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Analysis of Toll Switching Networks

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Two techniques are introduced for extending C. Y. Lee's method of switching network analysis to cases of toll-type networks. The methods avoid certain inconsistent independence assumptions which would otherwise be a source of inaccuracies. One method partitions the Lee graph in a special way, while the other uses a lemma that characterizes the generating function of an average network property. Examples are worked for three-stage networks and a model of the No. 4 ESS.

I. INTRODUCTION

In a well-known 1955 article,[1] C. Y. Lee introduced simplified methods for the analysis of switching network characteristics, such as blocking probability. Using a probability linear-graph (hereafter called Lee graph) to represent the network and an assumption of independent link occupancies, he described ways to quickly obtain approximate expressions in many cases of interest. As an example of possible inaccuracy, Lee pointed out a three-stage network that is known to be nonblocking but is assigned a nonzero blocking probability by his method.

The present work introduces two different techniques for extending Lee's method to avoid certain inconsistent independence assumptions, which are the source of the inaccuracies he noted. The extended methods will, for instance, reproduce M. Karnaugh's more accurate expression[2] for blocking probability of a three-stage network, but with less mathematical labor. When applied to a "generalized" three-stage network that can model the No. 4 ESS,[3] the techniques yield formulas that greatly simplify expressions currently in use. The appendix lists a computer routine to calculate blocking for the case of generalized three-stage networks.

II. FIRST METHOD

2.1 Generalized three-stage switching network

Consider first a "generalized" three-stage switching network, as indicated schematically in Fig. 1. The solid circles in the first and last

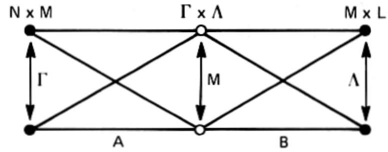

Fig. 1—Generalized three-stage switching network.

stages denote ordinary nonblocking switches, of sizes $N \times M$ and $M \times L$, respectively, such as crosspoint arrays or time-slot interchangers. The open circles in the middle stage, however, stand for switches that could block. Such a switch might, at a given time, have only a probability, rather than a certainty, of being able to connect a given pair of A- and B-links incident on it. This could model additional stages of switching network, such as the time-shared space-division stages in the No. 4 ESS.[3] An independent blocking probability Q is assigned to each middle-stage switch. We use the notation $\bar{Q} \equiv 1 - Q$ to indicate the corresponding transmission probability. The overbar denotes the probabilistic complement in all formulas that follow.

The Lee[1] graph in Fig. 2 shows all paths of the three-stage network that might be used to connect one call between a specified pair of terminations. Besides that pair, there are $E \equiv N - 1$ other input terminations at the first-stage switch and $F \equiv L - 1$ other output terminations at the last-stage switch. If we assign occupancy probabilities P and R, respectively, to input and output terminations other than the designated pair, the A- and B-links will have average occupancies $P_0 \equiv PE/M$ and $R_0 \equiv RF/M$.

If the first stage is a concentrator with $E > M$, it would be mathematically inconsistent to assume that the termination occupancies P were independent. In fact, that would imply a nonzero probability P^E of handling E calls on only M links. It would not be inconsistent to assume *a priori* that the A-link occupancies P_0 were independent when $E \geqq M$. A similar discussion applies to the last stage for the case $F > M$. Assuming that all probabilities P_0, Q, and R_0 are independent, the transmission probability for any single path becomes a product $\bar{P}_0\bar{Q}\bar{R}_0$ of transmission probabilities for each of its portions.

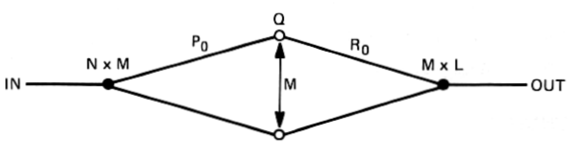

Fig. 2—Lee graph of generalized three-stage network.

