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Step Response of an Adaptive Delta Modulator

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N. S. Jayant has proposed a simple but effective form of adaptive delta modulation which uses two positive parameters, P and Q, to adjust the step size. The values $P = Q = 1$ describe linear delta modulation (LDM), and Jayant has recommended using $Q = 1/P$ and $1 < P < 2$. In this paper, we study the step response of this scheme for arbitrary P and Q. For each P and Q, we are able to define an integer n, the stability exponent for P and Q, such that the step response is unstable when $P^nQ > 1$, it converges to the new level when $P^nQ < 1$, and when $P^nQ = 1$, it eventually settles into a periodic $(2n + 2)$-step cycle, for almost all initial conditions. For $P \geqq 2$, and for some combinations of P and Q with P between 1.6 and 2, it is possible to have both $PQ < 1$ and $P^nQ \geqq 1$, so that $PQ < 1$ is not sufficient for convergence. When a system is convergent, but a minimum step size δ is imposed, the eventual periodic hunting will not necessarily resemble that of LDM, but will be bounded by δP^n.

I. INTRODUCTION

The basic concepts of delta modulation (DM) have been thoroughly discussed in several recent publications.[1,2] In its simpler forms, delta modulation is a method of digitally encoding an input signal $\mathbf{X} = \{x_i\}$ into binary pulses $\mathbf{C} = \{c_i\}$ (where each $c_i = \pm 1$) so that an approximation $\mathbf{Y} = \{y_i\}$ of $\mathbf{X}$ may be reconstructed from the pulses $\mathbf{C}$ by a simple decoding scheme. The signal $\mathbf{X}$, although presented to the encoder as a discrete-time sequence, will normally be a sampled (and perhaps digitized) version of a continuous-time analog signal. The encoder works by comparing each x_i with y_{i-1} through a feedback circuit to determine the sign of the subsequent pulse c_i, according to the equations

$$c_i = \text{sign}\,(x_i - y_{i-1})$$
$$m_i = c_i M_i, \qquad \text{where} \qquad M_i = |m_i| > 0$$
$$y_i = y_{i-1} + m_i.$$

Various forms of delta modulators differ primarily in the manner of determining the step-size M_i; of course, since only the pulses $\mathbf{C}$ are to be transmitted to the decoder, what is required is a rule for determining M_i from $\mathbf{C}$. In conventional *linear* delta modulation (LDM), the step-size M_i is taken to be a constant δ, independent of the pulses $\mathbf{C}$ (and the signal $\mathbf{X}$), so that each step $m_i = \pm\delta$, resulting in the familiar "staircase" appearance of $\mathbf{Y}$ under LDM. Since in this simplest form of DM, $\mathbf{Y}$ can change by only δ per step, no matter how far x_i is from y_{i-1}, $\mathbf{Y}$ has a very limited ability to keep up with $\mathbf{X}$ when $\mathbf{X}$ has a steep slope, which results in the condition known as slope overload. In contrast to LDM, *adaptive* delta modulation (ADM) permits M_i to be modified depending on $\mathbf{X}$, especially as the slope of the signal $\mathbf{X}$ changes. Since this relieves the slope-overload problem, such adaptation can result in better encoding, and several types of adaptive delta modulators have been described in the literature (for a survey, see Ref. 2).

In this paper, we are concerned with the particular ADM scheme devised by N. S. Jayant,[3] and with certain generalizations of this scheme which arise naturally in the course of the investigation. Jayant's one-bit-memory scheme has been characterized by Steele[2] as "instantaneously companded" (that is, having an "instantaneous" adjustment of the step-size M_i), and Steele refers to Jayant's ADM as "first order constant factor delta modulation." The method is "first order," since Jayant computes M_i using only c_{i-1} in addition to M_{i-1} and c_i; the "one-bit memory" is used to save c_{i-1}. When c_i and c_{i-1} are equal, so that $\mathbf{Y}$ has not yet crossed $\mathbf{X}$, there is a possibility of slope overload, so that M_i should be *increased,* and Jayant uses a "constant factor" $P \geqq 1$ so that $M_i = PM_{i-1}$ (and $m_i = Pm_{i-1}$) when $c_i = c_{i-1}$. To keep the step size from growing continuously with time, a second positive constant factor $Q \leqq 1$ is chosen, so that when c_i and c_{i-1} have *different* signs, indicating that $\mathbf{Y}$ has crossed $\mathbf{X}$, the step size is reduced: $M_i = QM_{i-1}$, so $m_i = -Qm_{i-1}$. (Jayant concluded that values of P and Q with $PQ = 1$ gave the best performance on segments of speech, and he especially recommended $P = \frac{3}{2} = 1.5$, $Q = \frac{2}{3}$.) We note that when $P = Q = 1$, we recover LDM, with $M_i = \delta$ and $m_i = \pm\delta$ for all i.

As even basic LDM has proved to be quite difficult to analyze (see Refs. 5 and 6 for some recent successful efforts), it is hardly surprising that there are few definite analytical conclusions concerning the behavior of Jayant's ADM. This is confirmed by Steele's comment that "An interesting feature of instantaneously adaptive [delta modulators] is their resistance to mathematical analysis····." Thus, in this paper, we restrict our attention to the comparatively simple problem of the step response of the approximating signal **Y** for Jayant's ADM, where by step response we mean the ultimate behavior of **Y** when **X** assumes a constant value $\bar{x}$, $x_j = \bar{x}$ for all $j \geqq i$.

For LDM, if **X** becomes constant, $x_j = \bar{x}$ for $j \geqq i$, then **Y** will eventually enter a "hunting" phase having a two-step period in which adjacent values of **Y** bracket $\bar{x}$ (see Fig. 1); for some k and all $j \geqq 0$,

$$y_{k+2j} = y_k \leqq \bar{x},$$
$$y_{k+2j+1} = y_k + \delta \geqq \bar{x}.$$

Thus, for LDM, **Y** will eventually get and remain no more than δ away from a constant signal **X**, which is a very desirable characteristic. This approximation error, which occurs because **Y** is discrete and cannot exactly match a constant or slowly varying signal **X**, is called "granular error" (or "quantization error"), in contrast to the "slope-overload" error which results from the inability of **Y** to keep up with a steeply climbing **X**. For LDM, a one-time compromise between these two types of error must be made in the choice of the sampling rate and step-size δ; then the granular error is known to be bounded by δ, but the slope-overload error can be severe for unexpectedly steep slopes. For ADM the step size can be varied with the signal, thus reducing the slope-overload error, but nature and magnitude of the granular error is less understood than for the LDM case, a situation which it is hoped that this paper will help resolve.

The question of the nature of the step response of Jayant's ADM was briefly discussed by Jayant in Section 2.3 of Ref. 3, but his conclusions were limited to the finding that in contrast to LDM, the characteristics of the "hunting" phase of the ADM, particularly the minimum step size and maximum error, were very dependent on the mag-

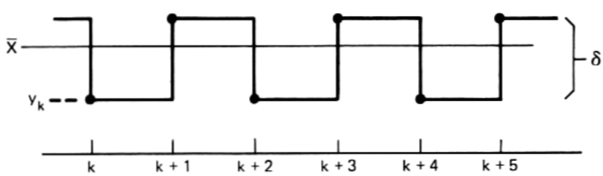

Fig. 1—Period-two (LDM) hunting.

nitude of the constant value $\bar{x}$ (with y_0 and m_0 held fixed). Figure 2, taken from Ref. 3, shows the behavior for $P = \frac{3}{2}$, $Q = \frac{2}{3}$, $y_0 = 0$, $m_0 = 1$, and $\bar{x} = 9, 10, 12$. Steele's analysis[2] showed that the *four*-step cycle exhibited in all three cases of Fig. 2 is exact and sustainable; as shown in Fig. 3, for some k and all $j \geqq 0$, the cycle is given by

$$\begin{aligned} y_{k+4j} &= y_k < \bar{x} \\ y_{k+4j+1} &= y_k + m < \bar{x} \\ y_{k+4j+2} &= y_k + m(P+1) > \bar{x} \\ y_{k+4j+3} &= y_k + mP > \bar{x}, \end{aligned}$$

where $m = m_{k+1} > 0$. Steele further indicated that this four-step periodic behavior is the typical ultimate step response of Jayant's ADM when $PQ = 1$. He also concluded that $PQ < 1$ was necessary for **Y** to converge to **X** for a step input, but he did not provide a complete proof, and he did not claim that $PQ < 1$ was sufficient for the decay of **Y** to a constant $\bar{x}$. (We note that when **Y** is in this four-step cycle, which is a "pure hunting" phase, the signal **X** is crossed only on alternate steps, and the signal value is typically not in the middle of the crossing step, calling into question assumptions used in Section IV of Ref. 3 and in Ref. 4.)

