of-frame for 0.5 ms, a pseudorandom signal with the proper frame format is switched in to provide a violation-free output. As soon as the VMR is back in-frame, the violation counter is reset and starts counting until the 1-second free-running release timer times out. If the counter does not overflow, the pseudorandom signal is then switched out. Thus, after a failure is restored, it takes anywhere from 0 to 1 second to switch out the pseudorandom signal.

2.2 Bit error rate versus parity violation rate

Since the digital transmission line performance objective is usually set in terms of the bit error rate, which cannot be directly measured in service, it is desirable to establish the relationship between the parity-violation rate and the bit-error rate. Let l be the number of information bits contained in each parity check. Then,

$$\begin{aligned} P\{\text{parity violation}\} = {} & P\{\text{odd number of bit errors in } l \text{ bits}\} \\ & \cdot P\{\text{the parity bit is correct}\} \\ & + P\{\text{even number of bit errors in } l \text{ bits}\} \\ & \cdot P\{\text{the parity bit is in error}\}. \end{aligned} \tag{1}$$

In what follows all random variables are in boldface type. Let the bit-error rate and the parity-violation rate be represented by $\boldsymbol{\varepsilon}$ and $\boldsymbol{v}$, respectively. For each realization of ϵ, (1) can be written as

$$v = (1 - b_l)(1 - \epsilon) + b_l\epsilon, \tag{2}$$

where b_l denotes the probability of having an even number of bit errors in l information bits. This event occurs if a correct first bit is followed by an even number of bit errors or if an incorrect first bit is followed by an odd number of bit errors. Therefore, for $l \geqq 1$,

$$b_l = (1 - \epsilon)b_{l-1} + \epsilon(1 - b_{l-1}), \qquad b_0 = 1. \tag{3}$$

Define the generating function[4]

$$B(S) = \sum_{l=0}^{\infty} b_l S^l \quad -1 < S < 1. \tag{4}$$

Multiplying (3) by S^l and adding over $l = 1, 2, \cdots$, we obtain

$$B(S) - 1 = (1 - \epsilon)SB(S) + \epsilon S(1 - S)^{-1} - \epsilon SB(S) \tag{5}$$

or

$$B(S) = \tfrac{1}{2}\{(1 - S)^{-1} + [1 - (1 - 2\epsilon)S]^{-1}\}. \tag{6}$$

Expanding into geometric series, we get

$$b_l = \frac{1 + (1 - 2\epsilon)^l}{2}, \tag{7}$$

which is equivalent but preferable to

$$b_l = \binom{l}{0} \epsilon^0 (1 - \epsilon)^l + \binom{l}{2} \epsilon^2 (1 - \epsilon)^{l-2} + \cdots.$$

Substituting (7) into (2)

$$v = \frac{1 - (1 - 2\epsilon)^l}{2} \times (1 - \epsilon) + \frac{1 + (1 - 2\epsilon)^l}{2} \times \epsilon. \tag{8}$$

Equation (8) establishes the relationship between the parity-violation rate and the bit-error rate. When $l\epsilon \ll 1$, it is easy to see that

$$v \approx (l + 1)\epsilon. \tag{9}$$

In the T4M frame format, $l = 192$. Therefore,

$$v \approx 193\epsilon. \tag{10}$$

Equation (10) is intuitively obvious because only errors occurring in the 192 information bits and the parity bit are counted by the VMR. Since a parity check is made every 196 bits, let $\epsilon' = v/196$, ϵ' can be considered as the measurable bit-error rate. It differs from ϵ by about 1.5 percent when (10) holds.

Figure 1 plots the parity violation rate versus the bit error rate based on (8) with $l = 192$, assuming the VMR stays in frame. We see that for bit-error rates below 10^{-3}, there is almost a one-to-one correspondence between a bit error and a parity violation. Above 10^{-3}, the VMR may go out of frame.

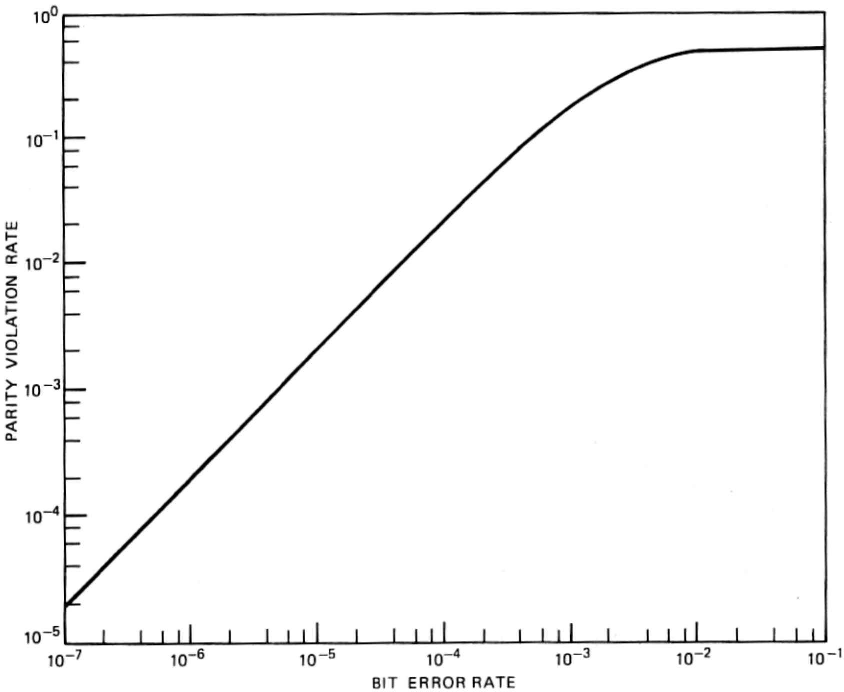

Fig. 1—Parity-violation rate vs bit-error rate.