Then the blocking probability for the network is just the product of blocking probabilities for each of the M independent paths.

$$\pi_B = (1 - \bar{P}_0\bar{Q}\bar{R}_0)^M = \overline{\bar{P}_0\bar{Q}\bar{R}_0}^M. \tag{1}$$

If there is expansion in the first stage, with $M > E$, it would be mathematically inconsistent to assume that the link occupancies P_0 were independent. Indeed, that would imply a nonzero probability P_0^M of busying all M links with at most E calls. It would not be inconsistent to assume that the termination occupancies P were independent when $M \geqq E$. A similar discussion applies to the last stage for the case $M > F$. In the boundary case $E = M$, assuming random connections through the switch, we find that each kind of independence assumption (link or termination) implies the other with $P = P_0$, and neither is inconsistent. Whether either assumption agrees with observed behavior of traffic is a difficult question that will not be explored here.

2.2 Toll-neutral case

To shorten our terminology, we will name the case $E > M$ "local," the case $E < M$ "toll," and the case $E = M$ "neutral." To emphasize the distinction, we cite the celebrated "Clos-type" network. Clos[4] showed that a pure three-stage network (the case $Q = 0$) will be nonblocking when $M > E + F$. But (1) can vanish only when P_0 and R_0 do. The contradiction arises from using the link independence assumptions in a toll case. The local case will not be considered in this study.

No matter what assumptions are appropriate, the event of having K busy A-links is the same as that of K busy input terminations, whence they have equal probability. In the toll case this is

$$\pi_A = \binom{E}{K} P^K(1 - P)^{E-K}, \tag{2}$$

since there are $\binom{E}{K}$ ways to choose K busies plus $E - K$ idles among E terminations, and each arrangement occurs with probability $P^K(1 - P)^{E-K}$, when termination occupancies P are independent. Assuming that all probabilities P, Q, and R_0 are independent, transmission probability for any single path containing an idle A-link is $\bar{Q}\bar{R}_0$, and blocking probability for all $M - K$ idles becomes $(1 - \bar{Q}\bar{R}_0)^{M-K}$. Multiplying by the probability of $M - K$ idles (2) and summing over all $0 \leqq K \leqq E$ yield

$$\pi_B = \sum_K \binom{E}{K} P^K(1 - P)^{E-K}(1 - \bar{Q}\bar{R}_0)^{M-K}, \tag{3}$$

the network blocking probability for the "toll-neutral" case $E \leqq M = F$,

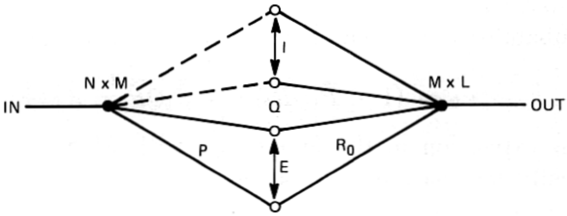

Fig. 3—Graph of toll-neutral network.

as opposed to the "neutral-neutral" case $E = M = F$ given by (1), in the form

$$\pi_B = (1 - \bar{P}\bar{Q}\bar{R})^M = \overline{\bar{P}\bar{Q}\bar{R}}^M. \tag{1a}$$

We have set $R_0 = R$ since $F = M$ and $P_0 = P$ since $E = M$.

By binomial theorem, the sum (3) is

$$\pi_B = (1 - \bar{Q}\bar{R}_0)^{M-E}(1 - \bar{P}\bar{Q}\bar{R}_0)^E, \tag{4}$$

but this may be derived by a more direct route. To see this, we note that there is a minimum of $I \equiv M - E$ idle A-links, no matter what the status of the E input terminations. Let us set aside I such idle A-links in Fig. 3, denoting them by dashed lines. This partitions Fig. 2 into two parallel graphs, the upper one with I independent paths and all A-links idle, and the lower one with E independent paths. The lower graph is just the neutral case so that, *mirabile dictu*, its A-links have independent occupancies P if its input terminations do. Thus, blocking probability for the lower graph is given by (1), with P_0 replaced by P and M by E. Blocking for the upper graph is just $(1 - \bar{Q}\bar{R}_0)^I$, as noted before (3). Network blocking (4) is then the product of these two terms.

