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Distinguishing Stable Probability Measures Part II: Continuous Time

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A sample function from one of two stable, stationary, independent-increment processes is observed for a finite time interval. For differing location, characteristic index, skewness, or scale, the probabilities measures induced by the process under either hypothesis are found to be mutually orthogonal. By suitably modifying the Lévy measure associated with each probability measure, continuous-time tests for differing characteristic indices, skewness, or scale parameters can be posed as nonsingular detection problems; distinguishing location remains a singular detection problem. For the nonsingular problems, the likelihood functional is found explicitly, and performance limitations are determined. As an alternative approach, the observed sample function is sampled at discrete time instants over a finite time interval, and the performance of log likelihood test is studied as a function of sample spacing with a fixed, total number of observations.

I. INTRODUCTION

In this paper, the work begun in Part I[1] on discrete-time hypothesis testing of stable probability measures is extended to continuous time. In contrast to the earlier work, analytic closed-form expressions are found for both the log likelihood functional and Chernoff-type upper and lower bounds on various error probabilities for the log likelihood test. As in Part I, the singular role played by the gaussian probability measure within the family of stable probability measures is emphasized, both in terms of the form of the log likelihood functional and the expressions for Chernoff-type bounds on error probabilities. The earlier work dealt with observing N samples from a stable process with one of two sets of parameters at time instants Δt apart; here, we fix the observation interval at duration T, and allow the number of observations to become infinite while the spacing between samples shrinks to zero ($N \to \infty$, $\Delta t \to 0$, such that $N \cdot \Delta t = T$).

Section II briefly reviews some properties of independent-increment processes and infinitely divisible distributions that were touched on in Part I. Section III draws on this tutorial material by considering an elementary hypothesis-testing problem for discriminating between two Poisson distributions with differing parameters.* Section IV briefly reviews some work by Newman[2,3] and Newman and the author[4,5] on calculating log likelihood functionals and Chernoff-type bounds on error probabilities for the path-space probability measures induced by independent-increment processes. These results are used in Section V to show if one or more of the parameters of the two stable-probability measures ($0 < \alpha < 2$) differs, then the two path-space measures are mutually orthogonal. Section VI develops one remedy to this so-called singular detection by modifying the Lévy measure of the two distributions to account for the real physical limitation that the process can only be observed to within an accuracy intrinsic in all measurement apparatus. Section VII considers a different but related issue, where the observed sample function is sampled at discrete time instants over a finite time interval, and the performance of the log likelihood test is studied as the sample spacing shrinks to zero; this allows one to trade off the sample spacing, or the rate at which samples are observed, for the total duration of the observation interval, or the total number of samples.†

The results developed here are novel in that one can immediately ascertain explicit bounds on the performance of the likelihood ratio test, while it is *not* clear how to do this after reading the literature (e.g., see Refs. 6 through 10). The method of proof here relies on probabilistic semigroup tools or on the explicit nature of the sample paths of an independent-increment process, and this appears to be novel when contrasted with such approaches as those referenced above.

II. MATHEMATICAL PRELIMINARIES ‡

Let $r_j(t)$ ($j = 0, 1$) be a scalar real-valued random process, with right continuous sample paths with left-hand limits everywhere defined. More explicitly, let $r_j(t)$ be the sum of a deterministic drift process, $\delta_j t$, and N independent Poisson processes (labeled by k, $1 \leqq k \leqq N$), where each Poisson process has rate λ_{jk} and hops of height h_{jk}. In words, $r_j(t)$ has simple jump discontinuities of heights h_{jk}, $1 \leqq k \leqq N$, at random times. The characteristic functional of

* The results in Sections III through VI were first announced in Proceedings of the 13th Annual Allerton Conference, University of Illinois, Champaign-Urbana, Illinois, October 1–3, 1975, pp. 234–239.

† The results in Section VII were first announced in Proceedings of the 1976 Johns Hopkins Conference on Information Sciences and Systems, Baltimore, Maryland, March 30–April 2, 1976, pp. 151–154.

‡ See Ref. 1, Section 3.1 and its list of references for much more information.