Even before the appearance of Steele's work, experimental results and preliminary analysis had given rise to the general supposition that

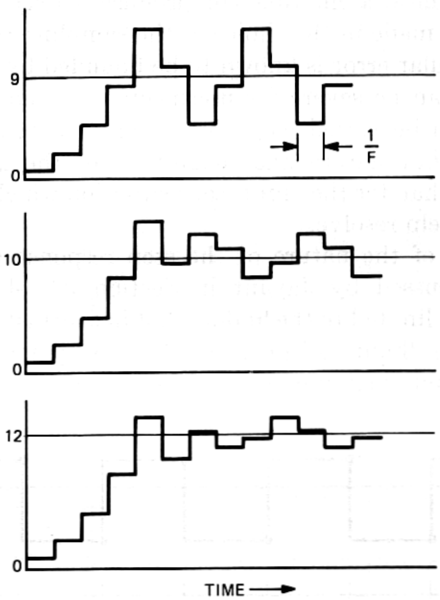

Fig. 2—$PQ = 1$ step responses (from Jayant[3]).

for a step input, (*i*) **Y** would be unstable when $PQ > 1$ (as it was for Jayant's speech data[3]), (*ii*) that when $PQ = 1$, **Y** would ultimately fall into the periodic four-step cycle, but with very large hunting amplitudes possible, and (*iii*) that for $PQ < 1$, **Y** would converge to the constant $\bar{x}$, with both step size and maximum hunting amplitude approaching zero. (Although having the step size get too small is considered undesirable in case **X** should begin to vary, it was generally thought that enforcing a well-chosen minimum step-size δ, as Jayant did in Ref. 3, would avoid this problem.) The question of convergence of **Y** for $PQ < 1$ is the most important of these, since as Steele and others have observed, using a value of PQ slightly less than 1, together with a minimum step size, would eliminate the problem of large-amplitude hunting cycles in **Y** during times when **X** was carrying no signal, while Jayant's results[3] indicate that for $PQ < 1$ but close to 1, the resulting penalty in signal-to-noise ratio during speech segments is negligible.

II. SUMMARY

Our findings on the step response of a P, Q delta modulator confirm that for almost all initial conditions, **Y** will be unstable when $PQ > 1$, and will eventually fall into the four-step cycle shown in Fig. 3 when $PQ = 1$. (We say "almost all" because for each P and Q with $PQ \geqq 1$, there is a set W of initial conditions, negligible in the sense of Lebesgue measure, for which **Y** converges to **X**. In Fig. 2, there would be convergence for $\bar{x} = 11.1625$, so that $y_0 = 0$, $m_0 = 1$, and $\bar{x} = 11.1625$ is a point of W.) More importantly, we find that $PQ < 1$ is *not* sufficient to insure that **Y** will converge to a step input **X**. Rather, in addition to those values of P and Q with $PQ < 1$ for which **Y** converges to **X**, there are values of P and Q with $PQ < 1$ for which **Y** is unstable, and also some combinations for which **Y** is eventually periodic, with a period even and greater than four. However, our results establish that

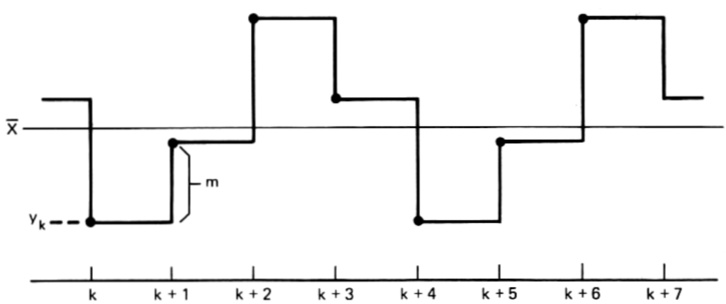

Fig. 3—Period-four ADM hunting.

when $PQ < 1$ and *either* $P \leqq 1.6$ *or* $PQ \geqq 1 - Q$, which are the cases of most practical interest at present, then the $PQ < 1$ conjecture is true, and **Y** converges to a step-input **X** for all initial conditions.

Our basic result is that for each P and Q, we can define an integer n, which we call the *stability exponent for P and Q*, such that the stability of the step response of **Y** depends not on the product PQ, as had been supposed, but on the product P^nQ. Thus, for almost all initial conditions, **Y** is unstable if $P^nQ > 1$, and is eventually periodic with period $2n + 2$ if $P^nQ = 1$; while for $P^nQ < 1$ (or whenever the initial conditions fall in W), **Y** converges to **X**. The generally expected findings for $PQ \geqq 1$ result from the fact that $n = 1$ when $PQ \geqq 1$.

It is useful to describe the stability exponent n in terms of P and PQ. If we define $F_k(P) = P(P - 1)/(P^k - 1)$, then n is the stability exponent for P and Q if and only if $F_{n+1}(P) \leqq PQ < F_n(P)$. Figure 4 shows the graphs of $F_k(P)$ for $k = 1, 2, 3, 4$. We see that $F_{k+1}(P) < F_k(P)$ for $P > 1$, so that n is well defined, and that $F_{k+1}(P)$ approaches zero with increasing k. Thus, n becomes unbounded as Q approaches zero.

The cases of most interest are those for which $PQ < 1$ and **Y** is not convergent, that is, when $F_{n+1}(P) \leqq PQ < F_n(P)$ *and* $P^nQ \geqq 1$. Since $F_{n+1}(P) \leqq P^{-n+1}$, $P^nQ \geqq 1$ implies $PQ \geqq F_{n+1}(P)$, so the bind-

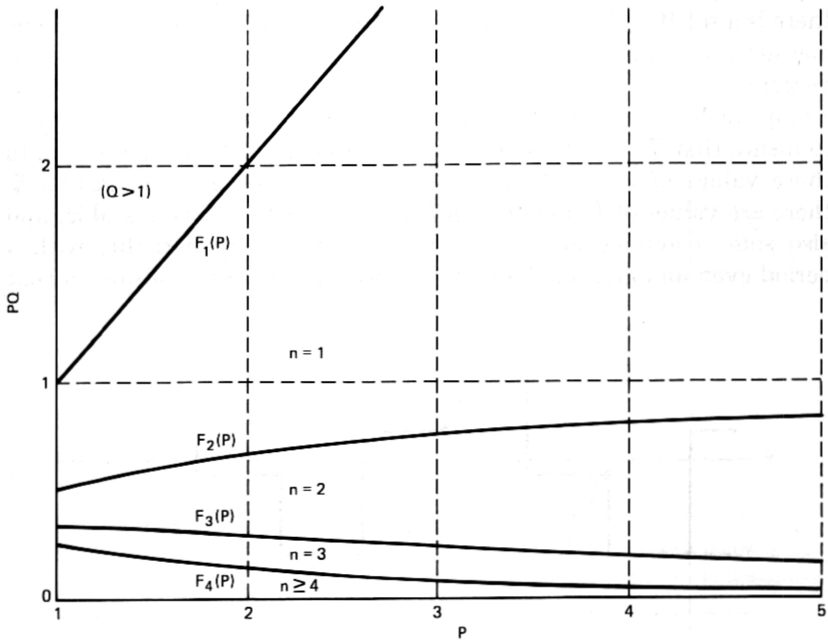

Fig. 4—Domains of the stability exponent n.