2.3 Toll-toll case

There remains only the "toll-toll" case $E \leqq M \geqq F$, with the assumption of independent occupancies P and R for input and output terminations. The argument leading to (2) may be repeated to yield the probability

$$\pi_{AB} = \binom{E}{X}\binom{F}{Y} P^X(1 - P)^{E-X}R^Y(1 - R)^{F-Y} \tag{5}$$

of having X busy A-links and Y busy B-links.

The pure three-stage switch ($Q = 0$) blocks only if none of the $M - X$ idle A-links matches any of the $M - Y$ idle B-links. There are $\binom{M}{Y}$ ways to arrange the idle B-links, but only $\binom{X}{M - Y}$ of these match all $M - Y$ idles to the X busy A-links. Thus, assuming all arrangements are equally probable, the mismatch blocking probability becomes

$$\pi_{XY} = \binom{X}{M - Y} \Big/ \binom{M}{Y} = X!Y!/M!(X + Y - M)! \qquad (6)$$

Multiplying (5) by (6) and summing over $0 \leqq X \leqq E$ and $0 \leqq Y \leqq F$ yields the overall network blocking probability

$$\pi_B = \sum_{X,\,Y} \pi_{AB}(X, Y)\pi_{XY}. \qquad (7)$$

Karnaugh[2] has performed this sum, using binomial theorem first on the X-sum to obtain

$$\pi_B = \binom{E}{M - F} \Big/ \binom{M}{F} \sum_Y \binom{E + F - M}{F - Y} P^{M-Y} R^Y (1 - R)^{F-Y} \quad (8)$$

and then on the Y-sum to yield the blocking

$$\pi_B = \frac{E!F!}{M!(E + F - M)!} P^{M-F} R^{M-E} (1 - \bar{P}\bar{R})^{E+F-M} \qquad (9)$$

in closed form. Now π_B in (9) appears to be the product of a "combinatorial" factor $\binom{E}{M-F}/\binom{M}{F}$ and a "probabilistic" one. A direct derivation will help to bring out their origins.

Again, there are at least $I \equiv M - E$ idle A-links and $J \equiv M - F$ idle B-links. We set these aside in the Lee graph in Fig. 4, denoting them by dashed lines, as before. No matching of dashed A- and B-links is shown in the figure, since this could not correspond to a blocked state of the network. There are E solid A-links, corresponding to the neutral case, so that we can assume independent occupancies $P_0 = P$ for them. Similarly, the F solid B-links will have independent occupancies R.

Figure 4 clearly partitions Fig. 2 into three parallel graphs. The top graph has I independent paths and its blocking is obviously R^I. The bottom graph has J independent paths, and its blocking is P^J. The middle graph has $H \equiv M - I - J = E + F - M$ independent paths, each with transmission probability $\bar{P}\bar{R}$, so that its blocking is

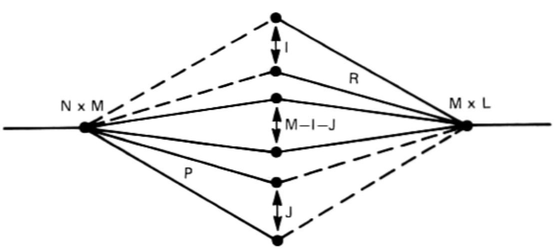

Fig. 4—Graph of toll-toll three-stage network.

$(1 - \bar{P}\bar{R})^{M-I-J}$. The product of these three blocking probabilities,

$$\pi_P = P^J R^I \overline{\bar{P}\bar{R}}^{M-I-J}, \tag{10}$$

is the blocking for the network configuration in Fig. 4 and is also the "probabilistic" portion of π_B in (9).

There are $\binom{M}{J}$ ways to arrange the dashed B-links, but only $\binom{M-I}{J}$ of these match all J dashed B-links to solid A-links. If all arrangements are equally likely, the quotient $\binom{E}{J}/\binom{M}{J}$ is the probability of the "blockable" configuration in Fig. 4. This accounts for the "combinatorial" portion of π_B in (9), which is in fact the blocking at full occupancy $P = R = 1$. Note that, if $M > E + F$, then $(E + F - M)!$ is infinite and π_B vanishes in (9), consistent with Clos' result.