$r_j(t)$ is easily seen to be $(t > s)$,

$$E[e^{iv[r_j(t)-r_j(s)]}] = \exp\left\{(t-s)\left[iv\delta_j + \sum_{k=1}^{N} \lambda_{jk}(e^{ivh_{jk}} - 1)\right]\right\}.$$

If we now pass to the limit of an infinite sum of Poisson processes, then the jump amplitudes $\{h_{jk}\}$ take on a continuum of values, and the characteristic functional becomes $(t > s)$

$$E[e^{iv[r_j(t)-r_j(s)]}] = \exp\left\{(t-s)\left[iv\delta_j + \int_{u\neq 0} (e^{ivu} - 1)d\nu_j(u)\right]\right\},$$

where ν is called the *Lévy measure* associated with the path-space measure of r_j, and generalizes the rate parameter set $\{\lambda_{jk}\}$; $(t-s)\int_{u\in A} d\nu_j(u)$ is the expected number of jumps of r_j whose amplitude falls in the set A, in a time interval of duration $(t-s)$. Lévy and Khinchin showed the following remarkable generalization of this heuristic development:

Theorem (Ref. 11, p. 76): *Let $r_j(t)$ be an R^n valued random process with independent increments. Then*

$$E(\exp\{iv^{TR}[r_j(t) - r_j(s)]\}) = \exp\left\{(t-s)\left[iv^{TR}\delta_j - \tfrac{1}{2}v^{TR}S_jv \right.\right.$$
$$\left.\left. + \int_{u\neq 0}\left(\exp(iv^{TR}u) - 1 - \frac{iv^{TR}u}{1+u^{TR}u}\right)d\nu_j(u)\right]\right\},$$

where $\delta_j \in R^n$, S_j is an $n \times n$ positive semidefinite matrix, and

$$\int_{u\neq 0} \frac{u^{TR}u}{1+u^{TR}u}\, d\nu_j(u) < \infty.$$

In words, any independent increment is the sum of three independent processes: (*i*) a purely deterministic drift process, completely specified by δ_j, (*ii*) a purely nondeterministic gaussian process with zero drift and almost surely continuous sample paths, specified by S_j, and (*iii*) a purely nondeterministic jump process with zero drift, a sum of independent Poisson processes with different rates and jump amplitudes, specified by ν_j.

Historically, the mathematical study of independent increment processes concentrated first on the purely gaussian case ($\nu_j = 0$); then on the purely stable case ($S_j = 0$, $d\nu_j = d\mu(\theta)dr/r^{\alpha+1}$, $0 < \alpha < 2$, where μ is a positive measure on the unit sphere in R^n and $[r, \theta]$ are polar coordinates in R^n); and lastly on the general case, building on the insight gained in the first two cases.[12] A second reason for wishing to study the gaussian and stable ($0 < \alpha < 2$) probability measures is that they arise naturally from studying limiting distributions of suitably scaled and translated sums of independent, identically distributed, random variables in the central limit theorem, and have found application in modeling noise in communication channels such as telephone lines.[13] These two reasons, as well as others, provide the major impetus for the study to follow. The richness of the structure of independent increment processes suggests they may find more and more application in model building as their properties become more widely known.

III. DISTINGUISHING POISSON PROCESSES

In this section, $r_j(t)$ $(j = 0, 1)$ is observed on the interval $[0, T)$, and is the sum of a purely deterministic drift process (specified by δ_j) and a purely nondeterministic Poisson process (specified by rate λ_j and jump amplitude h_j). What is the log likelihood functional, and what is its performance?

First, suppose the Poisson process has the same jump amplitude under either hypothesis, but the drifts differ. Then it is straightforward to show that the two probability measures P_0 and P_1, associated with r_j under hypothesis H_j, are mutually orthogonal, so (*i*) observing r over any finite interval, the log likelihood functional takes on the value $+\infty$ if H_1 is true, $-\infty$ if H_0 is true, and (*ii*) the probability of incorrectly choosing one hypothesis when the other is true is zero. The reason for this is clear on physical grounds: the Poisson component has constant sample functions with simple jump discontinuities at random times, while the drift process is continuous with constant slope. Thus, ignoring the jumps in the observation process, the slope of the continuous part of the sample path is δ_j, and to discriminate between the two hypotheses is now trivial. From this point on, therefore, it is assumed $\delta_1 = \delta_0$ and, without loss of generality, set $\delta_j = 0$ $(j = 0, 1)$.

