THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 55, No. 4, April 1976
Printed in U.S.A.

On the Design of All-Pass Signals With Peak Amplitude Constraints

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(Manuscript received November 24, 1975)

In this paper, the problem is discussed of designing a signal other than the standard impulse function to be used to test a digital system of limited dynamic range. The constraints on such a signal are that it must be all-pass, of limited duration (approximately), and peak-amplitude-limited so as to utilize the limited dynamic range of the system as far as possible. Stated another way, the goal is to spread out the energy in the signal as much as possible to reduce its peak amplitude and therefore to be able to pass higher energy signals through the system without clipping them. The class of all-pass signals (obtained as the impulse response of a variable order all-pass filter) was investigated for use as the test signal. The parameters of the all-pass filter of a given order were optimized to give an all-pass signal whose peak amplitude was the smallest possible. Filter orders from first to eighth order were designed and investigated. It was found that reductions in the peak signal level of up to 11.2 dB (relative to the signal level of an equivalent energy impulse) could be obtained for an eighth-order all-pass signal. Interpolated versions of these all-pass signals showed that the peak value of the interpolated waveform was only on the order of 6 dB. Thus, the use of an all-pass signal, rather than the standard impulse, for testing a digital system can result in about 1 bit extra dynamic range.

I. INTRODUCTION

The problem of designing digital signals for testing (e.g., evaluating the impulse response) digital systems is one which has received very little attention in the digital signal-processing literature. This is because the impulse function is used as the standard test signal for most systems. Although the impulse function is suitable for this purpose in a wide variety of digital systems, there are cases in which the use of the impulse function leads to problems. Generally, such systems are those that have limited dynamic range—e.g., digital hardware implementations of a system, or fixed-point, finite, precision, software implementation of a digital system. In this paper, the problem is considered of designing signals other than the standard impulse function to be used to test digital systems of limited dynamic range.

The desirable features of a test signal for digital systems are

(*i*) It must be an all-pass signal in that it must be capable of testing the system (i.e., determining the frequency response of the system) for any admissible frequency.

(*ii*) It should be of limited duration.

(*iii*) It should be peak-amplitude-limited, to give the maximum utilization of the limited dynamic range of the system.

The above features define a desirable test signal as one whose energy is spread out as much as possible to reduce the peak signal amplitude and therefore be able to pass higher energy signals through the system without clipping.

If we let $x(n)$ denote the test signal, then the requirements described above can be related to $x(n)$ and $X(e^{j\omega})$, the Fourier transform of $x(n)$, in the following manner. For the signal to be all-pass implies

$$|X(e^{j\omega})| = C, \qquad \text{all } \omega, \tag{1}$$

where C is an arbitrary constant value. If we let $C = 1$, then by Parseval's theorem we have

$$\frac{1}{2\pi}\int_0^{2\pi} |X(e^{j\omega})|^2 d\omega = 1 = \sum_{n=0}^{\infty} x^2(n), \tag{2}$$

i.e., the overall energy of the test signal is unity. For the signal to be of limited duration (at least approximately) requires

$$\sum_{n=0}^{N_1-1} x^2(n) = \gamma, \tag{3}$$

where $\gamma \approx 1$ and N_1 is the signal duration in samples. (The constraint of (3) has not been used directly in the work presented here, since it was found that it was satisfied by all the signals that were designed.) Finally, the constraint that the peak signal amplitude be as small as possible requires that $\max_n |x(n)|$ be minimized over the design parameters of the signal.

Besides the standard impulse function, the only other class of signals that is appropriate for a test function (i.e., that has the set of features described above) is the set of all-pass filter impulse responses. Such signals can be optimized to meet the design requirements by varying the parameters of the all-pass network to minimize the peak signal amplitude.

The purpose of this paper is to discuss the issues in the design of all-pass signals to be used to test a digital system. In Section II, the design methods used to optimize these all-pass signals are discussed. In Section III, considerations dealing with the interpolation of the resulting all-pass test signals are given. Finally, in Section IV a brief discussion of the effects of filtering these all-pass signals is given.