In this paper, only low-parity-violation rates are being studied. Thus, ϵ will be used in place of ϵ' for simplicity.

2.3 Statistics of the alarm interval

Let γ represent the transmission rate, $\mathbf{N}$ the number of independent violations incurred, and $\mathbf{T}$ the time spent to count the violations. By the Bernoulli trial assumption,

$$P\{\mathbf{N} = n | \mathbf{T} = t, \boldsymbol{\varepsilon} = \epsilon\} = \binom{\gamma t}{n} \epsilon^n (1 - \epsilon)^{\gamma t - n}. \tag{11}$$

In this paper, only conditional distributions are discussed in most cases. For simplicity, conditions such as $\boldsymbol{\varepsilon} = \epsilon$, $\mathbf{N} = n$, and $\mathbf{T} = t$ are not expressed explicitly when they are understood.

Since γt is large, by De Moivre-Laplace limit theorem, a normal approximation to the binomial distribution is applicable.

$$P\{\mathbf{N} \geqq n\} \approx 1 - \Phi\left(\frac{n - \gamma t \epsilon}{\sqrt{\gamma t \epsilon (1 - \epsilon)}}\right), \tag{12}$$

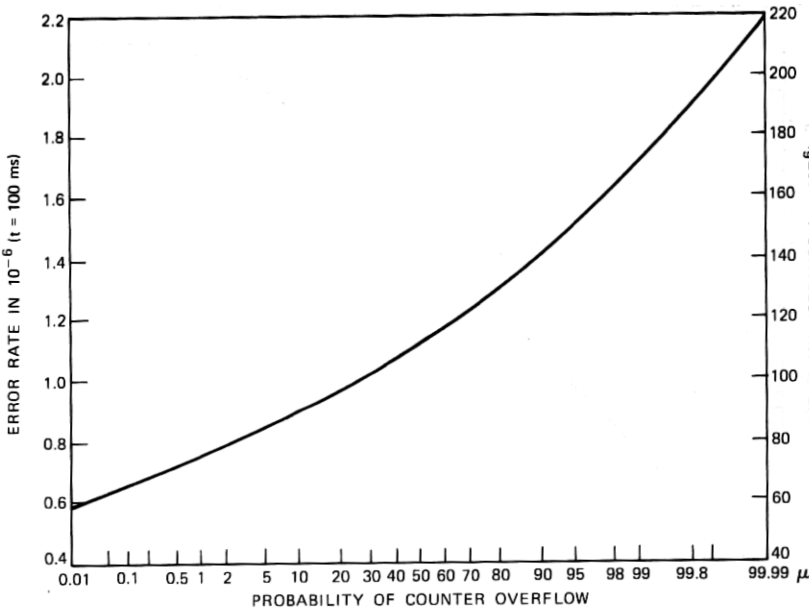

Fig. 2—Probability of counter overflow vs bit-error rate and time.

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \tag{13}$$

is the cumulative normal distribution. Let

$$\mu = P\{\mathbf{N} \geqq n\}, \tag{14}$$

μ is the probability of counter overflow given an error rate and a fixed timer. Figure 2 shows, on a probability scale, this probability as a function of the bit-error rate for $t_0 = 100$ ms. The same curve with a different ordinate also shows the probability as a function of time for $\epsilon = 10^{-6}$. It can be seen that when the error rate varies from $\epsilon_0/2$ to $2\epsilon_0$, the probability of counter overflow varies from 0.0001 to 0.9999. Thus, the threshold is very "hard."

Let $\mathbf{M}$ be the random variable such that the VMR alarm is generated at the $\mathbf{M}$th measuring period. Each period is 100 ms if the counter does not overflow. It is desirable, then, to determine the probability p_m, $m = 0, 1, 2, \cdots$, that the VMR will generate an alarm at the mth measuring period, given $\mathbf{T} = t$ and $\boldsymbol{\varepsilon} = \epsilon$. If 1 represents the event that during a measuring interval the counter overflows and 0 represents the opposite, the m periods must be of the form

$$\underbrace{X\ X\ X \cdots X}_{m-3}\ 0\ 1\ 1,$$

where the sequence of $m-3$ Xs does not have any 1 1 pair within it. Hence,

$$p_m = P\{\mathbf{M} = m\} \tag{15}$$
$$= P\{1\ 1 \text{ does not occur in a sequence of length } m-3\} \cdot p\{011\}$$
$$= \left(1 - \sum_{i=0}^{m-3} p_i\right)(1-\mu)\mu^2. \tag{16}$$

By definition, $p_0 = p_1 = 0$, $p_2 = \mu^2$, and $p_3 = (1-\mu)\mu^2$. Denote

$$q_m = P\{\mathbf{M} > m\}$$
$$= 1 - \sum_{i=0}^{m} p_i, \tag{17}$$

then

$$p_m = q_{m-3}(1-\mu)\mu^2. \tag{18}$$

Define the generating functions as

$$P(S) = \sum_{k=0}^{\infty} p_k S^k \qquad -1 \leqq S \leqq 1 \tag{19}$$

$$Q(S) = \sum_{k=0}^{\infty} q_k S^k \qquad -1 < S < 1. \tag{20}$$

Then,

$$(1-S)Q(S) = 1 - P(S), \tag{21}$$

as can be seen by comparing the coefficients of any S^k terms on each side. From (19), (18), and (20)

$$P(S) = \mu^2 S^2 + (1-\mu)\mu^2 S^3 Q(S). \tag{22}$$

Equations (21) and (22) give

$$P(S) = \frac{\mu^2 S^2 (1-\mu S)}{1 - S + \mu^2(1-\mu)S^3}. \tag{23}$$

From (23), the statistics of **M** can be derived. For instance, the mean and the variance are

$$E\{\mathbf{M}\} = \sum_{m=0}^{\infty} m p_m$$
$$= \lim_{S\to 1} P'(S)$$
$$= \frac{1+\mu}{\mu^2}. \tag{24}$$

$$\text{Var}\{\mathbf{M}\} = \lim_{S\to 1}[P''(S) + P'(S) - P'^2(S)]$$

$$= \frac{(1-\mu)(1+3\mu+\mu^2)}{\mu^4}. \tag{25}$$

Higher-order statistics of **M** can be similarly obtained. At the threshold error rate, ϵ_0, $\mu = \frac{1}{2}$, $E\{\mathbf{M}\} = 6$. Thus, the expected alarm time is approximately 600 ms. The threshold variance is 22, which is quite large.