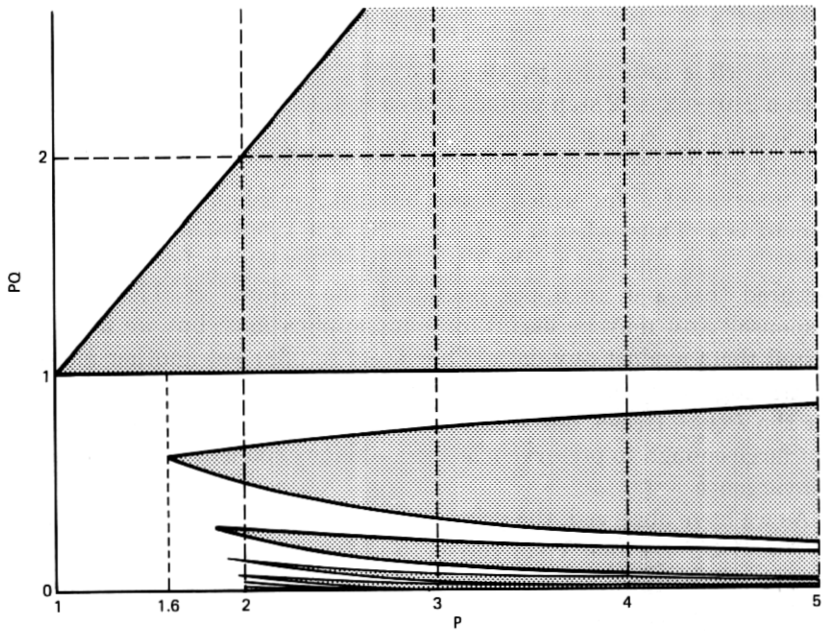

Fig. 5—Domains of unstable step response.

ing constraints are that $PQ < F_n(P)$ and $PQ \geqq P^{-n+1}$. In Fig. 5, those areas for which $P^{-n+1} < PQ < F_n(P)$ are shaded; they represent those values of P and PQ for which **Y** is unstable for almost all initial conditions. Looking particularly at the cases with $PQ < 1$, we see that when $P \leqq 1.6$, **Y** is never unstable, but even such seemingly safe cases as $P = 2$, $Q = 0.3$ fall in the shaded region. As P is made larger, which might be useful in some applications, the combinations for which **Y** is unstable become dominant, so that for $P = 4$, not only those values of Q above $\frac{1}{4}$ cause instability, but also all those between $\frac{1}{16}$ and $\frac{1}{5}$, as well as most values below $\frac{1}{16}$. The basic point of these examples is, of course, that it is not PQ which determines the stability of **Y**, but P^nQ.

The combinations for which $P^nQ = 1$ are interesting in that their step response is a straightforward generalization of that of Jayant's $PQ = 1$ ADM. Specifically, if we first decide on the stability exponent n, choose a $P \geqq 1$ which satisfies

$$P^{n+1} - 2P^n + 1 > 0,$$

and then set $Q = P^{-n}$ so that $P^nQ = 1$, we find that for almost all initial conditions, **Y** will eventually settle into a cycle of period $2n + 2$ steps. The $PQ = 1$ ADM thus appears as the $n = 1$ member of this family, while LDM may be viewed as the $n = 0$ case: $P^0Q = Q = 1$, with a $2 \cdot 0 + 2 = 2$ step period. For each n, the set of P which satisfies the inequality consists of an open interval $(p_n, +\infty)$, where p_n increases with n and approaches 2 from below; $(p_n, +\infty)$ is also exactly the interval of P for which $\mathbf{Y}$ can be unstable when the stability exponent is n. Thus, when $n > 1$, the $P^nQ = 1$ ADM is feasible primarily for $P \geqq 2$, in contrast to the $PQ = 1$ ADM, for which Jayant has conjectured that 2 is an upper bound on the optimal P. These "high-response" ADM may be useful for some applications, but we have not tested them against any data. They seem to offer yet another method of trading off granular error against slope overload. Of course, as for the $PQ = 1$ case, one would actually set P^nQ slightly less than 1, but large enough to preserve n as the stability exponent and thus insure convergence.

As we have observed, the primary current interest is in combinations of P and Q for which $\mathbf{Y}$ converges to a step input $\mathbf{X}$, so any practical system will provide for a minimum step-size δ. Thus, for a step input, the theoretically convergent $\mathbf{Y}$ will eventually encounter the minimum step size and become periodic, hunting about the constant $\bar{x}$. We have considered the step response of a P, Q delta modulator with minimum step size and stability exponent n, and we find that the eventual periodic behavior is exactly that of a P, $Q' = P^{-k}$ delta modulator with stability exponent k, where $0 \leqq k \leqq n$ and $P > p_k$, and where the value of k depends on the initial conditions. Thus, the hunting amplitude is bounded by $\delta P^k \leqq \delta P^n$. Moreover, all those k for which $0 \leqq k \leqq n$ and $P > p_k$ occur for initial conditions having positive Lebesgue measure. In particular, when $1 > PQ \geqq 1 - Q$, so that the stability exponent is $n = 1$, the four-step hunting cycle with range $\delta(1 + P)$ cannot be rejected. Thus, Steele's conclusion that the $k = 0$, LDM-type hunting is the only type that can occur when a minimum step size is imposed does not appear to be justified.

Our investigations also shed some light on the question of recognizing when the slope-overload condition is occurring. Since in the limit for $P^nQ = 1$, the sequence is n "forwards," one "reverse," etc., with only the nth forward crossing the signal, a sequence of n or fewer forwards should not be considered indicative of slope overload. But for $n + k$ forwards in a row, even if we decide to label k of them as slope overload, it is not clear which k of them: first? last? middle? Perhaps the magnitude of the error must be considered as well as crossings. On the other hand, for $P^nQ = 1$, distance *alone* cannot be used as the definition since the amplitude of the hunting can be quite large, depending on the initial conditions. For $P^nQ < 1$ with a minimum step size, much the same considerations apply, except that in this case, the error magnitude would be very useful in recognizing hunting.

III. ANALYSIS

We assume that $i \geqq 0$ for all i, and that "initial conditions" x_0, y_0, and m_0 are given. Since there are no bounds on **X** or **Y**, we may assume that $\bar{x} = 0$, and that the "step" in **X** occurs at $i = 1$, that is, that $x_i = \bar{x} = 0$ for $i \geqq 1$. The effects of the previous history of **Y** and **X** can be summarized in the selection of y_0 and m_0. The step response of **Y** for a P, Q delta modulator is then characterized by how well **Y** can approximate $\bar{x} = 0$ as a function of the parameters P and Q and the initial conditions y_0 and m_0.