What if the Poisson processes have different jump amplitudes? As soon as one or more jumps occur, it is possible to discriminate perfectly between the two processes, since the size of the jump h_j is associated with hypotheses H_j. To avoid this indeterminancy, it is assumed from this point on $h_1 = h_0 = 1$. Thus, P_0 and P_1, the probability measures associated with H_j, are mutually absolutely continuous.

Lemma 1: Let r_j be as just defined. Let

$$P_j'\left[r_j\left(\frac{k+1}{n}T\right)\middle|r_j\left(\frac{k}{n}T\right)\right], \qquad 0 \leqq k \leqq n-1;\ j = 0, 1$$

denote the conditional probability of r_j at time $[(k+1)/n]T$, given r_j

at time $(k/n)T$. *Then* $[r_j(0) = 0 \text{ a.s.};\ j = 0, 1]$,

$$(i)\quad \Lambda = \ln \frac{dP_1}{dP_0}(r) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \ln \frac{P_1'}{P_0'}(r).$$

$$(ii)\quad H_q(P_0, P_1) = \lim_{n\to\infty} [H_q(P_0', P_1')]^n.$$

Proof: The proof follows from standard limit theorems (Ref. 7, Lemma 1.1), Q.E.D.

We now explicitly evaluate the limits in Lemma 1:

Proposition 2: Given the conditions of Lemma 1,

$$(i)\quad \Lambda = \int_0^T \left\{ -(\lambda_1 - \lambda_0) + \sum_{k=0}^{N_r} \left[\ln \left(\frac{\lambda_1}{\lambda_0} \right) \right] \delta(t - t_k) \right\} dt$$

$$= \int_0^T \left[-(\lambda_1 - \lambda_0) dt + \ln \left(\frac{\lambda_1}{\lambda_0} \right) dr_t \right],$$

where N_r is the a.s. finite number of time instants $\{t_k\}$ where r_t changes state.

$$(ii)\quad H_q(P_0, P_1) = \exp\{-T[q\lambda_1 + (1 - q)\lambda_0 - \lambda_1^q \lambda_0^{1-q}]\}.$$

Proof:

(*i*) Given r_j at time $(k/n)T$, it will remain in that state in the next time interval (T/n) with probability $1 - \lambda_j T/n + o(T/n)$, and will increase by one with probability $\lambda_j(T/n) + o(T/n)$. The desired result now follows Lemma 1.

(*ii*) If r_j changes its state in the next time interval of duration (T/n), then

$$H_q(P_0', P_1') = \frac{T}{n} \lambda_1^q \lambda_0^{1-q} + o(T/n);$$

while if r_j stays in its present state in the next (T/n) time units, then

$$H_q(P_0', P_1') = 1 - \frac{T}{n}[q\lambda_1 + (1 - q)\lambda_0] + o(T/n).$$

$$\therefore\quad H_q(P_0', P_1') = \exp\left[-\frac{T}{n}(q\lambda_1 + (1 - q)\lambda_0 - \lambda_1^q \lambda_0^{1-q}) \right] + o(T/n),$$

$$\therefore\quad H_q(P_0, P_1) = \exp\{-T[q\lambda_1 + (1 - q)\lambda_0 - \lambda_1^q \lambda_0^{1-q}]\},$$

where the last step follows from Lemma 1.

Q.E.D.