Two standard methods are available to evaluate the probability coefficients p_m, $m = 1, 2, \cdots$. The first one is

$$p_m = \lim_{S\to 0} \frac{P^{(m)}(S)}{m!} \qquad m = 0, 1, 2, \cdots.$$

The second one is through partial fraction expansion of (23). Both methods require extremely tedious derivations. A simple alternative is presented in Appendix B which first expands the denominator of (23) as follows

$$\frac{1}{1 - S + (1-\mu)\mu^2 S^3} = \sum_{i=0}^{\infty} C_i S^i \tag{26}$$

with

$$C_0 = C_1 = C_2 = 1,$$

and

$$C_i = C_{i-1} - (1-\mu)\mu^2 C_{i-3} \qquad i \geqq 3. \tag{27}$$

From (19), (23), (26), and (27)

$$p_m = \mu^2 C_{m-2} - \mu^3 C_{m-3} \qquad m \geqq 3. \tag{28}$$

Equations (27) and (28) provide an attractive way to evaluate the probability coefficients p_m's. What is more, p_m can be obtained without first calculating p_{m-1}, p_{m-2}, etc. It is interesting to note that for any error rate,

$$p_2 = \mu^2$$

$$p_3 = p_4 = (1-\mu)\mu^2$$

$$p_m > p_{m+1} \qquad m \geqq 4.$$

Thus, the probability that the VMR will generate an alarm during the second measuring period is always the largest, regardless of the error rate. The probability decreases monotonically at later measuring periods.

The cumulative distribution function of $\mathbf{M}$ is

$$F_{\mathbf{M}}(m) = P\{\mathbf{M} \leqq m\}$$

$$= \sum_{k=1}^{m} p_k.$$

Since the duration of each measuring period is not greater than $t_0 = 100$ ms, the length of the timer,

$$P\{\text{VMR has generated an alarm in } mt_0 \text{ ms}\} \geqq \sum_{k=0}^{m} p_k.$$

This equation can be used to plot the lower bound of the alarm probability as a function of time.

2.4 Distribution of violation measuring time

The distribution of the measuring time $\mathbf{T}$, assuming $\mathbf{N} = n$ and $\boldsymbol{\varepsilon} = \epsilon$, is considered next. If we let $\mathbf{Y}$ be the number of error bits prior to the nth error, then $\mathbf{Y}$ has the negative binomial distribution

$$P\{\mathbf{Y} = y\} = \binom{y+n-1}{n-1} \epsilon^n (1-\epsilon)^y.$$

The time elapsed for the nth error to occur,

$$\mathbf{T} = \frac{\mathbf{Y} + n}{\gamma},$$

has the probability-density function (PDF)

$$P\{\mathbf{T} = t\} = \gamma \binom{\gamma t - 1}{n-1} \epsilon^n (1-\epsilon)^{\gamma t - n}. \tag{29}$$

Equation (29) is the distribution of the discrete violation measuring time $\mathbf{T}$ given that $\mathbf{N} = n$ and $\boldsymbol{\varepsilon} = \epsilon$. The T4M VMR has the additional condition $\mathbf{T} \leqq t_0 = 100$ ms; i.e., each measuring period is no greater than 100 ms. Let this censored random variable be denoted by $\mathbf{T}_c$. It is now desirable to find the distribution of $\mathbf{T}_c$, given that $\mathbf{N} = n$, $\boldsymbol{\varepsilon} = \epsilon$, and $\mathbf{T}_c \leqq t_0$. Unfortunately, this task is difficult to perform in the discrete sample space. However, since each information bit is 3.65 ns long while the $\mathbf{T}_c$ of interest is in milliseconds, the discrete censored random variable can be considered continuous for ease of calculation. From the Poisson theorem, (11) can be approximated by the Poisson distribution

$$P\{\mathbf{N} = n\} = \frac{e^{-\gamma \epsilon t}(\gamma \epsilon t)^n}{n!}. \tag{30}$$

Let $\mathbf{Y}_i$, $i = 1, 2, \cdots, n$ represent the time from the $(i-1)$th error to the ith error, then its PDF is given by

$$f_{\mathbf{Y}_i}(t) = \gamma\epsilon e^{-\gamma\epsilon t}.$$

Through the use of the characteristic functions, it is easy to see that the sum

$$\mathbf{T} = \sum_{i=1}^{n} \mathbf{Y}_i$$

has the gamma distribution

$$f_{\mathbf{T}}(t) = \frac{(\epsilon\gamma)^n}{(n-1)!} t^{n-1} e^{-\epsilon\gamma t} \qquad 0 \leqq t < \infty.$$

By successive integration by parts, it can be shown that

$$\int_{t_0}^{\infty} f_{\mathbf{T}}(t)dt = e^{-\epsilon\gamma t_0} \sum_{k=0}^{n-1} \frac{(\epsilon\gamma t_0)^k}{k!}.$$