Jayant's ADM calculates **Y** from **X** by the following equations:

$$c_i = \text{sign}\,(x_i - y_{i-1})$$

$$m_i = \begin{cases} Pm_{i-1} & \text{if} \quad c_i = c_{i-1} \\ -Qm_{i-1} & \text{if} \quad c_i = -c_{i-1} \end{cases}$$

$$y_i = y_{i-1} + m_i.$$

Since $(x_i - y_{i-1})$, c_i, and m_i will always have the same sign, we may summarize the first two equations as

$$m_i = \begin{cases} Pm_{i-1} & \text{if } (x_i - y_{i-1}) \text{ and } m_{i-1} \text{ have the same sign} \\ -Qm_{i-1} & \text{if they have different signs.} \end{cases}$$

There is ambiguity in this definition, as the sign of zero is not defined; that is, what value of c_i is chosen when $x_i = y_{i-1}$? Our later analysis indicates that the proper choice is $c_i = -c_{i-1}$ when $x_i = y_{i-1}$, so that equality is considered to be a "crossing." After making this convention, and after observing that $x_i = \bar{x} = 0$ implies sign $(x_i - y_{i-1}) = -\text{sign}\,(y_{i-1})$ for $i \geqq 1$, we obtain the equations

$$m_{i+1} = \begin{cases} Pm_i & \text{if } y_i \text{ and } m_i \text{ have different signs} \\ -Qm_i & \text{if they have the same sign (or if } y_i = 0) \end{cases}$$

$$y_{i+1} = y_i + m_{i+1}.$$

This is a two-state system whose state equations have a discontinuity at $y_i = 0$, but we can transform it into a single-state continuous system if we note that the conditions on the comparative signs of y_i and m_i may be expressed as a condition on the sign of their *ratio*, which is always defined since m_i is never zero.

We define the *error-step-size ratio* r_i by $r_i = y_i/m_i$. Then we have

$$\begin{aligned} r_{i+1} &= y_{i+1}/m_{i+1} = 1 + y_i/m_{i+1} \\ &= 1 + (y_i/m_i)(m_i/m_{i+1}) = 1 + r_i(m_i/m_{i+1}), \end{aligned}$$

where

$$\frac{m_{i+1}}{m_i} = \begin{cases} P & \text{if } y_i \text{ and } m_i \text{ have different signs} \\ -Q & \text{if they have the same sign (or } y_i = 0) \end{cases}$$
$$= \begin{cases} P & \text{if } y_i/m_i = r_i < 0 \\ -Q & \text{if } y_i/m_i = r_i \geqq 0; \end{cases}$$

so the state equation for the ratio may be written simply as

$$r_{i+1} = \begin{cases} 1 + r_i/P & \text{if} \quad r_i < 0 \\ 1 - r_i/Q & \text{if} \quad r_i \geqq 0. \end{cases}$$

Thus, the sequence of ratios r_i arises from repeated applications, beginning with $r_0 = y_0/m_0$, of the function $f(\cdot)$ given by

$$f(r) = \begin{cases} 1 + r/P & \text{if} \quad r < 0 \\ 1 - r/Q & \text{if} \quad r \geqq 0. \end{cases}$$

This function is graphed in Fig. 6 for $P = \frac{3}{2}$, $Q = \frac{2}{3}$. Note that $f(\cdot)$ is continuous at $r = 1$, and the continuity is not dependent on our choice of c_i when $x_i = y_{i-1}$, since $f(0) = 1$ simply says that $y_i = x_i + m_i$ when $x_i = y_{i-1}$, which is true no matter how one computes m_i from m_{i-1}. But an important observation is that a particular sequence of r_i's computed from $r_{i+1} = f(r_i)$, together with an initial step m_0, gives the complete sequence of m_i's, since a negative r_i indicates $m_i = Pm_{i-1}$, while an r_i which is positive *or zero* indicates $m_i = -Qm_{i-1}$. Thus, the convention on the sign of zero affects not the sequence of r_i's but the sequence of m_i's derived from it.

We shall henceforth restrict ourselves to combinations of P and Q for which $P > 1$ and $Q < 1$, since this is the only case (other than $P = Q = 1$) that is suitable for practical applications.

In our subsequent analysis we are primarily concerned with the function $f(\cdot)$, which describes how the ratio $r_i = y_i/m_i$ changes from one step to another. Since $f(r) \leqq 1$ for all r, except for r_0 no r_i can

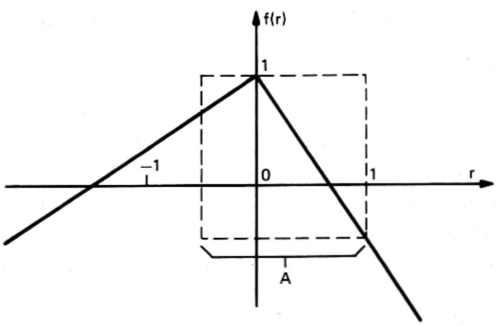

Fig. 6—The graph of $f(r)$ and A for $P = \frac{3}{2}$, $Q = \frac{2}{3}$.

exceed 1. Thus, after the first step we are not concerned with the behavior of $f(r)$ for $r > 1$.

We are not only interested in the change in the error-step-size ratio r_i during one step, which is given by $r_{i+1} = f(r_i)$, but also in the change over two steps, three steps, etc. The change in the ratio over j steps may be determined by applying the function j times, e.g., $r_{i+2} = f(r_{i+1}) = f(f(r_i))$, $r_{i+3} = f(f(f(r_i)))$, etc. The function obtained by applying $f(\cdot)$ j times we call the jth *iterate* of $f(\cdot)$, which we write $f^j(\cdot)$. Thus, we have $r_{i+j} = f^j(r_i)$, and by convention $f^1(r) = f(r)$ and $f^0(r) = r$.

Since $f(r) = 1 + r/P \geqq 1 + r$ when $P > 1$ and $r < 0$, when r is negative, the successive values of $f^j(r)$ will increase by at least 1 per step until finally one of the values $f^j(r)$ is nonnegative, that is, $0 \leqq f^j(r) \leqq 1$. This is just another way of saying that the signal $\mathbf{Y}$ will eventually cross zero on step y_{i+j} beginning at $r = r_i < 0$. But once $f^j(r)$ is in the interval $[0, 1]$, the next value of the ratio, namely $f^{j+1}(r)$, can be no smaller than $f(1) = 1 - 1/Q$, which we denote by q. If $f^{j+1}(r) < 0$, then the subsequent ratios will increase again until they reach $[0, 1]$, etc. Thus, the ratios can never escape the interval $[q, 1] = [1 - 1/Q, 1] = A$ once they enter, and we have proven:

Theorem 1: *If $q = f(1) = 1 - 1/Q < 0$ and $A = [q, 1]$, then for each r there is a j such that $f^j(r) \in A$, and $r_i \in A$ implies $r_k \in A$ for all $k \geqq i$.*

So the ultimate behavior of the ratios is determined by the function $f(\cdot)$ and its iterates on the interval $A = [q, 1]$, and thus by the graph of $f(\cdot)$ on the square $A \times A$, denoted by the dotted lines in Fig. 6.

We shall need more precise information on how many steps are necessary to go from a given ratio r to a zero crossing, or a nonnegative value of $f^j(r)$. We define $a_1 = 0$, $a_2 = -P$ and, in general, $a_{i+1} = a_i - P^i = -\sum_{j=1}^{i} P^j$. We further define $A_0 = [0, 1]$, and $A_i = [a_{i+1}, a_i)$ for $i \geqq 1$. Since $P > 1$, this set of intervals forms a disjoint cover of the range $(-\infty, 1]$ of $f(\cdot)$.

Theorem 2: *If $r \in A_j$, then j is the least integer such that $f^j(r)$ is nonnegative, so that $r \in A_j$ if and only if $r \leqq 1$ and exactly j steps produce a zero crossing of $\mathbf{Y}$. Also, the sequence $f^i(r)$ is increasing for $0 \leqq i \leqq j$.*

Proof: Since $f(a_{i+1}) = a_i$ for $i \geqq 1$, it follows that $f(A_{i+1}) = A_i$ for $i \geqq 1$. Thus, if $r \in A_j$, after $j - 1$ steps, $f^{j-1}(r) \in A_1$. Then, $f(A_1) = [0, 1) \subset [0, 1] = A_0$, so $f^j(r) \in [0, 1]$.