Recall from Part I that a crude bound on the total probability of error P_E for a log likelihood ratio test is provided by

$$\tfrac{1}{4}\min(\pi_1, \pi_0)H_{\frac{1}{2}}^2 \leqq P_E \leqq \sqrt{\pi_1, \pi_0}H_{\frac{1}{2}},$$

where π_j is the *a priori* probability hypothesis that j is true. Here,

$$H_{\frac{1}{2}} = \exp\left[-T(\sqrt{\lambda_1} - \sqrt{\lambda_0})^2/2\right]$$

and hence for fixed T, one would like to have the difference in the square roots of the rates as large as possible.

To gain further insight into $H_q(P_0, P_1)$, we rewrite it as the expectation of a third Poisson process. Let $x_q(t)$ be a Poisson process with rate $\lambda_1^q\lambda_0^{1-q}$, hops of height $+1$, and $x_q(0) = 0$ a.s. (intuitively, the probability measure P_q associated with x_q has support on the common support of P_0 and P_1).

Proposition 3:

$$\begin{aligned} H_q(P_0, P_1) &= \int dP_q \exp\left\{-\int_0^T D[x_q(t)]dt\right\} \\ &= E_{x_q}\left(\exp\left\{-\int_0^T D[x_q(t)]dt\right\}\right), \\ D(x_q) &= q\lambda_1 + (1-q)\lambda_0 - \lambda_1^q\lambda_0^{1-q}. \end{aligned}$$

Proof: The proof follows from the definition of D, P_q, and x_q.

Q.E.D.

To the best of our knowledge, this result is new, and will be generalized in the following section and elsewhere.[4,5] Its significance lies in the fact that there exists a large body of results in both the mathematics and physics literature for studying properties of expectations of multiplicative functionals of random processes, so called Feynman-Kac functionals; now we can immediately draw on this body of knowledge.

IV. DISTINGUISHING INDEPENDENT INCREMENT PROCESSES

In this section, the results of Section III are extended to arbitrary independent increment processes. Here, $r_t \in R^n$ is observed over $[0, T)$, and has one of two sets of parameters (δ_j, S_j, ν_j) $(j = 0, 1)$. As before, define for $0 < q < 1$,

$$dh_q(P_0, P_1) = \left(\frac{dP_1}{d\mu}\right)^q \left(\frac{dP_0}{d\mu}\right)^{1-q} d\mu,\ P_1, P_0 \ll \mu$$

$$H_q(P_0, P_1) = \int dh_q(P_0, P_1),$$

where H_q is the Kakutani product associated with P_0, P_1. Next, it is useful to define a nonnegative measure $j_q(\nu_0, \nu_1)$ [the generalization of the point measure at $+1$ with mass $q\lambda_1 + (1-q)\lambda_0 - \lambda_1^q\lambda_0^{1-q}$ in Section III],

$$dj_q(\nu_0, \nu_1) = qd\nu_1 + (1-q)d\nu_0 - dh_q(\nu_0, \nu_1),$$
$$J_q(\nu_0, \nu_1) = \int dj_q(\nu_0, \nu_1),$$

and J_q is nonnegative and may be infinite, since ν_0 or ν_1 or both may not be finite measures. If $J_q < \infty$, it is convenient to define

$$\delta_j' = \delta_j - \int_{u\neq 0} \frac{u}{1 + u^{TR}u}\, d\nu_j(u),$$
$$\delta_q = q\delta_1 + (1-q)\delta_0 - \int_{u\neq 0} \frac{u}{1 + u^{TR}u}\, dj_q(\nu_0, \nu_1).$$

Finally, if $S_1 = S_0 = S$, and $J_q < \infty$, a third independent increment process $x_q(t)$ is defined with parameters $[\delta_q, S, h_q(\nu_0, \nu_1)]$.

Theorem 4: For P_0 and P_1 not to be mutually orthogonal, it is necessary and sufficient for the following three conditions to hold:

(*i*) $J_q(\nu_0, \nu_1) < \infty$
(*ii*) $S_1 = S_0 = S \geqq 0$
(*iii*) $\delta_q \in$ *range* (S).