2.4 Generalized toll-toll case

Even if Z of the $M - X$ idle A-links match Z of the $M - Y$ idle B-links in (5), a generalized three-stage network may still block, with probability Q^Z, for P, Q, and R independent. There are $\binom{M}{Y}$ ways to arrange the $M - Y$ idle B-links, but only $\binom{M-X}{Z}\binom{X}{M-Y-Z}$ ways to match just Z of them to $M - X$ idle A-links and the rest to the X busy A-links. For equally likely arrangements, then, the probability of a Z-match becomes

$$\pi_Z = \binom{M-X}{Z}\binom{X}{M-Y-Z}\bigg/\binom{M}{Y}. \tag{11}$$

The overall network blocking probability now is the sum over X, Y, and Z of the product of (5), (11), and Q^Z,

$$\pi_B = \sum_{X,Y,Z} Q^Z \pi_{AB}(X, Y)\pi_Z(X, Y). \tag{12}$$

As an example,[3] $M = 128$ and $L = N = 105$ in the No. 4 ESS so that $E = F = 104$ and $I = J = 24$. This makes π_B in (12) the sum of 302,845 nonzero terms.

As discussed previously, a more direct derivation of blocking probability π_B may be prosecuted for the generalized three-stage network. We set aside I idle A-links and J idle B-links as dashed lines in the Lee graph, Fig. 5. This time there may be some matching of V dashed A- and B-links, for $0 \leqq V \leqq I, J$. Figure 5 partitions Fig. 2 into four parallel graphs with $I - V$, V, $J - V$, and $H + V = M - I - J + V$ independent paths, respectively, to yield blocking probabilities of $(1 - \bar{Q}\bar{R})^{I-V}$, Q^V, $(1 - \bar{P}\bar{Q})^{J-V}$, and $(1 - \bar{P}\bar{Q}\bar{R})^{H+V}$ as discussed earlier.

There are $\binom{M}{J}$ ways to arrange the J dashed B-links but only $\binom{I}{V}\binom{E}{J-V}$ ways to match just V of them to the I dashed A-links and

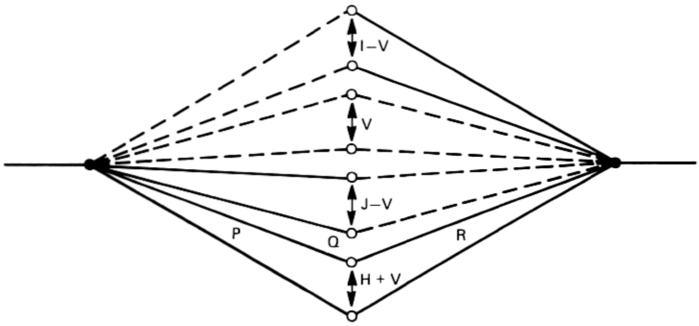

Fig. 5—Graph of toll-toll generalized three-stage network.

the rest to the E solid A-links. Thus, the probability of a V-match is

$$\pi_V = \binom{I}{V}\binom{E}{J-V}\Big/\binom{M}{J} = \frac{I!J!E!F!/M!V!}{(I-V)!(J-V)!(H+V)!} \tag{13}$$

if all arrangements have equal likelihood. This is multiplied by the blocking probability for a network with V matches, as in Fig. 5, and summed over V to yield the overall network blocking probability

$$\pi_B = \sum_V \pi_V Q^V (1-\bar{P}\bar{Q})^{J-V}(1-\bar{Q}\bar{R})^{I-V}(1-\bar{P}\bar{Q}\bar{R})^{H+V}. \tag{14}$$

III. SECOND METHOD

For the example of the No. 4 ESS, π_B in (14) is the sum of 25 non-zero terms. It does not appear to be possible to perform the V-sum and reduce (14) to a single term; however, we shall see that π_B does have a simple one-term generating function, as well as one-term operator-product expressions.