Corollary: *For every $r_0 = y_0/m_0$, the ratios eventually enter and remain in A.*

Proof: For $r \leqq 1$, we have $f^j(r) \in [0, 1] \subset A$, while for $r > 1$, $f(r) \leqq 1$ so that $f(r) \in A_j$ for some j, so that $f^{j+1}(r) \in [0, 1]$.

We can now define n, the *stability exponent for P and Q*, as the largest value of j such that A_j intersects A; that is, it is the maximum number of steps from a ratio r *in* A to a zero crossing. Clearly, n is determined by the fact that $q < 0$, so that $q \in A_n$ for some $n > 0$, and this n is the stability exponent. More explicitly, P and Q must satisfy

$$a_{n+1} \leqq q < a_n$$

or

$$-\sum_{j=1}^{n} P^j \leqq 1 - 1/Q < -\sum_{j=1}^{n-1} P^j$$

or

$$\sum_{j=0}^{n} P^j \geqq 1/Q > \sum_{j=0}^{n-1} P^j.$$

To obtain the conditions cited in the summary, we invert and multiply by P to obtain

$$F_{n+1}(P) \leqq PQ < F_n(P),$$

where

$$F_k(P) = P \Big/ \sum_{j=0}^{k-1} P^j = \frac{P(P-1)}{P^k - 1}.$$

Another way of expressing this condition is

$$\frac{P^n - 1}{P - 1} < \frac{1}{Q} \leqq \frac{P^{n+1} - 1}{P - 1},$$

so multiplying by $(P - 1)Q$ and adding Q gives

$$P^nQ < P + Q - 1 \leqq P^{n+1}Q.$$

Thus, the stability exponent n is the largest n such that P^nQ is strictly less than the quantity $P + Q - 1$. Note that $Q < 1$ implies $P^nQ < P + Q - 1 < P$, so that $P^{n-1}Q < 1$ whenever n is the stability exponent for P and Q.

Theorem 3: If n is the stability exponent for P and Q, and $P^nQ < 1$, then $\boldsymbol{Y}$ converges for all initial conditions, that is, both m_i and y_i tend to zero with increasing i.

Proof: Once the ratios enter A, no more than n negative ratios can occur without an intervening nonnegative ratio. Thus, as m_i evolves by multiplication of P's and $(-Q)$'s, each $-Q$ can be grouped with at most n P's with no P's left over. Since $P^nQ < 1$, the absolute value of m_i will be decreasing by a factor bounded away from 1 at least every

$(n+1)$ steps, and hence going to zero. Each time a ratio is nonnegative, which occurs at least once every $(n+1)$ steps, **Y** has just crossed zero, so y_i must go to zero along with m_i.

The next theorem is the basic result of the theory of Jayant's adaptive delta modulation. It states that not only is the stability exponent n the *maximum* number of successive negative ratios that can occur once the ratios enter A, but that for *almost all* initial conditions (initial ratios r_0), a sequence of n negative ratios all in A will eventually occur. (Here by "almost all" we mean that the set of initial conditions for which this is false has Lebesgue measure zero—it can be covered by a family of open intervals of arbitrarily small total length.) This result is the key to the analysis for $P^nQ \geqq 1$.

Theorem 4: Let $B_n = A \cap A_n = [q, a_n)$ and let B be the set of $r \in A$ such that $f^j(r) \in B_n$ for some j (so that n successive negative ratios eventually occur). Then B is open (as a subset of A) and has Lebesgue measure $\mu(B) = 1/Q = 1 - q$, the length (and Lebesgue measure) of A. Thus, $A \backslash B$ (the points of A not in B) is a measurable set of Lebesgue measure zero.

Proof: B_n is open in A, and B can be written as

$$B = \bigcup_{i=0}^{\infty} \{r \mid f^i(r) \in B_n\}.$$

Since each $f^i(\cdot)$ is a continuous function from A into A, each set in the union is open, so B itself is open. Thus, B and its complement $A \backslash B$ are measurable. Clearly, if S is a subset of B, and S' is a subset of A such that $f(S') = S$, then S' is a subset of B also. In addition, if $f(\cdot)$ is linear with slope $1/s$ on S', $f(S') = S$, and S and S' are measurable, then $\mu(S') = |s| \cdot \mu(S)$. For each i, $0 \leqq i \leqq n$, let $B_i = A_i \cap B$, so that each B_i is measurable with measure $\mu(B_i)$. Now $f(\cdot)$ maps A_0 linearly onto A, with slope $-1/Q$, so $f(\cdot)$ must map B_0 linearly onto B, and $\mu(B_0) = Q \cdot \mu(B)$. Similarly, for each i such that $n - 1 > i > 0$, $f(\cdot)$ maps A_{i+1} linearly onto A_i with slope $1/P$, so $f(\cdot)$ must map B_{i+1} linearly onto B_i, and $\mu(B_{i+1}) = P \cdot \mu(B_i)$. When $i = 0$, $f(\cdot)$ maps A_1 linearly onto $A_0 \backslash \{1\}$, so $\mu(B_1) = P \cdot \mu(B_0 \backslash \{1\})$; but since $\{1\}$ has measure zero, $\mu(B_1) = P \cdot \mu(B_0)$ also. Thus, for $0 \leqq i < n$, we have $\mu(B_i) = P^i \cdot \mu(B_0)$. But since B is the disjoint union of the B_i, we have

$$\mu(B) = \sum_{i=0}^{n} \mu(B_i) = \mu(B_n) + \sum_{i=0}^{n-1} P^i \cdot \mu(B_0)$$

$$= (a_n - q) + \mu(B_0) \cdot \sum_{i=0}^{n-1} P^i.$$

Since $\mu(B_0) = Q \cdot \mu(B)$, and $\sum_{i=0}^{n-1} P^i = 1 - a_n$,

$$\mu(B_0)/Q = (a_n - q) + (1 - a_n)\mu(B_0)$$

or

$$\mu(B_0)(1/Q - 1 + a_n) = \mu(B_0)(a_n - q) = a_n - q.$$

Since $q < a_n$ (this relies on the convention that $m_{i+1} = -Q \cdot m_i$ when $y_i = 0$), we have $\mu(B_0) = 1$, so $\mu(B) = 1/Q = 1 - q = \mu(A)$. Thus $\mu(A \backslash B) = 0$.

Corollary: Let W be the set of real numbers r such that $f^i(r) \notin B_n$ for all i, i.e., once $f^j(r)$ is in A, no sequence of n successive negative ratios ever occurs. Then W has Lebesgue measure zero.

Proof: Let $W_0 = A \backslash B$ and for all i, let W_i be the set of r for which $f^i(r) \in W_0$. Since each $f^i(\cdot)$ is piecewise linear, each W_i has measure zero, so $W = \bigcup_{i=0}^{\infty} W_i = \{r \,|\, f^i(r) \in W_0 \text{ for some } i\}$ has measure zero. But since W_0 is the set of $r \in A$ such that $f^i(r) \notin B_n$ for all i, W is the set of (unrestricted) r such that $f^i(r) \notin B_n$ for all i.

We note that W is nonempty for all $P > 1$ and $Q < 1$, since $f(\cdot)$ has a fixed-point $w = Q/(Q+1) \in (0, 1)$, and w and all its preimages (r such that $f^i(r) = w$ for some i) will be in W. In addition, for all $i \geqq 2$, $f^i(\cdot)$ will have fixed points in addition to w, and many of these fixed points and their preimages will be in W also.