II. DESIGN TECHNIQUES FOR ALL-PASS SIGNALS

The signal design problem is one of choosing the parameters (the filter coefficients) in the implementation of an Nth-order all-pass filter to minimize the peak amplitude of the resulting impulse response. For the actual implementation of most all-pass filters, it is generally convenient to consider the cascade realization which is of the form

$$X(z) = \prod_{i=1}^{N_s} H_i(z), \tag{4}$$

where N_s is the number of sections in the cascade and $H_i(z)$ are the individual sections, which generally are either first-order or second-order sections. A first-order all-pass section has the system function

$$H_i(z) = \frac{-a + z^{-1}}{1 - az^{-1}}, \tag{5}$$

whereas a second-order all-pass section has the system function

$$H_i(z) = \frac{b_i - c_i z^{-1} + z^{-2}}{1 - c_i z^{-1} + b_i z^{-2}}. \tag{6}$$

The design problem is thus to choose the all-pass parameters (a, b_i, c_i) to minimize the peak signal amplitude in the impulse response of the filter.

For the first-order case, the parameter a can be analytically determined. In this case, the difference equation is

$$x(n) = u_0(n-1) - au_0(n) + ax(n-1), \tag{7}$$

where

$$u_0(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise,} \end{cases}$$

or

$$\begin{aligned} x(n) &= 0 & n < 0 \\ x(0) &= -a \\ x(1) &= (1 - a^2) \\ x(n) &= (1 - a^2)a^{n-1}, & n \geqq 2. \end{aligned} \tag{8}$$

Since $|a| < 1$ for stability, it is seen from (8) that the largest possible samples are $x(0)$ and $x(1)$. Thus, to minimize the larger of $|x(0)|$ and $|x(1)|$ requires a choice of a such that

$$|x(0)| = |x(1)| \tag{9}$$

or

$$|a_{\min}| = |1 - a_{\min}^2|. \tag{10}$$

The solution to (10) gives $a_{\min} = 0.618$.

For optimization of higher-order all-pass filters, no analytical solution could be found. Thus, an optimization method was used to obtain the desired solutions. In particular, a nonlinear unconstrained optimization method developed by Powell[1] was used in which the evaluation of derivatives was not required. The maximum peak amplitude of the all-pass signal can be minimized by minimizing the function

$$G = \lim_{\rho \to \infty} \left[\sum_{n=0}^{\infty} |x(n)|^{\rho} \right]^{1/\rho}. \tag{11}$$

In practice, however, the function of (11) is not unimodal or smooth, and thus it is not practical to find the optimum choice of parameters without a good starting point (initial choice of parameters) for the optimization routine. To obtain such starting points, (11) was used as the objective function for a value of $\rho = 4$. A variety of randomly chosen starting points was used to obtain the best solutions for $\rho = 4$. The $\rho = 4$ solutions were then used as starting points to determine the optimum $\rho = \infty$ solutions.

The parameters that were varied within the optimization program were the b_i's and c_i's of the second-order sections within the cascade and the a for a first-order section (used whenever the order of the all-pass filter was odd). The advantage of using the cascade realization is that it is simple to ensure stability of the resulting filter. Additionally, instabilities occurring during the optimization program because of poles drifting outside the unit circle were easily detected and corrected with minimal computational effort.

Using the Powell optimization method, the optimum all-pass signals of order 1 to 8 were designed. Table I gives values of the optimum all-pass filter parameters and the resulting peak signal level for each of these cases. It is seen in this table that the peak signal level falls from 0.618 to 0.275 as the all-pass filter order varies from first to eighth order. Further, it can be seen that progressive increases in the order of the all-pass filter result only in very modest reductions of the peak signal level beyond a second-order filter. Figures 1 and 2 show the positions of the poles and zeros of the optimum all-pass filters and their group delay responses for each of the filters of Table I.

Table I — Filter coefficients for optimum all-pass filters with peak amplitude constraints

Filter Order	Maximum Signal Level	a	b_1	c_1	b_2	c_2	b_3	c_3	b_4	c_4
1	0.618	0.6180	—	—	—	—	—	—	—	—
2	0.500	—	0.5	1.0	—	—	—	—	—	—
3	0.428	0.8698	0.4915	0.6961	—	—	—	—	—	—
4	0.386	—	0.8008	0.5137	−0.4823	0.3491	—	—	—	—
5	0.3380	0.6734	0.6081	0.4462	0.7996	−0.5253	—	—	—	—
6	0.3183	—	0.6151	1.4640	0.8228	0.3748	−0.6290	0.0197	—	—
7	0.2895	−0.5183	0.8339	−0.4254	0.7668	1.5407	0.8735	0.4700	—	—
8	0.2748	—	0.8149	1.2308	−0.4970	−0.1060	0.8621	−0.2135	0.7870	1.5727