3.1 The generating function

It is possible to perform the X-, Y-, and Z-sums in (12), and thus reduce π_B to the simpler form (14). While tedious, this exercise is also highly instructive. Writing out (5) and (11) at length in (12) yields

$$\pi_B = \sum_{X,Y,Z} P^X Q^Z R^Y \frac{(M-X)!}{(E-X)!}\alpha^{E-X}\frac{(M-Y)!}{(F-Y)!}\beta^{F-Y}\pi_{XYZ}, \tag{15}$$

where $\alpha \equiv \bar{P}$, and $\beta \equiv \bar{R}$ and the last factor is

$$\pi_{XYZ} \equiv \frac{E!F!/M!Z!}{(M-X-Z)!(M-Y-Z)!(X+Y+Z-M)!}$$

$$= \frac{E!F!}{M!^2}\binom{M}{Z}\binom{M-Z}{X}\binom{X}{M-Y-Z}. \tag{16}$$

Substituting the following identities into (15),

$$\frac{(M-X)!}{(E-X)!}\,\alpha^{E-X} = \partial_\alpha^{M-E}\alpha^{M-X} = \partial_\alpha^{I}\alpha^{M-X} \tag{17}$$

$$\frac{(M-Y)!}{(F-Y)!}\,\beta^{F-Y} = \partial_\beta^{M-F}\beta^{M-Y} = \partial_\beta^{J}\beta^{M-Y}, \tag{18}$$

where ∂_α and ∂_β are the α- and β-derivatives, yields

$$\pi_B = \frac{E!F!}{M!^2}\,\partial_\alpha^I\partial_\beta^J \sum_{X,Z} P^X Q^Z \binom{M}{Z}\binom{M-Z}{X} S_Y \alpha^{M-X} \tag{19}$$

$$S_Y \equiv \sum_Y \binom{X}{M-Y-Z} R^Y \beta^{M-Y}. \tag{20}$$

We should note that, formally, α and β are independent variables, which are set equal to $\bar{P}$ and $\bar{R}$ only after all differentiations are performed. Thus, ∂_α and ∂_β do not act on P and R.

By binomial theorem, the sum in (20) is just

$$S_Y = \beta^Z R^{M-X-Z}(\beta + R)^X,$$

which reduces (19) to

$$\pi_B = \frac{E!F!}{M!^2}\,\partial_\alpha^I\partial_\beta^J \sum_Z Q^Z\beta^Z \binom{M}{Z} S_X \tag{21}$$

$$S_X \equiv \sum_X \binom{M-Z}{X} P^X \alpha^{M-X} R^{M-X-Z}(\beta + R)^X. \tag{22}$$

Similarly, the sum in (22) may be performed to get $S_X = \alpha^Z\gamma^{M-Z}$ for $\gamma \equiv \alpha R + \beta P + PR$ and reduce (21) to

$$\pi_B = \frac{E!F!}{M!^2}\,\partial_\alpha^I\partial_\beta^J \sum_Z \binom{M}{Z} \alpha^Z\beta^Z Q^Z \gamma^{M-Z}. \tag{23}$$

The sum in (23) is just $(\alpha\beta Q + \gamma)^M$, by a third application of binomial theorem, whence it becomes

$$\begin{aligned}\pi_B &= \frac{E!F!}{M!^2}\,\partial_\alpha^I\partial_\beta^J(\alpha\beta Q + \alpha R + \beta P + PR)^M \\ &= \frac{E!}{M!}\,\partial_\alpha^I(\alpha Q + P)^J(\alpha\beta Q + \gamma)^F. \end{aligned} \tag{24}$$

The forms (24) are about the closest we can get π_B to a simple closed expression. Leibniz' rule for the derivatives of a product will give (24) the form

$$\pi_B = \sum_V \pi_V Q^V(\alpha Q + P)^{J-V}(\beta Q + R)^{I-V}(\alpha\beta Q + \gamma)^{H+V}, \tag{25}$$

with π_V as in (13). Substituting $\alpha = \bar{P}$ and $\beta = \bar{R}$ will now reduce (25) to (14), since, for example,

$$\alpha Q + P = \bar{P}(1 - \bar{Q}) + 1 - \bar{P} = 1 - \bar{P}\bar{Q} \tag{26}$$

$$\begin{aligned} \alpha\beta Q + \gamma &= \beta(\alpha Q + P) + R(\alpha + P) \\ &= \bar{R}(1 - \bar{P}\bar{Q}) + 1 - \bar{R} = 1 - \bar{P}\bar{Q}\bar{R}. \end{aligned} \tag{27}$$