Theorem 5: With n the stability exponent for P and Q, on $A_n = [a_{n+1}, a_n)$ the function $f^{n+1}(\cdot)$ is linear with slope $-(P^nQ)^{-1}$ and has a fixed point $z \in [q, a_n) = B_n$.

Proof: Since $f(A_{i+1}) \subset A_i$ and $f(\cdot)$ is linear on each A_i, $f^j(\cdot)$ is linear on each A_i for $j \leqq i + 1$. Clearly, $f^n(a_{n+1}) = 0$ and $f^n(a_n) = 1$, so $f^{n+1}(a_{n+1}) = 1$ and $f^{n+1}(a_n) = f(1) = q = 1 - 1/Q$. The slope of $f^{n+1}(\cdot)$ on A_n is thus $(q - 1)/(a_n - a_{n+1}) = (-1/Q)/P^n = -(P^nQ)^{-1}$. Since $q \in A_n$ by definition, $q = f^{n+1}(a_n) < a_n$, but since $f^{n+1}(\cdot)$ has negative slope, $f^{n+1}(q) > f^{n+1}(a_n) = q$. Thus, $f^{n+1}(q) > q$, $f^{n+1}(a_n) < a_n$, and so $f^{n+1}(\cdot)$ has a fixed point z between q and a_n.

Theorem 6: If $P^nQ > 1$, then $f^{n+1}(B_n) \subset B_n$, that is, if $r_i \in B_n = [q, a_n)$, then $r_{i+(n+1)k} \in B_n$ for all $k \geqq 0$. Thus, except for $r_0 \in W$, the ratios eventually enter B_n and return to B_n every $n + 1$ steps thereafter. Moreover, the ratios falling in B_n converge to the fixed point z of $f^{n+1}(\cdot)$.

Proof: $f^{n+1}(a_n) = q$, and the absolute value of the slope of $f^{n+1}(\cdot)$ on A_n is $(P^nQ)^{-1} < 1$ so $|f^{n+1}(q) - f^{n+1}(a_n)| < |q - a_n|$ and so $f^{n+1}(B_n) = (q, f^{n+1}(q)] \subset (q, a_n) \subset B_n$. Each $f^{(n+1)k}(B_n)$ is an interval containing z, and each increase in k (each $n+1$ steps) reduces the length of the interval by a factor $(P^nQ)^{-1} < 1$, so for each $r \in B_n$ we have $f^{(n+1)k}(r)$ approaching z with increasing k. Thus, except for initial conditions in W, the ratios not only eventually enter B_n (by the corollary to Theorem 4) but return there every $n+1$ steps, each time coming closer to z.

Corollary: If n is the stability exponent for P and Q and $P^nQ > 1$, then for all initial conditions which are not in W, the signal $\mathbf{Y}$ is unstable. Also, if $r_i \in B_n$, then $M_{i+j} > M_i$ for all $j > 0$, where $M_i = |m_i|$.

Proof: Once r_i is in B_n, every $n+1$ steps M_i increases by a factor of $P^nQ > 1$; hence the step size increases without bound.

The next theorem and its corollary establish the nature of the stable, periodic step response which is characteristic of the Jayant family of delta modulators.

Theorem 7: If n is the stability exponent for P and Q and $P^nQ = 1$, then $f^{2n+2}(\cdot)$ is the identity on B_n, and if y_i and m_i are such that $r_i = y_i/m_i \in B_n$, then whenever $j \geqq i$, $k \geqq 0$, and $l = (2n+2)k$, we have $y_{j+l} = y_j$ and $m_{j+l} = m_j$, so that $\mathbf{Y}$ becomes periodic with period $2n+2$ steps. Thus for all initial conditions which are not in W, $\mathbf{Y}$ eventually settles into a periodic $(2n+2)$-step cycle.

Proof: If $P^nQ = 1$, then the slope of $f^{n+1}(\cdot)$ is -1, so that $f^{n+1}(q) = a_n$ in addition to $f^{n+1}(a_n) = q$. Thus, $f^{2n+2}(a_n) = a_n$, $f^{2n+2}(q) = q$, so $f^{2n+2}(\cdot)$ is the identity on $[q, a_n]$ and hence on $B_n = [q, a_n)$ itself. Thus, when $r_j \in B_n$, $r_{j+2n+2} = r_j$. But by Theorem 2 we know that among the $2n+2$ successive values of r_{j+i} there are $2n$ negative ones and 2 nonnegative ones, so that $m_{j+2n+2} = P^{2n}(-Q)^2 m_j = (-P^nQ)^2 m_j = m_j$. Thus, $y_{j+2n+2} = y_j$ as well. The connection with W is made as in previous theorems.

Theorem 8: If $P^nQ \geqq 1$ and $r_0 \in W$, then y_i and m_i both converge to 0, i.e., for initial conditions in W, $\mathbf{Y}$ is neither unstable nor periodic but converges to $\mathbf{X}$.

Proof: For all initial conditions, the ratios eventually enter and remain in A, but if $r_0 \in W$, then all ratios in A fall in the A_i with $i < n$. Thus, at most, $n-1$ successive negative ratios can occur; hence, each $-Q$ can be grouped with no more than $n-1$ P's with no P's left over. But $P^iQ < 1$ for $i < n$ even if $P^nQ > 1$, so at intervals of no more than n steps m_i will be reduced in absolute value by a factor bounded away from 1; hence m_i will converge to zero, and with it $\mathbf{Y}$, since a zero crossing will occur at least every n steps.

We can now relate our findings to the general supposition on the stability of Jayant's delta modulator: that is, that the system is unstable, periodic, or convergent according to whether PQ exceeds, equals, or is less than 1. We see that the general supposition is in fact correct when $PQ \geqq 1 - Q$ and $r_0 \notin W$.

Theorem 9: *If $PQ \geqq 1 - Q$, then the stability exponent for P and Q is 1. Thus,* $\mathbf{Y}$ *converges to* $\mathbf{X}$ *when $1 - Q \leqq PQ < 1$ (or when $PQ \geqq 1$ and $r_0 \in W$), settles into a four-step cycle when $PQ = 1$ and $r_0 \notin W$, and is unstable when $PQ > 1$ and $r_0 \notin W$.*

Proof: All we must show is that $q = 1 - 1/Q \geqq a_2 = -P$, so that $q \in A_1$. But dividing $1 - Q \leqq PQ$ by $-Q$ yields $q \geqq -P$ as required. The rest follows from our earlier theorems, taking $n = 1$.

The most unexpected results of our analysis are the existence of both unstable combinations of P and Q with $PQ < 1$ and Jayant-type delta modulators that satisfy $P^nQ = 1$ and are eventually periodic with a $2n + 2$ step period when $n > 1$ (and $r_0 \notin W$). The next three theorems establish that since n depends on P and Q, in order to attain $P^nQ \geqq 1$ we must have $P > p_n$, where $p_1 = 1$, $p_2 \approx 1.62$, $p_i < p_{i+1}$, and $\lim_{i\to\infty} p_i = 2$. Thus, for $P \geqq 2$, all values of n are realizable, while for $P \leqq p_2 \approx 1.62$, only the $n = 1$ value will allow $P^nQ \geqq 1$. (The sequence $\{p_i\}$ that we define here comes up again in our subsequent analysis of a P, Q delta modulator with a minimum step size.)

Theorem 10: *If $P^kQ \geqq 1$, then $q \geqq a_{k+1}$, so the stability exponent for P and $Q \geqq P^{-k}$ cannot exceed k.*

Proof: Since $q = 1 - 1/Q \geqq 1 - P^k$, all we need show is that $1 - P^k \geqq a_{k+1} = -\sum_{i=1}^{k} P^i$, or $P^k \leqq \sum_{i=0}^{k} P^i$, which always holds. Thus, if $q \in A_n = [a_{n+1}, a_n)$, then $a_n > q \geqq a_{k+1}$ so $n \leqq k$.