If these three conditions are satisfied, then

$$\text{(a)}\ \Lambda(r_t) = \int_0^T \left[\int_{u\neq 0} \ln \frac{d\nu_1}{d\nu_0}(u) d_u r_t - \int_{u\neq 0} (d\nu_1 - d\nu_0) dt \right] + \delta_q^{TR} S^{-1}[r_T - j_T - \tfrac{1}{2}(\delta_0' + \delta_1')],$$

where $d_u r_t$ assigns a point mass at time instants where $r_t - r_{t-} = u$, i.e., where r_t hops with amplitude u, and j_t is the jump process component of r_t.

$$\text{(b)}\ H_q(P_0, P_1) = \exp\left[-TJ_q(\nu_0, \nu_1) - \frac{T}{2} q(1-q)\delta_q^{TR} S^{-1}\delta_q \right].$$

Proof (*sketched*)*: The proof is broken into two parts, one part dealing with the jump process, the other with the gaussian process (including drift). The part dealing with the gaussian component is classical,[6] and yields conditions (*ii*) and (*iii*), above. The main method employed in showing condition (*i*) for the jump-process component is to approximate the jump process by a sum of independent Poisson processes with different rates and jump amplitudes. As more and more Poisson processes are included in this sum, it can be shown that the approxi-

* From a detailed proof in Ref. 5.

mation converges weakly to the actual jump process. The Kakutani inner product of the probability measures of the approximations is simply the product of the Kakutani inner product associated with Poisson processes of the same jump amplitude (but possibly different rates); again, the delicate part of the proof is to show this approximation converges to the actual Kakutani inner product of the path-space probability measures of the two independent increment processes.

The program is to use this theorem in the remainder of this paper to exhibit the log likelihood functional and ascertain bounds on its performance in hypothesis testing for stable processes. Skorokhod[7–9] has obtained conditions (*ii*) and (*iii*) in Theorem 2, and instead of condition (*i*) obtained two conditions which must hold:

$$\int_{|g-1|>\frac{1}{2}} (g-1)d\nu_0 < \infty \quad \text{and} \quad \int_{|g-1|\leqq\frac{1}{2}} (g-1)^2 d\nu_0 < \infty,$$

where $g = (d\nu_1/d\nu_0)$; it is easy to show these two requirements are equivalent to $J_{\frac{1}{2}}(\nu_0, \nu_1) < \infty$. Hence, these conditions appear simpler than those of Skorokhod. Moreover, it is obvious how to use J_q to determine performance limitations, while it is not obvious at first glance how to apply Skorokhod's work. Also, the method of proof is different and may be easier to follow.

Finally, it is instructive to rewrite H_q as a Feynman-Kac type of functional of x_q:

Proposition 5: Let x_q be a stationary independent increment process with parameters (δ_q, S, h_q) as defined previously. Then,

$$H_q(P_0, P_1) = E_{x_q}\left[\exp\left(-\int_0^T D(x_q)dt\right)\right]$$
$$= \int dP_q \exp\left[-\int_0^T D(x_q)dt\right],$$

where

$$D(x_q) = q(1-q)\delta_q^{TR} S^{-1}\delta_q/2 + \int_{u\neq 0} dj_q.$$

Proof: The proof follows immediately from the definitions of D, x_q, P_q. Q.E.D.

Again, note that

$$D(x_{\frac{1}{2}}) = \frac{1}{2}\left[(\delta_q^{TR}/2)S^{-1}(\delta_q/2) + \int_{u\neq 0}\left(\sqrt{\frac{d\nu_1}{d\mu}} - \sqrt{\frac{d\nu_0}{d\mu}}\right)^2 d\mu\right], \quad \nu_1, \nu_0 \ll \mu$$

can be immediately used to provide a crude upper and lower bound on the total probability of error. As in the Poisson case, one desires the differences in the square roots of the Lévy measures (suitably defined) as large as possible, for good performance.