Thus, the censored random variables $\mathbf{T}_c$ has the PDF

$$f_{\mathbf{T}_c}(t) = \begin{cases} f_{\mathbf{T}}(t) + \delta(t-t_0)e^{-\epsilon\gamma t_0} \displaystyle\sum_{k=0}^{n-1} \frac{(\epsilon\gamma t_0)^k}{k!} & t \leqq t_0 \\ 0 & t > t_0, \end{cases} \tag{31}$$

where $\delta(t - t_0)$ is the delta function. Its characteristic function $T(\omega)$ is

$$T_c(\omega) = \frac{(\epsilon\gamma)^n}{(\epsilon\gamma - j\omega)^n} \left(1 - e^{-(\epsilon\gamma - j\omega)t_0} \sum_{k=0}^{n-1} \frac{(\epsilon\gamma - j\omega)^k t_0^k}{k!}\right) + e^{-(\epsilon\gamma - j\omega)t_0} \sum_{k=0}^{n-1} \frac{(\epsilon\gamma t_0)^k}{k!}.$$

The mean η_{t_c} is given by

$$\eta_{t_c} = \left.\frac{dT_c(\omega)}{jd\omega}\right|_{\omega=0} = \frac{n}{\epsilon\gamma} - \frac{1}{\epsilon\gamma} e^{-\epsilon\gamma t_0} \sum_{k=1}^{n} \frac{k(\epsilon\gamma t_0)^{n-k}}{(n-k)!}. \tag{32}$$

The variance $\sigma_{t_c}^2$ can be evaluated similarly. The first term on the right of (32) is the mean value of $\mathbf{T}$. The second term is present because of the additional restriction $\mathbf{T} \leqq t_0$. At the alarm-error-rate threshold, $\eta_{t_c} \approx 97$ ms, $n/\epsilon\gamma = 100$ ms, the contribution of the second term is about 3 ms.

2.5 Distribution of the alarm time

Let $\mathbf{T}_0$ represent the time it takes the VMR to generate an alarm at a given error rate. It is desired to find the PDF of $\mathbf{T}_0$. Let $\mathbf{T}_i$, $i = 1, 2, \cdots, \mathbf{M}$, represent the time from the $(i-1)$th to the ith measuring interval, neglecting the 3-ms waiting time. The PDF of $\mathbf{T}_i$ is given in (31). The alarm time is then

$$\mathbf{T}_0 = \sum_{i=1}^{\mathbf{M}} \mathbf{T}_i.$$

Note that $\mathbf{T}_0$ is the sum of a random number of random variables.[4,5] Through the use of conditional probability, since $\mathbf{M}$ and $\mathbf{T}_i$'s are independent, the PDF of the random sum has a compound distribution

$$f_{\mathbf{T}_0}(t) = \sum_{m=0}^{\infty} p_m f_{\mathbf{T}_c}^{(m)}(t),$$

where p_m is given in (28) and $f_{\mathbf{T}_c}^{(m)}(t)$ is the m-fold convolution of $f_{\mathbf{T}_c}(t)$ with itself. The characteristic function of $\mathbf{T}_0$ is

$$T_0(\omega) = \sum_{m=0}^{\infty} p_m [T_c(\omega)]^m. \tag{33}$$

The right side of (33) is the Taylor expansion of $P(S)$ obtained in (23) with S replaced by $T_c(\omega)$. Thus,

$$T_0(\omega) = P[T_c(\omega)]. \tag{34}$$

The mean and the variance of $\mathbf{T}_0$ are

$$\begin{aligned} \eta_{t_0} &= \frac{1+\mu}{\mu^2}\,\eta_{t_c} \\ \sigma_{t_0}^2 &= \frac{1+\mu}{\mu^2}\,\sigma_{t_c}^2 + \frac{(1-\mu)(1+3\mu+\mu^2)}{\mu^4}\,\eta_{t_c}^2, \end{aligned} \tag{35}$$

where η_{t_c} is given in (32). Equation (35) is used to plot Fig. 3 which shows the mean alarm time versus the error rate. It can be seen that the mean alarm time decreases very fast as the error rate increases. The total alarm probability after time t is

$$\begin{aligned} P\{\mathbf{T}_0 \leqq t\} &= \int_0^t f_{\mathbf{T}_0}(t)dt \\ &= \sum_{m=0}^{\infty} p_m \int_0^t f_{\mathbf{T}_c}^{(m)}(t)dt. \end{aligned} \tag{36}$$

2.6 Waiting time distribution

In the above analyses, the 3-ms waiting intervals have not been taken into account. The waiting timer is triggered after each counter overflow. The distribution of the waiting periods is studied next. Let $\mathbf{W}$ be the number of times the waiting timer is triggered before a VMR alarm is generated, assuming that $\mathbf{M} = m$. The last three measuring periods before a VMR alarm should be 011 [notations are defined before (15)] and the waiting timer is definitely triggered once. Let

A_l = the event that 11 does not occur in l measuring intervals.