Theorem 11: *We can choose a Q such that $P^nQ \geqq 1$, where n is the stability exponent for P and Q, if and only if P satisfies $P^{n+1} - 2P^n + 1 > 0$. Equivalently, n is the stability exponent for P and $Q = P^{-n}$ ($P^nQ = 1$) if and only if $P^{n+1} - 2P^n + 1 > 0$.*

Proof: If $P^nQ \geqq 1$, then $q = 1 - 1/Q \geqq 1 - P^n$. By the definition of n, $1 - P^n \leqq q < a_n = -\sum_{j=1}^{n-1} P^j$, so $P^n > \sum_{j=1}^{n-1} P^j = (P^n - 1)/(P - 1)$. But then $P^n(P - 1) = P^{n+1} - P^n > P^n - 1$, and $P^{n+1} - 2P^n + 1 > 0$. Since each of these steps can be reversed, if $P^{k+1} - 2P^k + 1 > 0$, then setting $Q = P^{-k}$, we have $q < a_k$, so $n \geqq k$. Since q is strictly less than a_k and $\partial q/\partial Q > 0$, there is an open interval of values of $Q \geqq P^{-k}$ for which $n \geqq k$. But by Theorem 10, $n \leqq k$

when $P^kQ \geqq 1$, so for these values of Q we have $n = k$ and $P^nQ = 1$ or $P^nQ > 1$, respectively.

Theorem 12: For each $k \geqq 1$, let $\mathcal{P}_k$ be the set of $P > 1$ which satisfy $P^{k+1} - 2P^k + 1 > 0$. Then, each $\mathcal{P}_k$ is an open half-line $(p_k, +\infty)$, where $p_k < p_{k+1} < 2$ and $\lim_{k\to\infty} p_k = 2$.

Proof: For $k = 1$, the requirement is simply that $(P - 1)^2 > 0$, so $p_1 = 1$. For $k \geqq 2$, differentiating $g(P) = P^{k+1} - 2P^k + 1$ gives $g'(P) = (k + 1)P^k - 2kP^{k-1}$, whose only zero besides $P = 0$ is $P = 2k/(k + 1)$, which lies between 1 and 2 and approaches 2 with increasing k. Since $g(1) = 0$, $g'(1) = 1 - k < 0$, and $g(2) = 1$, $g(P)$ has a zero p_k between $2k/(k + 1)$ and 2, and $g(P) > 0$ for $P \geqq 2$. Thus, $P > p_k$ implies $g(P) > 0$, and $1 < P < p_k$ implies $g(P) < 0$. Since $2k/(k + 1) < p_k < 2$, p_k approaches 2 with increasing k. Since $g(p_{k+1}) = p_k - 1 > 0$, $p_{k+1} > p_k$, so the sequence $\{p_k\}$ converges monotonically to 2.

In fact, since $g(2) = 1$ and $g'(2) = 2^k$, a good approximation for p_k is $2 - 2^{-k}$. For $k = 2, 3, 4$, the approximations are 1.75, 1.875, 1.9375 and the actual values 1.6180, 1.8393, 1.9275.

We have previously observed that the periodicity that occurs when $P^nQ = 1$ is undesirable in practical systems, since it may result in **Y** having significant power when **X** is zero or close to it. This problem is aggravated by the fact that the amplitude of the periodic hunting is unpredictable and can be quite large. To overcome this problem, Steele and others have suggested setting P^nQ slightly less than 1, so as to make the **Y** converge to **X**, and using a minimum step size, which we call δ, to prevent the step size from getting so close to zero during long stretches of zero (or constant) signal **X** that **Y** cannot quickly respond when **X** begins to vary. Indeed even when studying the case $PQ = 1$, Jayant used a minimum step size, although it was seldom binding (see Fig. 3 of Ref. 3).

In our final three theorems we treat the case of a P, Q delta modulator with a minimum step-size δ, so that when $M_i < \delta/Q$ and a zero crossing occurs, instead of the next step having magnitude $M_{i+1} = QM_i < \delta$, we set $M_{i+1} = \delta$. Thus, $M_i \geqq \delta$ for all i. We note that if **Y** would be unstable or periodic in the absence of a minimum step size, then the step sizes may never be reduced to the point that the minimum becomes binding. If a step of size δ does occur, however, with $M_i = \delta$ and $r_{i-1} \in [0, 1] = A_0$, we show that **Y** eventually becomes periodic with a $2J + 2$ step cycle, where $0 \leqq J \leqq n$ (the stability exponent for P and Q) and $P > p_J$; with the exception of the case $P^nQ > 1$, $r_i \in B_n \subset A_n$, for which **Y** is unstable and a step of size δ never reoccurs. Thus, in contradiction to Steele's conclusion, the step response of a P, Q delta modulator with minimum step size does not reduce to the LDM case, but is fully as complex as the $P^nQ = 1$ case with no minimum. However, it is true that if $P^nQ < 1$, or $P^nQ \geqq 1$ and $r_0 \in W$, the minimum step size will eventually occur and the hunting amplitudes be thereafter bounded by $P^n\delta$.

Theorem 13: *If $r_i < 0$, then $r_{i+1} = f(r_i)$; if $r_i \geqq 0$ and $M_i \geqq \delta/Q$, then $r_{i+1} = f(r_i)$; and if $r_i \geqq 0$ and $\delta \leqq M_i < \delta/Q$, then $f(r_i) < r_{i+1} \leqq 1$. Thus, for all initial conditions, $r_i \in A$ for some i, and if $r_i \in A$ then $r_j \in A$ for all $j \geqq i$.*

Proof: When $r_i \leqq 0$, we have $m_{i+1} = Pm_i$, so the minimum is not relevant, and when $r_i \geqq 0$ and $M_i \geqq \delta/Q$, we have $m_{i+1} = -Qm_i$, so the minimum is not binding. Thus, for these cases, $r_{i+1} = f(r_i)$. But when $r_i \geqq 0$ and $M_i < \delta/Q$, we have $f(r_i) = 1 - r_i/Q$ but $r_{i+1} = y_{i+1}/m_{i+1} = 1 + (m_i/m_{i+1})(y_i/m_i) = 1 + (m_i/m_{i+1})r_i$. Since $m_i/m_{i+1} < 0$, we can write this $r_{i+1} = 1 - (M_i/M_{i+1})r_i$. But $M_{i+1} = \delta$, $M_i < \delta/Q$ so $M_i/M_{i+1} < (\delta/Q)/\delta = 1/Q, 0 \leqq 1 - r_{i+1} = r_i(M_i/M_{i+1}) < r_i/Q$, and so $1 \geqq r_{i+1} > 1 - r_i/Q = f(r_i)$. Thus, the evolution of r_i for $r_i < 0$ is given by $f(\cdot)$, so $r_i \in A$ and $r_i < 0$ implies $r_{i+1} \in A$; while if $r_i \in [0, 1]$, $q \leqq f(r_i) \leqq r_{i+1} \leqq 1$ so $r_{i+1} \in A$ in this case also.

For the next two theorems, we assume that a minimum step size has occurred, with $M_i = \delta$, and that $r_{i-1} \in A_0$ so that $r_i \in A$. Since $r_i \in A$, we must have $r_i \in A_J$ for some J, $0 \leqq J \leqq n$, where n is the stability exponent for P and Q. For almost all cases of interest, steps of size δ will continue to occur at least every J steps, and $\mathbf{Y}$ will be periodic; the sole exception, which we dispose of first, is when $P^nQ > 1$ and $J = n$, in which case $\mathbf{Y}$ is unstable and a step of size δ *never* reoccurs.