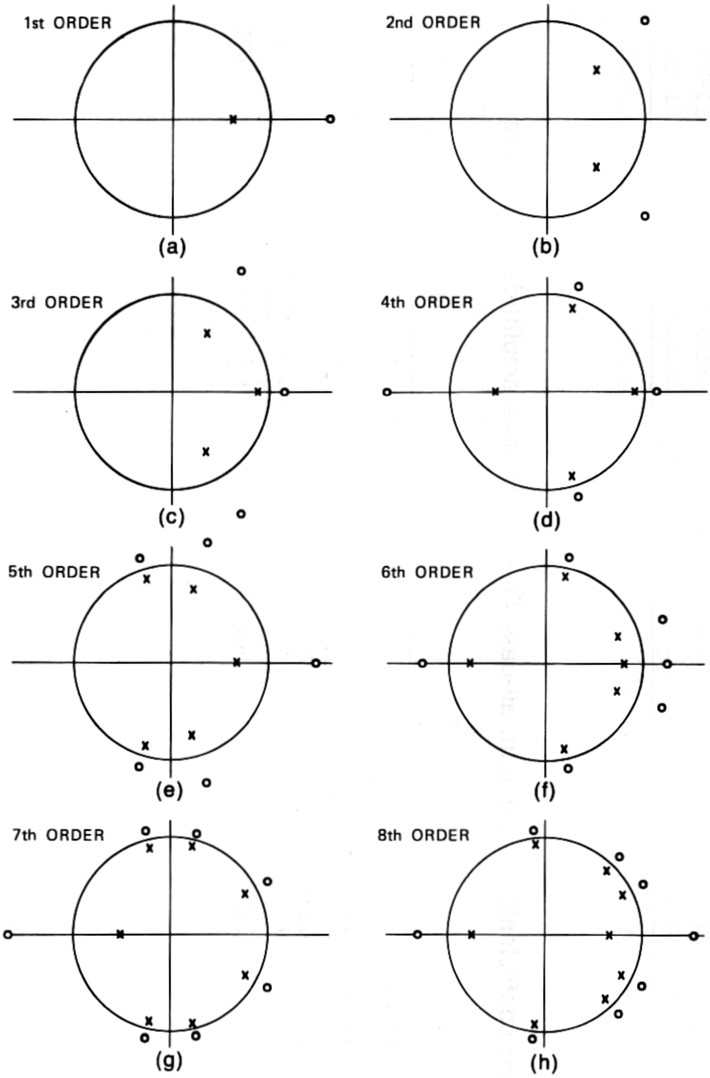


Fig. 1—Positions of the poles and zeros of the optimized all-pass signals of order 1 to 8.

An interesting property of this class of signals is that the optimum all-pass filter is not unique. This result is readily seen since the simple replacement of z by z^{-1} in the z transform leads to a multiplication of the signal by $(-1)^n$, which does not affect the signal magnitude at all. Thus, each pole and zero of Fig. 1 could equally be shown reflected about the imaginary z axis and still be a valid optimum solution.