V. DISTINGUISHING STABLE PROCESSES

Let $x_j(t)$ $(j = 0, 1)$ be a scalar real-valued representation of a stable $(0 < \alpha < 2)$, stationary, independent increment process, $t \in [0, T)$, with characteristic functional $[x_j(0) = 0$ a.s.; $j = 0, 1]$.*

$$E(e^{ivx_j(t)}) = \exp\left\{t\left[i\delta_j v + \int_{u\neq 0}\left(e^{ivu} - 1 - \frac{ivu}{1+u^2}\right)d\nu_j(u)\right]\right\}$$

$$\nu_j(u) = \begin{cases}\nu_j^- = c_-^j|u|^{-\alpha_j} & u < 0\\ \nu_j^+ = -c_+^j u^{-\alpha_j} & u > 0.\end{cases}$$

Following Section IV, it is clear that $J_q(\nu_0, \nu_1)$ diverges (to $+\infty$), from simply substituting in the explicit form for ν_j and carrying out the calculations. Hence, P_0 and P_1 are mutually orthogonal if one or more of the parameters differ, the log likelihood functional is either $+\infty$ or $-\infty$ on hypothesis one or zero, respectively, and the probability of incorrectly choosing one hypothesis when the other is true is zero.

Since $J_q(\nu_0, \nu_1)$ diverges because ν_j diverges as $|u| \to 0$, this suggests that being able to observe the process perfectly, down to jumps of vanishingly small amplitudes, may be the mathematical reason for singular detection; but therein lies the flaw: it may well be physically impossible (the mathematical model is inadequate) to achieve this. Frost[10] apparently first popularized this idea in the engineering literature; here we reach the same conclusions by entirely different methods. Sections VI and VII deal with two distinct methods for overcoming these flaws in the mathematical model.

VI. DISTINGUISHING PSEUDO-STABLE PROCESSES

Let $x_j(t)$ $(j = 0, 1)$ be a scalar real-valued representation of a stationary independent increment process just as in Section V, except that the Lévy measure is now written as

$$\nu_j(u) = \begin{cases} c_-^j|u|^{-\alpha_j} & u < -L\\ \lambda^j(u) & -L < u < 0\\ \lambda_+^j(u) & 0 < u < R\\ -c_+^j u^{-\alpha_j} & R < u,\end{cases}$$

where

$$\int_{-L}^{0-}\frac{u}{1+u^2}\,d\lambda_-^j(u) < \infty, \quad \int_{0+}^{R}\frac{u}{1+u^2}\,d\lambda_+^j(u) < \infty,$$

and

$$\delta_j' = \delta_j - \int_{u\neq 0}\frac{u}{1+u^2}\,d\nu_j(u).$$

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* The case $\alpha = 2$ is well known[6–10] and, for brevity, is not included here.

λ_-^j, λ_+^j are absolutely continuous with respect to Lebesgue measure, and ν_j is nondecreasing on $(-\infty, 0-)$, nonincreasing on $(0+, \infty)$. The limit, as both R and L approach zero, of a sequence of such processes can be shown to converge weakly to a stable process, and hence these processes are christened pseudo stable processes. Here L and R quantify that the fact that no negative jumps can be observed with amplitude less than L, no positive jumps can be observed with amplitude less than R. Both the properties of the sample functions and the one-dimensional distributions are radically different here from stable processes: (*i*) pseudo-stable process sample functions are of bounded variation w.p.1, with only finitely many nonzero jumps in any finite time interval; stable process sample functions are of either unbounded $(1 < \alpha < 2)$ or bounded $(0 < \alpha < 1)$ variation w.p.1, with the set of time instants at which nonzero jumps occur being dense in any finite time interval, and (*ii*) the set of one-dimensional distributions of pseudo-stable processes is clearly not closed under convolution, which was the defining property of stable distributions, but the asymptotic tail behavior is the same, since

$$\Pr\left[x_j(t=1) > x\right] \sim 0\left(\int_x^\infty d\nu_j(u)\right),$$

$$\Pr\left[x_j(t=1) < -x\right] \sim 0\left(\int_{-\infty}^{-x} d\nu_j(u)\right).$$

For this special case, it is straightforward to show that $J_q(\nu_0, \nu_1) < \infty$, and hence condition (*i*) of Theorem 2 is satisfied. However, δ_q is not in general in the range of $S(=0)$, and again singular detection is possible. The reason is clear on physical grounds (cf. Section III, the Poisson case): the slope of the sample paths of $x_j(t)$ is δ_j', ignoring the jump discontinuities, and hence it is trivial to discriminate between two pseudo-stable processes with different drifts. Two approaches are available: either let S be nonzero, which we do not pursue here because this seems ad hoc, having introduced L, R, already, or make the drifts match, $\delta_1' = \delta_0'$, which we assume from this point on.