This event occurs if the counter does not overflow in the first measuring interval, followed by the event A_{l-1}, or the counter overflows in the

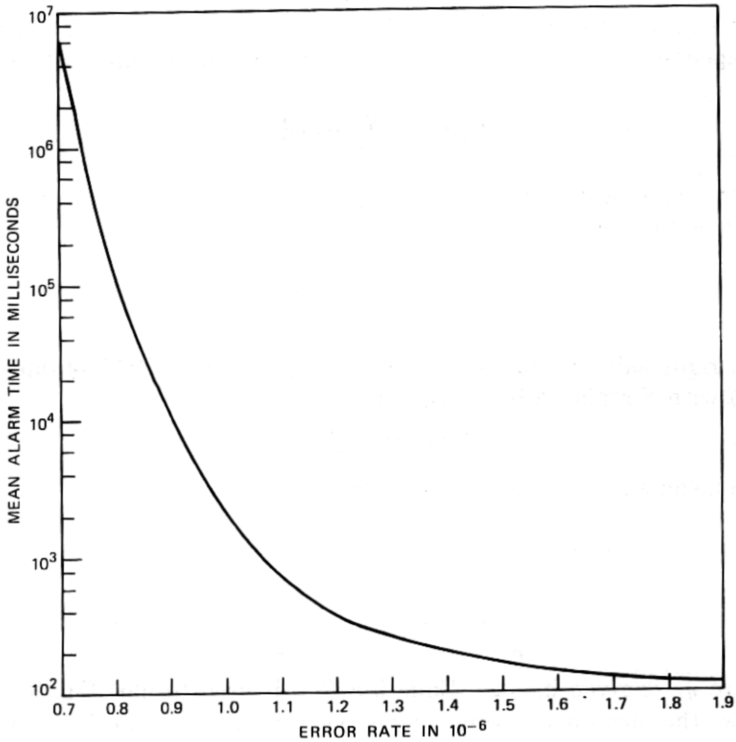

Fig. 3—Expected alarm time vs error rate.

first interval but not the second interval, followed by the event A_{l-2}. Let

$$a_l = P\{A_l\},$$

then

$$a_l = \mu(1 - \mu)a_{l-2} + (1 - \mu)a_{l-1} \qquad l \geqq 2. \tag{37}$$

The generating function $A(S)$ is defined as

$$A(S) = \sum_{l=0}^{\infty} a_l S^l \qquad -1 \leqq S \leqq 1.$$

Multiplying (37) by S^l and summing from $l = 2$ to infinity,

$$\sum_{l=2}^{\infty} a_l S^l = \mu(1 - \mu)S^2 \sum_{l=2}^{\infty} a_{l-2}S^{l-2} + (1 - \mu)S \sum_{l=2}^{\infty} a_{l-1}S^{l-1}. \tag{38}$$

Since $a_0 = a_1 = 1$, (38) can be written as

$$A(S) - S - 1 = \mu(1 - \mu)S^2A(S) + (1 - \mu)S[A(S) - 1],$$

then

$$A(S) = \frac{1 + \mu S}{1 - (1 - \mu)S - \mu(1 - \mu)S^2}.$$

Therefore, each $P\{A_l\}$, $l \geqq 2$, can be obtained by the method indicated in Appendix B. Let $\mathbf{X}$ be the number of times the waiting timer is triggered in l measuring periods, given that the event A_l is true, then

$$P\{\mathbf{X} = j | A_l\} = \frac{P\{\mathbf{X} = j, A_l\}}{P\{A_l\}}. \tag{39}$$

Let $P\{\cdot | 0\}$ denote the conditional probability assuming the counter does not overflow in the first measuring interval, and $P\{\cdot | 1, 0\}$ denote the conditional probability assuming the counter overflows in the first but not the second interval. The numerator of (39) can be written as

$$\begin{aligned} P\{\mathbf{X} = j, A_l\} &= P\{\mathbf{X} = j, A_l | 1\} P\{1\} + P\{\mathbf{X} = j, A_l | 0\} P\{0\} \\ &= \mu P\{\mathbf{X} = j, A_l | 1\} + (1 - \mu) P\{\mathbf{X} = j, A_l | 0\}. \end{aligned} \tag{40}$$

However,

$$\begin{aligned} P\{\mathbf{X} = j, A_l | 1\} &= P\{\mathbf{X} = j, A_l | 1, 1\} P\{1 | 1\} \\ &\qquad + P\{\mathbf{X} = j, A_l | 1, 0\} P\{0 | 1\} \\ &= 0 + (1 - \mu) P\{\mathbf{X} = j - 1, A_{l-2}\} \end{aligned} \tag{41}$$

$$P\{\mathbf{X} = j, A_l | 0\} = P\{\mathbf{X} = j, A_{l-1}\}. \tag{42}$$

Insert (41) and (42) into (40); then,

$$\begin{aligned} P\{\mathbf{X} = j, A_l\} &= \mu(1 - \mu) P\{\mathbf{X} = j - 1, A_{l-2}\} \\ &\qquad + (1 - \mu) P\{\mathbf{X} = j, A_{l-1}\}. \end{aligned} \tag{43}$$

Let

$$p_{j,l} = P\{\mathbf{X} = j, A_l\}.$$

Equation (43) can be written as

$$p_{j,l} = \mu(1 - \mu) p_{j-1,l-2} + (1 - \mu) p_{j,l-1}.$$

Following the derivation of (38), we obtain

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} p_{j,l} S_1^j S_2^l &= \mu(1 - \mu) S_1 S_2^2 \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} p_{j-1,l-2} S_1^{j-1} S_2^{l-2} \\ &\qquad + (1 - \mu) S_2 \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} p_{j,l-1} S_1^j S_2^{l-1}. \end{aligned} \tag{44}$$

Define the bivariate generating function $A(S_1, S_2)$ as

$$A(S_1, S_2) = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} p_{j,l} S_1^j S_2^l.$$

Since

$$p_{j,l} = 0 \qquad j > l,\ j = 1, 2, \cdots$$
$$p_{0,l} = (1-\mu)^l \qquad l = 0, 1, 2, \cdots$$
$$p_{1,1} = \mu,$$

eq. (44) can be reduced to

$$A(S_1, S_2) - \frac{1}{1-(1-\mu)S_2} - \mu S_1 S_2 = \mu(1-\mu)S_1 S_2^2 A(S_1, S_2) + (1-\mu)S_2\left[A(S_1, S_2) - \frac{1}{1-(1-\mu)S_2}\right].$$

Therefore,

$$A(S_1, S_2) = \frac{1+\mu S_1 S_2}{1-(1-\mu)S_2-\mu(1-\mu)S_1 S_2^2}. \tag{45}$$

An iterative expression similar to that given in Appendix B can be obtained for the evaluation of $p_{j,l}$, and, hence, $P\{\mathbf{X} = j|A_l\}$ according to (39). Note that $P\{\mathbf{X} = j|A_l\} = 0$ for $j > l/2 + 1$.