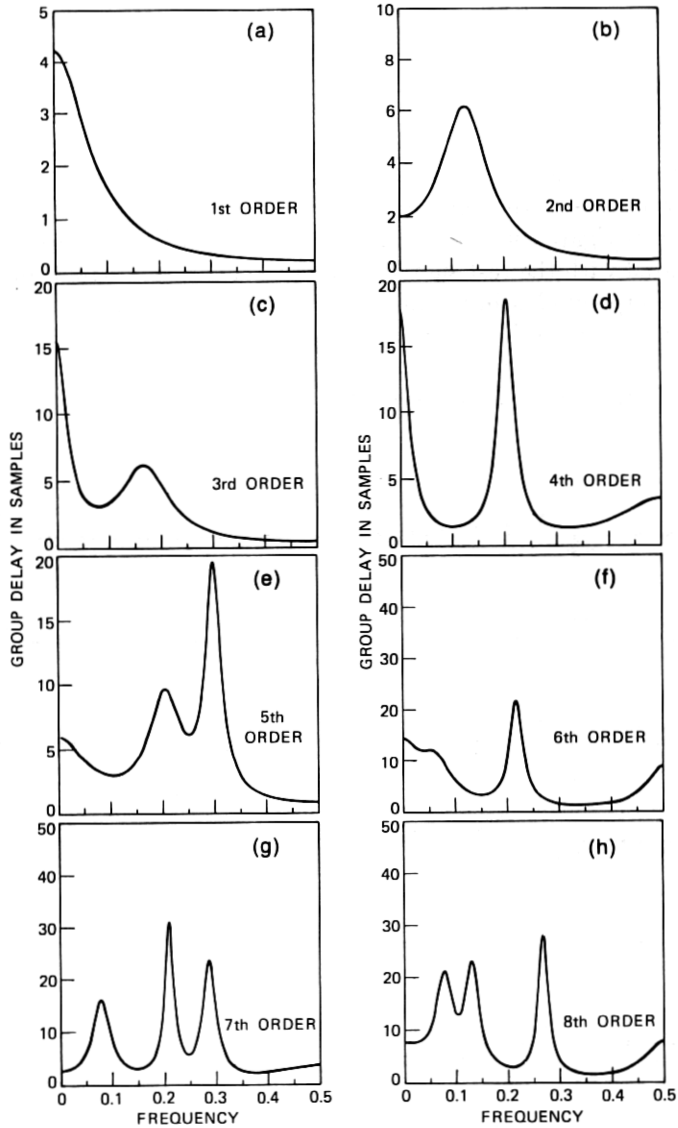


Fig. 2—Group delay responses of the optimized all-pass signals of order 1 to 8.

III. INTERPOLATION OF THE OPTIMUM ALL-PASS SIGNALS

The results of the preceding section indicate that reductions in the peak level of the optimized all-pass signal on the order of 4 to 1 can be obtained with an eighth-order filter. This result can be somewhat misleading, however, since the continuous waveform (from which the signal samples could be derived) could peak up between samples—i.e., the actual reduction in signal level could be a fortuitous result obtained by sampling the waveform at the most opportune sampling intervals. If this were the case, and the test signal was used as input to a network which approximated a noninteger delay, the output signal could be of higher amplitude than the input signal simply because of the interpolative properties of the network.

To investigate the true peak amplitude of the continuous waveform associated with the test signal, each of the eight test signals of Table I were interpolated using a 20-to-1 interpolator implemented using the methods described by Crochiere and Rabiner.[2,3] Figure 3 and Table II show the results of interpolating the test signals. Figure 3a shows both the test signal samples as well as the interpolated waveforms (dotted