The log likelihood functional is thus

$$\Lambda(r_t) = \int_0^T dt\left[\int_{-\infty}^{-L} \ln\,(d\nu_1/d\nu_0)dr_t + \int_R^\infty \ln\,(d\nu_1/d\nu_0)dr_t\right] + \int_0^T dt\left[\int_{-\infty}^{-L}(-d\nu_1 + d\nu_0) + \int_R^\infty(-d\nu_1 + d\nu_0)\right],$$

where, for simplicity, it was assumed

$$\lambda_-^j = c_-^j L^{-\alpha_j}, \quad \lambda_+^j = -c_+^j R^{-\alpha_j}.$$

As expected, the form of the log likelihood functional is quite sensitive to whether $\alpha = 2$ or $0 < \alpha < 2$ (e.g., see Refs. 7–10 for $\alpha = 2$).

To obtain upper and lower bounds on the probabilities of an error of the first or second kind, and on the total probability of error, the Kakutani inner product H_q must be calculated. Assuming $\lambda_-^j = c_-^j L^{\alpha_j}$, $\lambda_+^j = -c_+^j R^{-\alpha_j}$, the result is

$$H_q(P_0, P_1) = \exp[-TJ_q(\nu_0, \nu_1)],$$

$$J_q(\nu_0, \nu_1) = q(c_-^1 L^{-\alpha_1} + c_+^1 R^{-\alpha_1}) + (1-q)(c_-^0 L^{-\alpha_0} + c_+^0 R^{-\alpha_0}) - \frac{(\alpha_0 c_-^0)^{1-q}(\alpha_1 c_-^1)^q}{q\alpha_1 + (1-q)\alpha_0} L^{-q\alpha_1-(1-q)\alpha_0} - \frac{(\alpha_0 c_+^0)^{1-q}(\alpha_1 c_+^1)^q}{q\alpha_1 + (1-q)\alpha_0} R^{-q\alpha_1-(1-q)\alpha_0},$$

$$H_q(P_0, P_1) = E_{x_q}\left\{\exp\left[-\int_0^T D(x_q)dt\right]\right\},$$

$$D(x_q) = J_q.$$

In summary, discriminating between Wiener processes ($\alpha = 2$) with different variances leads to singular detection, while if the variances are identical then the detection problem is nonsingular.[6–10] Discriminating between stable processes ($0 < \alpha < 2$) with one or more different parameters leads to singular detection. If the Lévy measure is modified to be a finite measure, then if the drifts differ, singular detection occurs, while if the drifts are identical, then the detection problem is nonsingular.

VII. DISTINGUISHING SAMPLED STABLE PROCESSES

The previous sections show that it is quite easy to find examples of continuous time singular-detection problems. In this section, it is assumed that N samples of a stable process with one of two sets of parameters are observed, and we wish to study the effect of choosing the sample spacing and the total length of the observation interval on the Kakutani inner product H_q; the goal is to make H_q as small as possible.

Attention is confined solely to scalar processes from this point on. The distribution of $x_j[(k+1)\Delta t] - x_j(k\Delta t)$ is given by $P_j(\Delta t; \delta_j, S_j, \nu_j)$. The Kakutani inner product of the new two discrete time distributions is

$$H_q(T, \Delta t) = \left[\int (dP_1/d\mu)^q (dP_0/d\mu)^{1-q} d\mu\right]^{T/\Delta t}.$$