An example is given below. When $\epsilon = \epsilon_0$, $\mu = \frac{1}{2}$, it was shown in (24) that on the average six measuring periods are required for the VMR to generate an alarm. During the last three periods (011), the waiting timer is triggered once. It is desirable to find the distribution of **X** in the first three periods. From (37) and (45),

$$P\{A_3\} = 1 - 2\mu^2 + \mu^3$$
$$p_{03} = (1-\mu)^3$$
$$p_{13} = 3\mu(1-\mu)^2$$
$$p_{23} = \mu^2(1-\mu)$$
$$p_{33} = 0$$
$$E\{\mathbf{X}|A_3\} = \sum_{i=0}^{3} i \times P\{\mathbf{X} = i|A_3\} = \frac{\mu(1-\mu)(3-\mu)}{1-2\mu^2+\mu^3}.$$

At the threshold, $u = \frac{1}{2}$,

$$E\{\mathbf{X}|A_3\} = 1.$$

Thus, in the first three measuring intervals, the waiting timer is expected to be triggered once. In the last three intervals (011), the waiting timer is definitely triggered once. Hence, if the alarm occurs at the sixth measuring interval, then

$$E\{\mathbf{W}|\mathbf{M} = 6\} = 2. \tag{46}$$

Equation (46) says when $\epsilon = \epsilon_0$, the waiting timer shall be, on the average, triggered twice before an alarm is generated.

2.7 Statistics of alarm release and oscillation

To avoid oscillatory alarms near the error threshold, a release timer with duration $d > t_0$ is used to measure the violation rate when the VMR is in the alarm condition. The alarm is released only after the release timer times out and the counter does not overflow. Let ν be the probability of counter overflow during the measuring period d. From (12)

$$\nu \approx 1 - \Phi\left(\frac{n - \gamma d\epsilon}{\sqrt{\gamma d\epsilon(1 - \epsilon)}}\right).$$

Let $\mathbf{K}$ represent the number of measuring periods before the VMR stops alarming; i.e., the VMR will release the alarm at the $(\mathbf{K} + 1)$th period. Then,

$$\begin{aligned} h_k &= P\{\mathbf{K} = k\} \\ &= (1 - \nu)\nu^k \qquad k = 0, 1, 2, \cdots. \end{aligned} \tag{47}$$

Thus, $\mathbf{K}$ is governed by a geometric distribution with generating function

$$H(s) = \frac{1 - \nu}{1 - \nu S}. \tag{48}$$

The distribution of the alarm-release time $\mathbf{D}$ (assuming the error rate remains constant) will be derived first. Let $\mathbf{D}_i$ represent the time from the $(i - 1)$th to the ith counter overflow during the alarm state. The distribution of $\mathbf{D}_i$ is given by (29) and its generating function is

$$D_i(s) = \frac{\epsilon S^{1/\gamma}}{1 - (1 - \epsilon)S^{1/\gamma}}. \tag{49}$$

The alarm-release time is again given by a random sum

$$\mathbf{D} = \sum_{i=0}^{\mathbf{k}} \mathbf{D}_i + d, \tag{50}$$

where by definition, $\mathbf{D}_0 = 0$. Since $\mathbf{K}$ and the $\mathbf{D}_i$'s are independent, the generating function of $\mathbf{D}$ is

$$\begin{aligned} D(s) &= S^d \sum_{k=0}^{\infty} h_k \left(\frac{\epsilon S^{1/\gamma}}{1 - (1 - \epsilon)S^{1/\gamma}}\right)^{kn} \\ &= S^d H[D_i(S)]. \end{aligned} \tag{51}$$

The PDF of $\mathbf{D}$ is the compound distribution

$$f_{\mathbf{D}}(x) = \sum_{k=1}^{\infty} h_k \gamma \binom{\gamma(x - d) - 1}{kn - 1} \epsilon^{kn}(1 - \epsilon)^{\gamma(x-d)-kn}. \tag{52}$$

The mean and the variance of $\mathbf{D}$ are

$$\eta_d = \frac{\nu}{1 - \nu} \times \frac{n}{\epsilon\gamma} + d \tag{53}$$

$$\sigma_d^2 = \frac{\nu}{1 - \nu} \times \frac{n}{\epsilon^2\gamma^2} \times \left(\frac{n}{1 - \nu} + 1 - \epsilon\right). \tag{54}$$