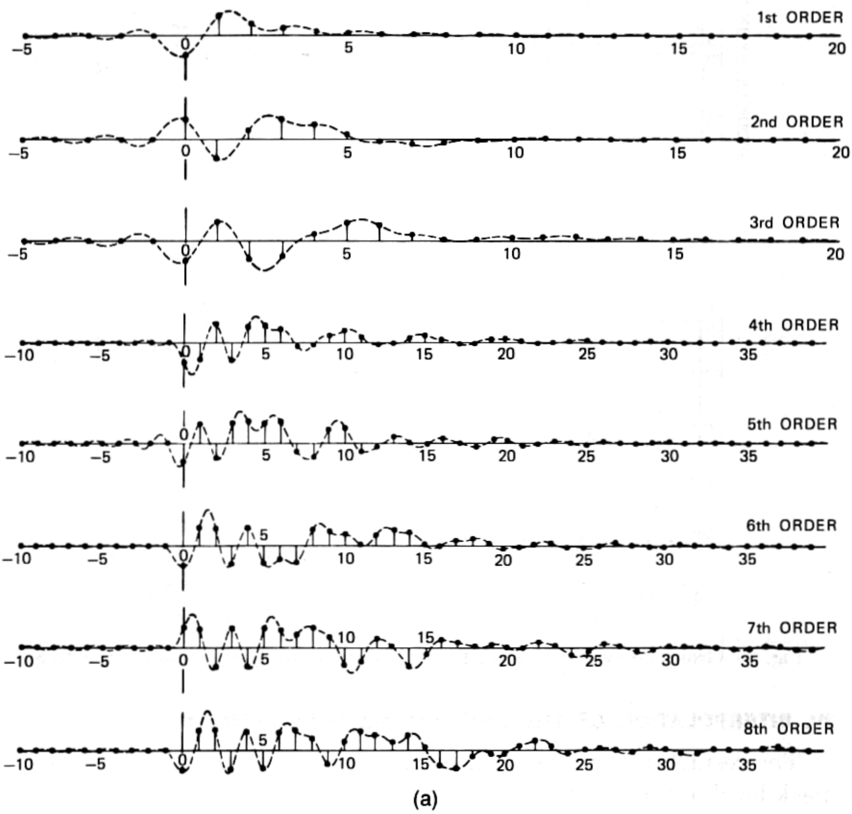
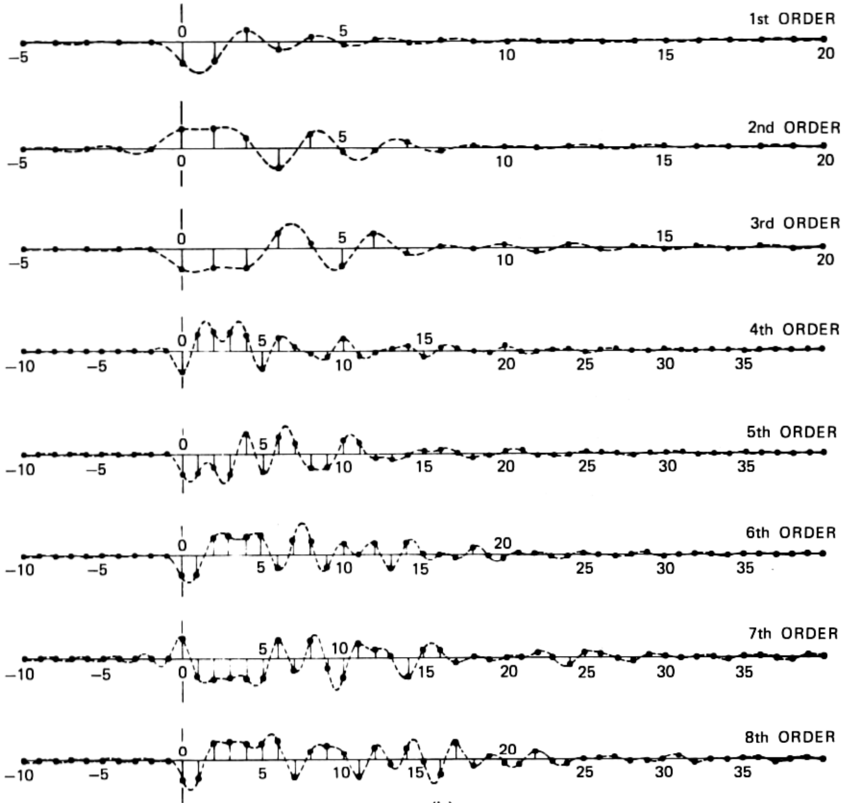


Fig. 3—Samples and interpolated waveforms of (a) the all-pass signals for orders 1 to 8 and (b) the all-pass signals modulated by $(-1)^n$.



(b)

Fig. 3 (continued).

lines) associated with the signals. Figure 3b shows the alternate set of peak-limited waveforms formed by multiplication of the signals in Fig. 3a by $(-1)^n$. Although each test signal attains its peak amplitude at a number of different sampling instants, its interpolated waveform generally shows a distinct maximum amplitude. Table II also shows that the peak interpolated waveform amplitude ranged from 0.766 for the first-order signal to 0.421 for the seventh-order signal. Thus, in terms of the interpolated waveform, on the order of a 2-to-1 reduction in peak signal level was obtained for these test signals.

One more observation can be obtained from Fig. 3 and that is that the test signals, although generated as the output of a recursive structure, damp out in level extremely rapidly and could be considered finite duration signals. It was found that 128 samples of the test signal were sufficient for obtaining 16-bit test signals to full 16-bit accuracy.

Table II — Comparison of signal level of peak-limited signals to a unit sample, all with unit signal energy