Theorem 14: *If $M_i \geqq \delta$, $r_i \in A \cap A_n = B_n$, and $P^nQ > 1$, then $r_{i+j} = f^j(r_i)$ and $M_{i+j} > \delta$ for all $j > 0$, so that Theorem 6 and its corollary apply and $\mathbf{Y}$ is unstable.*

Proof: If $M_i \geqq \delta$ and $r_i \in B_n$, then by Theorem 13, $r_{i+n} = f^n(r_i) \in A_0$, and $M_{i+n} = P^nM_i$. But $P^nQ > 1$, so $M_{i+n} > M_i/Q \geqq \delta/Q$, and $M_{i+n+1} = QM_{i+n} = P^nQM_i \geqq \delta P^nQ > \delta$. Thus, $r_{i+n+1} = f^{n+1}(r_i) \in B_n$, $M_{i+n+1} \geqq \delta P^nQ$, and so $r_{i+(n+1)k} \in B_n$ and $M_{i+(n+1)k} \geqq \delta(P^nQ)^k$ for all $k \geqq 0$.

The next theorem characterizes the ultimate behavior of the P, Q delta modulator with minimum step size for the more interesting cases —those not covered by Theorem 14. Thus, we assume that $M_i = \delta$ and $r_i \in A$, with $r_i \in A_J$, where $P^JQ \leqq 1$. Without loss of generality, we choose signs so that $m_i = M_i = \delta$ and $y_i - \delta = y_{i-1} \leqq 0$ (we continue to assume $\bar{x} = 0$, i.e., **X** is identically zero). Since $r_i \in A_J$, we have $r_{i+J} \in A_0$ so $r_{i+J+1} \in A_K$ for some K, $0 \leqq K \leqq n$. To simplify the notation, we set $l = 2J + 2$.

Theorem 15: *If* $m_i = \delta$, $r_i \in A \cap A_J$, $P^J Q \leqq 1$, *and* $r_{i+J+1} \in A_K$, *then* $K \leqq J$. *If* $K = J$, *then* $P > p_J$, *and* $y_{i+l} = y_i$, $m_{i+l} = m_i$, *and* **Y** *is periodic with period* $2J + 2$ *and maximum amplitude* $\delta P^J \leqq \delta P^n$. *Moreover, for each* j *such that* $0 \leqq j \leqq n$ *and* $P > p_j$, *the set of initial conditions which produce a* $(2j + 2)$*-step period has positive Lebesgue measure.*

Proof: When $J \geqq 2$, we have $y_{i+1} = y_i + \delta P < 0$, $m_{i+1} = \delta P$; $y_{i+2} = y_i + \delta(P + P^2)$, $m_{i+2} = \delta P^2$; and, in general (even for $J = 0, 1$), we have $y_{i+J} = y_i + \delta \sum_{j=1}^{J} P^j \geqq 0$, $m_{i+J} = \delta P^J$. Since $P^J Q \leqq 1$, $\delta P^J \leqq \delta / Q$ so that $m_{i+J+1} = -\delta$. If $K \geqq J$, then

$$y_{i-1+l} = y_{i+J} - \delta \sum_{j=0}^{J} P^j = y_i - \delta = y_{i-1} \leqq 0,$$

so that $K \leqq J$; thus, $K \geqq J$ implies $K = J$, so we have proven that $K \leqq J$. If $K = J$, then we have seen that $y_{i-1+l} = y_{i-1}$; also, $m_{i+l} = \delta$ since $m_{i-1+l} = -\delta P^J$ and $P^J \leqq 1/Q$. Thus, $y_{i+l} = y_{i-1+l} + \delta = y_{i-1} + \delta = y_i$, and $m_{i+l} = \delta = m_i$, so **Y** is periodic with period $l = 2J + 2$. To show that $P > p_J$ when $K = J$ and **Y** has period $l = 2J + 2$, we observe that by the definition of J and K $(=J)$ we have (see Fig. 7,

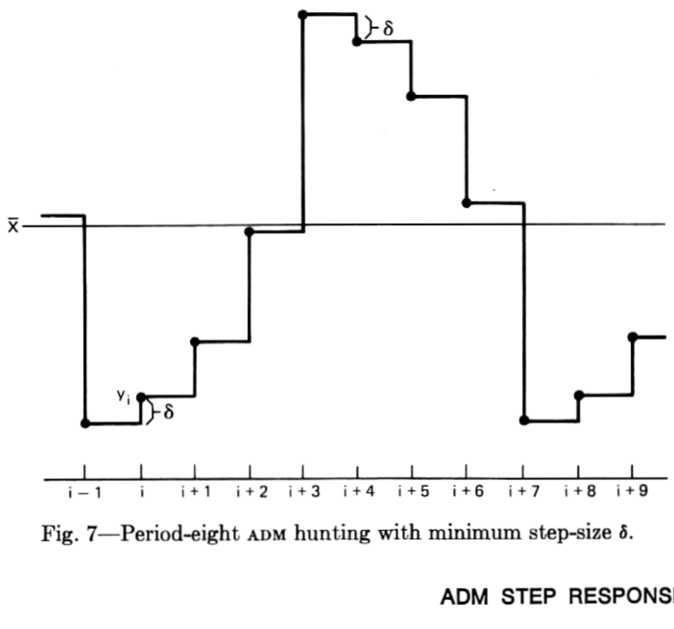

Fig. 7—Period-eight ADM hunting with minimum step-size δ.

with $J = 3$)

$$y_{i+J-1} < 0, \qquad y_{i+J} \geqq 0,$$
$$y_{i-2+l} > 0, \qquad y_{i-1+l} \leqq 0,$$

so that

$$y_{i+J-1} = y_{i-1} + \delta \sum_{j=0}^{J-1} P^j < 0$$

$$y_{i-2+l} = y_{i-1+l} + \delta P^J = y_{i-1} + \delta P^J > 0,$$

so that

$$y_{i-1} + \delta \sum_{j=0}^{J-1} p^j < y_{i-1} + \delta P^J,$$

so

$$\frac{P^J - 1}{P - 1} < P^J$$

from which

$$P^{J+1} - 2P^J + 1 > 0.$$

But this is the defining condition for $P > p_J$. To show that each j satisfying $0 \leqq j \leqq n$ and $P > p_j$ comes up with positive measure, it is only necessary to observe that choosing y_0, m_0 such that $\delta \leqq -m_0 \leqq \delta/Q$ and

$$-\delta P^J < y_0 < -\delta \sum_{j=1}^{J-1} P^j$$

will realize the $2J + 2$ step period analyzed above with $i = 1$.

We note that once a minimum step occurs, the series of "reversal numbers" (of which the J and K are two adjacent elements) is monotone decreasing ($K < J$) until it repeats itself ($K = J$), after which it is constant, and $\mathbf{Y}$ is periodic. This monotonicity holds only *after* δ occurs; when there is no minimum step size, there is no monotonicity, except that when $P^nQ \geqq 1$ an occurrence of $J = n$ will result in nothing but n's thereafter. What we have shown is:

Corollary: *If* δ *is the minimum step size and* $M_i = \delta$ *where* $r_i \in A \cap A_j$, *then unless* $P^nQ > 1$ *and* $j = n$, *within* $(j + 1)^2$ *steps* $\mathbf{Y}$ *will become periodic with period* $2J + 2$, *where* $0 \leqq J \leqq j$.

Proof: Until the reversal numbers become constant, at least every $j + 1$ steps a new, lower reversal number occurs, and there are only $j + 1$ possible such numbers; thus, within $(j + 1)^2$ steps the minimum number J is obtained and $\mathbf{Y}$ is periodic.

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