What is particularly illuminating in the preceding calculation is step (24). It says that $\pi_B(I, J)$, the blocking probability for a network with I excess A-links and J excess B-links, is a differential coefficient of "something." To make this idea more precise, we construct the corresponding generating function G. Multiplying (24) by $\binom{M}{I} U^I \binom{M}{J} W^J$ and then summing over all $0 \leqq I, J \leqq M$ yield the form

$$\begin{aligned} G &\equiv \sum_{I,J} \binom{M}{I} U^I \binom{M}{J} W^J \pi_B(I, J) \\ &= \sum_{I,J} \frac{U^I}{I!} \frac{W^J}{J!} \partial_\alpha^I \partial_\beta^J (\alpha\beta Q + \alpha R + \beta P + PR)^M. \end{aligned} \tag{28}$$

By Taylor's theorem, the second line is just

$$\begin{aligned} G &= [(U + \bar{P})(W + \bar{R})Q + (U + \bar{P})R + (W + \bar{R})P + PR]^M \\ &= [(U + 1)(W + 1) - (U + \bar{P})(W + \bar{R})\bar{Q}]^M \\ &= (U + 1)^M (W + 1)^M \left[1 - \left(1 - \frac{P}{U + 1}\right)\left(1 - \frac{R}{W + 1}\right)\bar{Q}\right]^M. \end{aligned} \tag{29}$$

Thus, $\pi_B(I, J)$ can be obtained by expanding (29) in powers of U and W, then inspecting the coefficient of $U^I W^J$. In practice, we get the coefficient by differentiating I times in U and J times in W, then setting $U = W = 0$. But this is exactly equivalent to the steps leading from (24) to (14).

3.2 Fundamental lemma

In the preceding section, it was shown that blocking probability π_B could be obtained quickly and directly from a generating function G, rather than through an argument involving a four-way partition of the Lee graph. If we could now obtain G directly, without steps (5), (11), (12), and (15) to (24), a great deal of effort might be saved. The structure of G is indeed determined by the following lemma, which has surprisingly little to do with switching networks.

We idealize the network as a "black box," as in Fig. 6, on which a certain set of M "trunks" are terminated. These are assumed to have

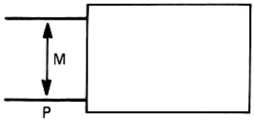

Fig. 6—Network idealized as black box.

independent occupancy probabilities P, except that some number $I \leqq M$ of the trunks are "dead" or disconnected, with zero occupancies. Associate with the black box some quantity $\Lambda(K)$, a function of the number K of busy terminations among the $E = M - I$ "active" trunks. By using (2), we find the average value of Λ to be

$$\pi(I, P) \equiv \sum_K \binom{M-I}{K} P^K (1-P)^{M-I-K} \Lambda(K). \tag{30}$$

We have assumed that the M trunks are interchangeable, in the sense that Λ, and hence π, does not depend on which I of the M are dead.

Fundamental Lemma: The generating function for $\pi(I, P)$ is written in terms of $\pi(0, P)$ as

$$G \equiv \sum_I \binom{M}{I} U^I \pi(I, P) = (U+1)^M \pi\left(0, \frac{P}{U+1}\right). \tag{31}$$

Formal Proof: Substitute (30) into (31) and use binomial theorem to perform the sum over $0 \leqq I \leqq M$.

Informal Proof: Suppose that each trunk has an independent probability λ of being active, and hence $\bar{\lambda} = 1 - \lambda$ of being dead. Then the average value of π is just

$$\hat{\pi} = \sum_I \binom{M}{I} \lambda^{M-I}(1-\lambda)^I \pi(I, P). \tag{32}$$

On the other hand, each trunk carries an average load of λP erlangs, independent of the others, so that $\hat{\pi} = \pi(0, \lambda P)$. We now define $U \equiv \bar{\lambda}/\lambda$, which yields $\lambda = 1/(U+1)$, and set $G = (U+1)^M \hat{\pi}$ to transform (32) into (31). The model of "dead trunks" is invoked only to validate the mathematical relation (31), of course, and need not accord with observed behavior.