Filter Order	Peak-Limited Signal				Waveform (Interpolated) of Peak-Limited Signal				Waveform (Interpolated) of Peak-Limited Signal with $(-1)^n$ Modulation			
	Max.	Min.	Ratio	Ratio dB	Max.	Min.	Ratio	Ratio dB	Max.	Min.	Ratio	Ratio dB
1	0.618	−0.618	0.618	−4.18	0.766	−0.633	0.766	−2.32	0.377	−0.924	0.924	−0.68
2	0.500	−0.500	0.500	−6.02	0.595	−0.522	0.595	−4.52	0.533	−0.501	0.533	−5.46
3	0.428	−0.428	0.428	−7.37	0.465	−0.632	0.632	−3.98	0.545	−0.502	0.545	−5.27
4	0.375	−0.386	0.386	−8.27	0.505	−0.627	0.627	−4.06	0.586	−0.391	0.586	−4.65
5	0.338	−0.334	0.338	−9.42	0.526	−0.391	0.526	−5.57	0.474	−0.503	0.503	−5.97
6	0.301	−0.318	0.318	−9.95	0.585	−0.345	0.585	−4.65	0.525	−0.442	0.525	−5.60
7	0.290	−0.290	0.290	−10.75	0.497	−0.367	0.497	−6.07	0.338	−0.457	0.457	−6.79
8	0.273	−0.275	0.275	−11.21	0.544	−0.313	0.544	−5.29	0.387	−0.421	0.421	−7.51

IV. APPLICATION OF PEAK-LIMITED SIGNALS AS TEST SIGNALS

One application of the above class of peak-limited signals is for use as test signals for systems of limited dynamic range. By spreading the signal energy among many samples, a test signal of greater total energy than an impulse can be used without exceeding the dynamic range of the system. This then enhances the signal-to-noise ratio (s/n) of the measurement.

For a system that has approximately a linear-phase response, s/n improvements of the orders shown in Table II can be expected. If the system has considerable phase distortion, the amount of s/n enhancement may be less. In an extreme case, a system could act as a "matched filter" to a particular test signal and compress all the signal energy back into a single sample. In this case, no s/n improvement would be possible with that test signal, although other peak-limited test signals in this class might be useful.

To investigate the use of the peak-limited signals as test signals, we chose a system that consists of a complex modulator, a decimator, an interpolator, and another complex modulator. The system was implemented on a 16-bit computer, and the decimator and interpolator were designed as discussed in Refs. 2 and 3. The net function of the above system is that of a bandpass filtering operation. It represents a useful type of system for speech-processing applications (e.g., vocoders).

The frequency response of the system is shown in Fig. 4a. It was measured by exciting the system with the peak-limited signal for $N = 7$ and taking the Fourier transform of the output. The largest peak amplitude signal which could be used without overflow was 16384, or 2^{14}. Similarly, the largest impulse that could be used as a test signal was 2^{14}. The frequency response measurement in this case was essentially equivalent to that using the peak-limited signal (see Fig. 4a). The reason for this is apparent. The 16-bit system has a large dynamic range (about 90 dB) compared to the frequency response of the filter (about 45 dB). Obviously, the use of peak-limited signals is not warranted.

We next considered a 12-bit implementation of the same system.* This would very likely be the available word length of a practical hardware implementation or small minicomputer implementation. In this case, the dynamic range of the system is about 66 dB, and we can expect that roundoff noise will affect the frequency response measurement. The largest magnitude impulse that could be used to test this

* This was simulated on the 16-bit system by not allowing the use of the four most significant bits.