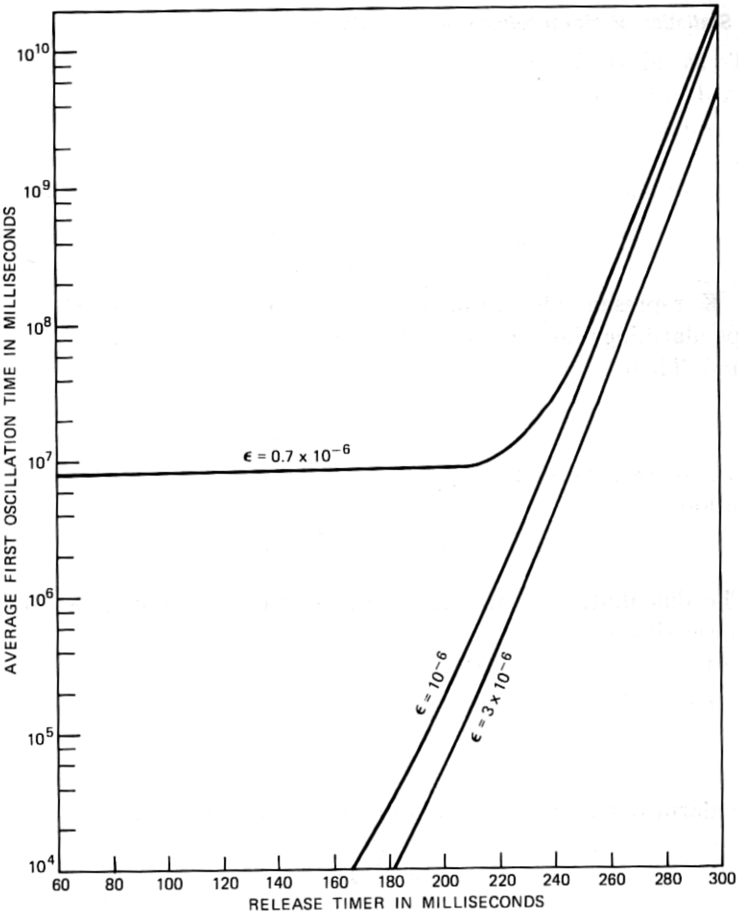

Fig. 4—Average first-oscillation time vs duration of release timer.

The first oscillation time $\mathbf{T}_a$, i.e., the time it takes for an alarming VMR to release and then generate another alarm, assuming the error rate remains constant, is

$$\mathbf{T}_a = \mathbf{D} + \mathbf{T}_0. \tag{55}$$

Its PDF is simply the convolution

$$f_{\mathbf{T}_a}(t) = f_{\mathbf{D}}(t) * f_{\mathbf{T}_0}(t).$$

The mean and the variance of $\mathbf{T}_a$ are

$$\eta_{t_a} = \eta_d + \eta_{t_0}. \tag{56}$$

$$\sigma_{t_a}^2 = \sigma_d^2 + \sigma_{t_0}^2. \tag{57}$$

Equation (56) is employed to plot Fig. 4 which shows the expected first oscillation time versus the duration of the release timer d with the error rate as the parameter. When $\epsilon = 10^{-6}$, if $d = 330$ ms, alarm oscillation is expected to occur once in 187 years; if $d = 1$ second, alarm oscillation is extremely unlikely to occur.

The time from an initial alarm state to the ith alarm oscillation is $i \times \mathbf{T}_a$, whose distribution can be easily obtained from that of $\mathbf{T}_a$.

2.8 Pseudorandom signal switching statistics

As described in 2.1, after the pseudorandom signal is switched in, if the VMR is back in-frame, immediately the violation counter is reset and starts counting again until the free-running release timer of duration d times out. Since the in-frame condition can occur anytime within the interval 0 to d, the time spent to count the violations is uniformly distributed between 0 and d. When the release timer times out, the number of violations counted is a mixture distribution obtained through randomization[5] of the parameter t in (30)

$$P\{\mathbf{N} = n \mid \boldsymbol{\varepsilon} = \epsilon\} = \int_0^d \frac{e^{-\gamma \epsilon t}(\gamma \epsilon t)^n}{n!} \times \frac{1}{d}\, dt$$

$$= \frac{1}{\gamma \epsilon d}\left(1 - e^{-\gamma \epsilon d} \sum_{i=0}^{n} \frac{(\gamma \epsilon t)^i}{i!}\right). \tag{58}$$

The PDF $f_\epsilon(\epsilon)$ of the error rate $\boldsymbol{\varepsilon}$ is usually unknown. If $f_\epsilon(\epsilon)$ is given or can be estimated empirically, (58) can be randomized by $f_\epsilon(\epsilon)$.

$$P\{\mathbf{N} = n\} = \int_0^{\cdot} P\{\mathbf{N} = n \mid \boldsymbol{\varepsilon} = \epsilon\} f_\epsilon(\epsilon) d\epsilon, \tag{59}$$

where the upper integration limit is determined by the domain of ϵ. From (59), the probability $P\{\mathbf{N} < n\}$ that the counter does not overflow, i.e., the pseudorandom signal will be switched out, can be evaluated.

2.9 Generalizations

All the above derivations are general enough so that if one requires the counter to overflow consecutively more than twice (with the waiting timer triggered each time the counter overflows) before an alarm is generated, the results can be easily extended. For example, if the VMR generates an alarm after k consecutive counter overflows, then (24) becomes

$$E\{\mathbf{M}\} = \frac{1 - \mu^k}{(1 - \mu)\mu^k},$$

and the probability coefficients are

$$p_m = 0 \qquad m = 0, 1, \cdots, k-1$$
$$p_k = \mu^k$$
$$p_{k+1} = p_{k+2} = \cdots = p_{2k} = (1-\mu)\mu^k$$
$$p_m > p_{m+1} \qquad m \geqq 2k+1$$

III. DISCUSSION

As discussed in the introduction, one of the main functions of the VMR is to generate alarms when it detects that the line performance is below a predetermined objective. However, the digital line performance objective is usually set in terms of a threshold bit-error rate which cannot be directly measured in service. Equation (8) establishes the relationship between the bit-error rate and the parity-violation rate for a digital line employing parity-checking digits. Figure 1 shows that for the parity-check structure used in the T4M system and for bit-error rates below 10^{-3}, there is almost a one-to-one correspondence between a bit error and a parity violation. This implies that the parity-checking scheme is effective in determining digital transmission line performance.