An obvious application of the lemma is to let the black box be a switching network and π be its blocking probability π_B for some pair of terminations. As an example, consider the "toll-neutral" case of the generalized three-stage network, with $E \leqq M = F$. Let the M trunks be terminated on the same first-stage switch as the input termination of the pair whose blocking is sought. Then $I = 0$ corresponds to the neutral case $E = M$, for which (1) is valid with $P_0 = P$. Now the generating function becomes

$$G = (U+1)^M \left[1 - \left(1 - \frac{P}{U+1}\right)\bar{Q}\bar{R}_0\right]^M$$
$$= [U + 1 - (U + \bar{P})\bar{Q}\bar{R}_0]^M. \tag{33}$$

Differentiating I times at $U = 0$ yields the blocking

$$\pi_B(I, P) = \partial_U^I G/I! \binom{M}{I}$$
$$= (1 - \bar{Q}\bar{R}_0)^I (1 - \bar{P}\bar{Q}\bar{R}_0)^{M-I}, \tag{34}$$

which is the same formula as (4). A bare minimum of knowledge about the network structure, just formula (1), was thus sufficient to determine blocking probability.

The "toll-toll" case, $E \leqq M \geqq F$ of the generalized three-stage network, requires an iterated form

$$G \equiv \sum_{I,J} \binom{M}{I} U^I \binom{\mu}{J} W^J \pi(I, P; J, R)$$
$$= \sum_I \binom{M}{I} U^I \pi\left(I, P; 0, \frac{R}{W+1}\right)(W+1)^\mu$$
$$= (U+1)^M (W+1)^\mu \pi\left(0, \frac{P}{U+1}; 0, \frac{R}{W+1}\right) \tag{35}$$

of the lemma. For $\mu = M$ trunks terminated on the same last-stage switch as the output termination of the pair whose blocking is sought, we see that $I = J = 0$ is the neutral case $E = M = F$, with π_B given by (1a). Thus, substituting (1a) into (35) yields a generating function

$$G = (U+1)^M (W+1)^M \left[1 - \left(1 - \frac{P}{U+1}\right)\left(1 - \frac{R}{W+1}\right)\bar{Q}\right]^M, \tag{36}$$

which is the same as (29). The blocking becomes

$$\pi_B = \partial_U^I \partial_W^J G/I! \binom{M}{I} J! \binom{M}{J} = \partial_W^J g/J! \binom{M}{J} \tag{37}$$

$$g = (W+1)^M \left[1 - \left(1 - \frac{R}{W+1}\right)\bar{Q}\right]^I \cdot \left[1 - \bar{P}\left(1 - \frac{R}{W+1}\right)\bar{Q}\right]^{M-I}, \tag{38}$$

and applying Leibniz' rule for the derivatives at $W = 0$ again yields (14).

Another approach is to observe that we have already "differentiated off" I terminations in the "toll-neutral" case. Applying the lemma once more to (34), we can construct the generating function g in (38) at once and then "remove" the J spare output terminations as in (37) to obtain blocking probability (14). Again, only (1) was needed to specify the particular network under consideration.

3.3 Operator formulation

Lemma (31) may be "solved" for $\pi(I, P)$ by making a formal expansion of its right side in terms of ∂, the P-derivative. Observing that

$$e^{aP\partial}P^n = \sum_l \frac{(aP\partial)^l}{l!} P^n = \sum_l \frac{(an)^l}{l!} P^n = (e^a P)^n \tag{39}$$

and hence $A^{P\partial}f(P) = f(AP)$, the generating function is

$$G = (U+1)^{M-P\partial}\pi(0, P) = \sum_{I=0}^{\infty} \binom{M - P\partial}{I} U^I \pi(0, P). \tag{40}$$

Equating powers of U in (31) and (40) yields the identity

$$\pi(I, P) = \binom{M - P\partial}{I} \pi(0, P) \bigg/ \binom{M}{I}, \tag{41}$$

which we write out as an operator product

$$\begin{aligned} \pi(I, P) &= \left(1 - \frac{P\partial}{M - I + 1}\right) \cdots \left(1 - \frac{P\partial}{M - 1}\right)\left(1 - \frac{P\partial}{M}\right) \pi(0, P) \\ &= \left(1 - \frac{P\partial}{M - I + 1}\right) \pi(I - 1, P). \end{aligned} \tag{42}$$

This has a simple and natural interpretation: "deloading operator" $1 - P\partial/E$ serves to "remove" one of the E remaining terminations represented in the expression $\pi(M - E, P)$ for any average network property π.