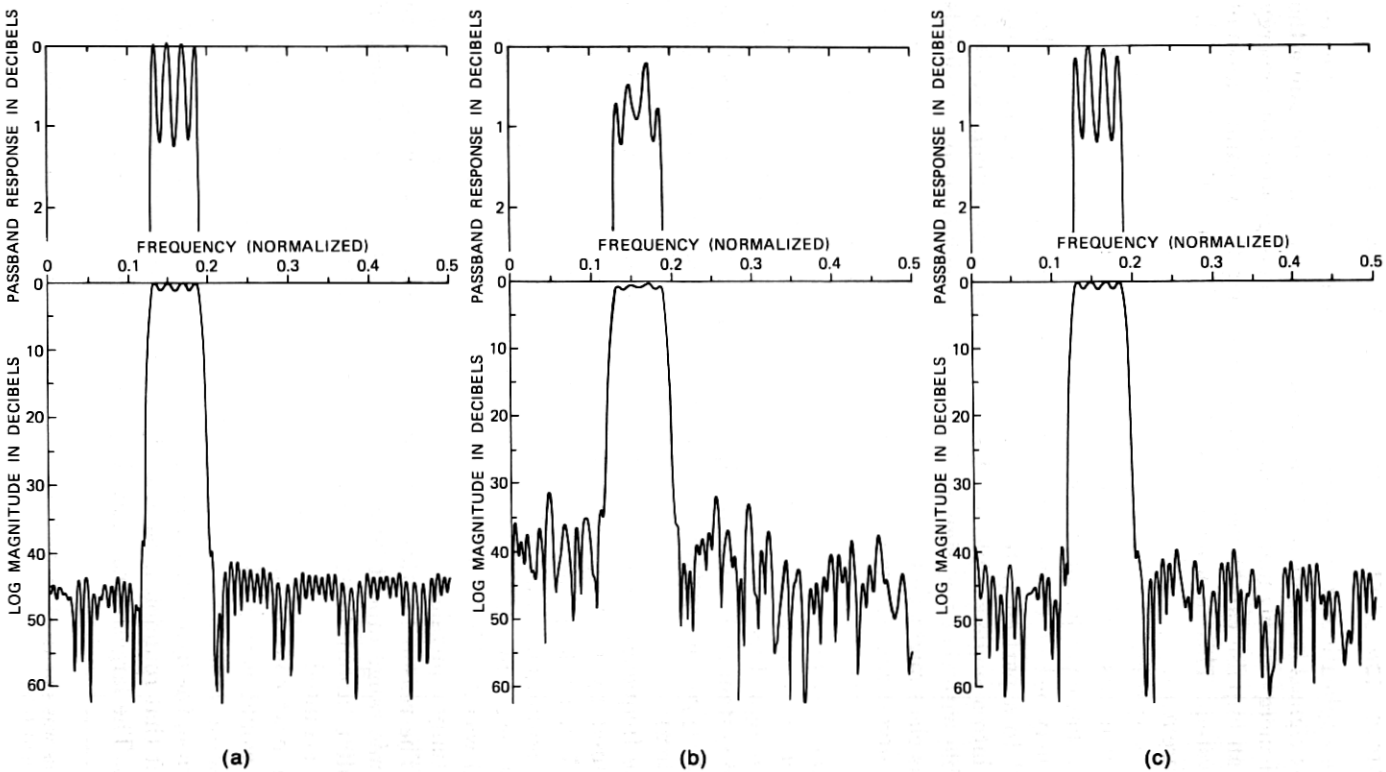


Fig. 4—Frequency response measurement using (a) an $N = 7$ all-pass test signal in a 16-bit system, (b) an impulse test signal in an equivalent 12-bit system, and (c) an $N = 7$ all-pass test signal in the 12-bit system.

system without overflow was 1024, or 2^{10}. A measurement of the frequency response based on this impulse response is shown in Fig. 4b. It is apparent that the roundoff noise has degraded the measurement considerably. The passband response has been distorted, and the peak stopband signal rejection measures only 31 dB compared to 41 dB in Fig. 4a.

Figure 4c shows the frequency response measurement of the same 12-bit system based on the peak-limited signal for $N = 7$. The maximum amplitude that could be used for this signal was 2^{10} and, as can be seen from Table II, it contains 10.75 dB more signal energy than an impulse of the same amplitude. In comparing Figs. 4a, b, and c, it is clear that the use of the peak-limited signal has improved the frequency response measurement of the 12-bit system. The measurement of the stopband rejection is on the order of 40 dB, or 9 dB better than in Fig. 4b. The passband response looks more like the essentially noiseless measurement in Fig. 4a.

V. CONCLUSIONS

It has been shown that a class of peak-limited and essentially finite duration signals can be generated by optimizing the $\rho = \infty$ norm of the impulse responses of the class of all-pass networks. Signals were generated for all-pass filter orders from $N = 1$ to $N = 8$. It was demonstrated that this class of signals is useful as test signals for systems of limited dynamic range. Improvements of up to 11 dB in s/n enhancement were found to be possible.

REFERENCES

1. M. J. Powell, "An Efficient Method for Finding the Minimum of a Function of Several Variables Without Calculating Derivatives," Computer J., *7*, 1964, pp. 155–162. (Computer code available in J. L. Kuester and J. H. Mize, *Optimization Techniques with Fortran*, New York: McGraw-Hill, 1973.)
2. R. E. Crochiere and L. R. Rabiner, "Optimum FIR Digital Filter Implementations for Decimation, Interpolation, and Narrow Band Filtering," IEEE Trans. on Acoustics, Speech, and Signal Processing, *ASSP-23*, No. 5 (October 1975), pp. 444–456.
3. R. E. Crochiere and L. R. Rabiner, "Further Considerations in the Design of Decimators and Interpolators," IEEE Trans. Acoustics, Speech, and Signal Processing, *ASSP-24*, No. 4 (August 1976).