When the T4M VMR parity violations exceed a specified threshold in two consecutive measuring intervals, an alarm is generated. This is normally followed by an automatic transfer of the failed line to a spare line if the latter is available. In general, each spare line will protect several service lines to reduce system cost. Thus, a so-called "hard" alarm threshold, which clearly distinguishes between error rates slightly above and below the threshold, is desirable because it is unlikely to cause an alarm at error rates below the threshold. In this case, the spare line will be available to protect more serious failures on other service lines. It also takes less time for a VMR with a hard threshold to generate alarms when the error rates are above the threshold. Equation (14) gives the probability of the parity-violation counter overflow as a function of the error rate and the duration of the measuring interval. Figure 2 is a plot of (14) and exhibits the desirable hard threshold characteristics. As the error rate varies from 0.6×10^{-6} to 2×10^{-6}, the probability of counter overflow changes from 0.0001 to 0.999.

When a catastrophic failure occurs on a line, its VMR should generate an alarm as soon as possible so that an automatic transfer to a spare line can take place without trunk disconnection. When an error rate just above the threshold is detected, little harm will be done if the VMR takes longer to announce an alarm. Equation (35) obtains the mean alarm time as a function of the error rate. From Fig. 3 it can be seen that the mean alarm time of the T4M VMR reduces very fast with increasing error rates. Note that if an error rate of 0.7×10^{-6} (slightly below the threshold) persists for hours, eventually an alarm will be generated because the alarm threshold is not infinitely hard.

The amount of hysteresis required in releasing an alarm is an important part of VMR design. The release timer should be long enough so that oscillation between alarm request and alarm release is unlikely to occur. It should also be short enough so that alarms are not unnecessarily prolonged. Equation (56) gives the expected oscillation time as a sum of the mean alarm time and the mean release time, both of which are functions of the error rate and the length of the release timer. Figure 4 shows that when the duration of the release timer is greater than three times that of the parity-violation measuring interval, alarm oscillation is not likely to occur at any constant error rates. This is due to the fact that the mean alarm time is large for error rates below the alarm threshold while the mean release time is long for error rates above the alarm threshold.

APPENDIX A

The T1 Outstate (1.544 Mb/s) VMR counts 16 bipolar violations (violations occurring within a 0.3-ms interval are counted only once) in 85 ms to generate an alarm. The T2 (6.312 Mb/s) VMR generates a low-error alarm if it counts 32 bipolar violations in 5 seconds (violations occurring within a 3.2-μs interval are counted only once). Since the error rates of interest are near the threshold, it can be assumed that no two violations occur "close" to each other. The 3A-RDS (44.736 Mb/s) VMR generates an alarm if it counts 31 parity violations in 2 seconds. These alarm rules are simpler than that for the T4M VMR, hence, the alarm statistics of these VMRs are also easier to derive. For the VMR of each system, a probability of counter overflow μ can be derived as in (14). This probability is also the probability of alarm. The three alarm rules have identical mathematical models; hence, no separate discussions are necessary.

Let $\mathbf{M}$ represent the number of elapsed measuring periods before the VMR generates an alarm; i.e., the VMR will generate an alarm at the $(\mathbf{M} + 1)$th period. Then,

$$\begin{aligned} p_m &= P\{\mathbf{M} = m\} \\ &= \mu(1 - \mu)^m \qquad m = 0, 1, 2, \cdots. \end{aligned}$$

$\mathbf{M}$ is governed by a geometric distribution. Most other statistics discussed in Section II can be derived similarly.

Because of the simplicity of the geometric distribution, given a probability of alarm p, the number of elapsed periods k before the probability p is reached can be obtained explicitly,

$$p \leqq \sum_{m=0}^{k} p_m$$

$$= \sum_{m=0}^{k} \mu(1-\mu)^m$$

$$= 1 - (1-\mu)^{k+1}.$$

Therefore,

$$k \leqq \frac{ln(1-p)}{ln(1-\mu)} - 1.$$

In each of the first k measuring intervals, the counter will not overflow when the measuring timer times out. Thus, the total alarm probability as a function of elapsed time can be plotted easily as opposed to evaluating (36) for the T4M VMR.

APPENDIX B

This appendix derives an iterative expression to calculate the probability coefficients p_i's discussed in Section 2.3. Specifically, given that

$$P(S) = \sum_{i=0}^{\infty} p_i S^i \tag{60}$$

and

$$P(S) = \frac{\sum_{j=0}^{m} a_j S^j}{1 + \sum_{i=1}^{n} b_i S^i} \tag{61}$$

it is desired to obtain the p_i's in terms of the a_i's and the b_i's. Let the denominator of (61) be expanded as follows

$$\frac{1}{1 + \sum_{i=0}^{n} b_i S^i} = \sum_{i=0}^{\infty} C_i S^i.$$

C_i, $i = 0, \cdots, n-1$, can be determined through long division or by comparing the coefficients of the S^i's in

$$1 = \left(1 + \sum_{i=1}^{n} b_i S^i\right)\left(\sum_{i=0}^{\infty} C_i S^i\right).$$

For $i \geqq n$,

$$C_i = -\sum_{k=1}^{n} b_k C_{i-k}.$$

Therefore

$$P(S) = \sum_{j=0}^{m} a_j S^j \times \sum_{i=0}^{\infty} C_i S^i. \tag{62}$$

Compare (60) and (62), p_i, $i = 0, 1, \cdots, m - 1$ can be determined easily. For $i \geqq m$

$$\begin{aligned} p_i &= \sum_{h=0}^{m} a_h C_{i-h} \\ &= \sum_{h=0}^{m} a_h \left[- \sum_{k=1}^{n} b_k C_{i-h-k} \right] \\ &= - \sum_{h=0}^{m} \sum_{k=1}^{n} a_h b_k C_{i-h-k}. \end{aligned}$$

p_i can be calculated by computer without knowing p_{i-1}, p_{i-2}, etc.

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