Of course, the validity of the mathematical manipulations above must still be demonstrated. This hinges upon establishing convergence of the operator expansion in (40), and hence the sum in I. But we note that each factor of the form $n - P\partial$ will annihilate the corresponding power P^n in $\pi(0, P)$. Thus, if π is a polynomial of degree M at most, as in (30), it is annihilated completely in all terms of (40) for which I exceeds M, and the expansion indeed converges, since the sum is finite. In practical calculations of blocking probabilities or other formulas, the operator formulation (42) may not be as convenient to use as direct differentiation of the generating function.

IV. SUMMARY AND DISCUSSION

Two techniques are introduced to extend C. Y. Lee's method to switching networks of "toll" types defined in the text. Basically, these have some expansion in the first stage or concentration in the last stage, or both. The link independence assumptions of Lee's method are inconsistent in such cases, which causes some inaccuracy.

The first of the two extension techniques partitions the Lee graph into two or more smaller graphs, so that the independence assumptions will be consistent within various portions. This is done by setting apart the proper number of links that are known to be idle. The first technique is applied to examples of three-stage networks, yielding a result of Karnaugh's and a simplification of the expression used for computing blocking for the No. 4 ESS. To focus attention on the methods, all examples worked out are blocking probabilities, but other network averages may be treated similarly.

The second, and more general, technique makes use of a lemma that characterizes the generating function of an average property for a whole family of networks. In a sense, this technique is a counterpart of the first, since it operates by attaching a sufficient number of idle "phantom" terminations to the network to make the link independence assumptions valid, instead of setting aside idle links. Then the generating function can be constructed from the resulting "neutral" case, and the lemma, or an equivalent operator formulation, allows removal of the excess terminations. Two of the three-stage network examples are reworked to make a comparison of the two techniques.

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APPENDIX

The FORTRAN subroutine listed in Table I calculates the blocking probability π_B given by (14) for the "toll-toll" case of the generalized three-stage switching network. Cancelling among the nine factorials in (13) yields an efficient calculation and minimizes the chance of underflow or overflow. The routine is accessed by:

CALL BLOCK (I, P, J, R, M, Q, PI)

Table I — Blocking probability subroutine

```
      SUBROUTINE BLOCK (I, P, J, R, M, Q, PI)
      K = M - I - J
      L = MAX0(0, I + J - M)
      PQ = 1.0 - (1.0 - P)*(1.0 - Q)
      RQ = 1.0 - (1.0 - R)*(1.0 - Q)
      SQ = 1.0 - (1.0 - R)*(1.0 - Q)*(1.0 - P)
      A = 1.0
      IF(L.GT.0) A = Q**L
      A = A*PQ**(J - L)*RQ**(I - L)*SQ**(K + L)
      B = A
      C = Q*SQ/(PQ*RQ)
      D = 1.0
      LOW = L + 1
      LIM = MIN0(I, J)
      IF(LIM - L) 30, 20, 10
   10 DO 15 N = LOW, LIM
      A = A*C*FLOAT((I - N + 1)*(J - N + 1))/FLOAT((K + N)*N)
      B = B + A
   15 D = D*FLOAT(K + L + N)/FLOAT(M + N - LIM)
   20 PI = B*D
   30 RETURN
      END
```

and, referring to Fig. 1, the arguments are:

$I = M - E = M - N + 1$ minimum number of idle A-links
$J = M - F = M - L + 1$ minimum number of idle B-links
M number of center-stage switches
P average occupancy probability of input terminations
Q blocking probability of center-stage switches
R average occupancy probability of output terminations
PI returns the calculated value of π_B.

Independence is assumed for all P, Q, and R.

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