For $\Delta t \downarrow 0$ or $T \to \infty$, with $(T/\Delta t) \equiv N$ fixed in both cases, fixed, that H_q can approach one, some number between zero and one [say

e^{-kN}, where $k = k(\delta_1, \delta_0, S_1, S_0, \nu_1, \nu_0)]$, or zero. It is obvious that if the two continuous time independent increment process path-space measures are *not* mutually orthogonal, then the only approach to reducing H_q is to fix Δt and increase T. However, if the two continuous time independent increment processes have mutually orthogonal path-space measures, then it is possible to reduce H_q by decreasing Δt with $(T/\Delta t)$ fixed. To state the result, a lemma is needed:

Lemma 6. If μ is an infinitely divisible probability measure, with $\nu(u) \sim 0(|u|^{-\alpha})$ as $|u| \to 0$, $0 < \alpha < 2$, then

$$\int \exp (ivx) d\mu(x) = \exp [-S|v|^\alpha + D(v)], \qquad 0 < \alpha \leqq 2,$$

where if

$$\delta - \int_{u \neq 0} \frac{u}{1+u^2} d\nu(u) = 0,$$

then

$$\operatorname*{Lim}_{|v| \to \infty} D(v)/|v|^\alpha = 0, \qquad 0 < \alpha \leqq 2;$$

otherwise,

$$D(v) = iv\delta + D'(v), \qquad \operatorname*{Lim}_{|v| \to \infty} D'(v)/|v|^\alpha = 0, \qquad 1 \leqq \alpha \leqq 2.$$

Proof: The proof follows from properties of ν, and is found in Ref. 5.
Q.E.D.

The main result can now be stated:

Proposition 7: For $0 < q < 1$, with a zero-drift gaussian component ($\alpha = 2$) present in either x_1, or x_0, or both, if $(T/\Delta t)$ is fixed

$$\operatorname*{Lim}_{\Delta t \downarrow 0} H_q(T, \Delta t) =$$

(a) 1 iff $S_1 = S_0 > 0$.
(b) Exp $(-kN)$ iff $S_1 \neq S_0$, $S_1 > 0$, and $S_0 > 0$,
$k = \ln [q(S_0/S_1)^{1-q} + (1-q)(S_1/S_0)^q]^{\frac{1}{2}}$.
(c) 0 iff $S_1 \neq 0 = S_0$ or $S_1 = 0 \neq S_0$.

If a zero-drift nongaussian stable $(0 < \alpha < 2)$ component is present in either r_1 or r_0, then

$$\operatorname*{Lim}_{\Delta t \downarrow 0} H_q(T, \Delta t) =$$

(d) 1 iff $\alpha_1 = \alpha_0$, $S_1 = S_0$.
(e) Exp $(-kN)$ iff $\alpha_1 = \alpha_0$, $S_1 \neq S_0$, $S_1 > 0_1$, $S_0 > 0$.
(f) 0 iff $\alpha_1 \neq \alpha_0$.

Proof: The proof follows from scaling arguments, and is found in Ref. 5.

Q.E.D.

If a gaussian component is present in both processes, then decreasing the sampling interval has no effect on decreasing H_q, and T must be increased to decrease H_q. However, if no gaussian component is present in one or the other of the processes, or if $\alpha_1 \neq \alpha_0$, then it is possible to decrease H_q by decreasing Δt with $(T/\Delta t)$ fixed.

Analogous results for $T \to \infty$ with $(T/\Delta t)$ fixed are presented in Ref. 5, as well as some results on the rate at which H_q approaches its limiting value.

Related work on nonuniformly sampling a continuous time independent increment process with one of two drift parameters is available in the literature.[14] A typical result is that sampling two stable processes with identical characteristic index, skewness, and scale, but differing drifts, is a singular detection problem if

$$\sum_{j=0}^{\infty} (t_{j+1} - t_j)^{2[1-(1/\alpha)]}$$

diverges, where $\{t_j\}$ are the sampling epochs,

$$\sum_{j=0}^{\infty} (t_{j+1} - t_j) = T.$$

Thus, spacing the samples apart by $t_{j+1} - t_j \propto j^{-m} (m > 1)$ results in singular detection, but $(t_{j+1} - t_j) \propto e^{-mj} (m > 0)$ may not.

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