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A Note on the Capacity of the Band-Limited Gaussian Channel

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In this paper we reexamine results of a previous paper[1] *in which the capacity of the continuous-time channel with bandwidth* W, *average signal power* P_0, *and additive gaussian noise with flat spectral density* N_0 *was shown to be approximately* $W \ln (1 + P_0/N_0W)$ *under a number of physically consistent assumptions.*

When one of the models in Ref. 1 is modified by techniques suggested by Slepian in his 1974 Shannon Lecture,[2] *the channel capacity turns out to be exactly* $W \ln (1 + P_0/N_0W)$.

I. INTRODUCTION

In his 1974 Shannon Lecture,[2] D. Slepian introduced still another way of resolving the well-known paradoxes that arise when band-limited signals are studied in a physical "real world" context. One such paradox results from the fact that a mathematically band-limited function is determined for all time by its values in an arbitrarily small temporal interval—a highly nonphysical situation. An essential element in Slepian's resolution of these paradoxes is the recognition of the role of measurement accuracy in the determination of signals. To incorporate this into his mathematical model, he introduces the following concept. Two signals $s_1(t)$, $s_2(t)$, $-\infty < t < \infty$ *are really indistinguishable at level* ϵ if

$$\|s_1 - s_2\|^2 \leqq \epsilon, \tag{1}$$

where

$$\|f\|^2 \triangleq \int_{-\infty}^{\infty} f^2(t)dt$$

is the "energy" of the function of $f(t)$. He then says that a signal $g(t)$, $-\infty < t < \infty$, is *bandlimited to* $(-W, W)$ *at level* ϵ if $u_1(t)$ and $u_2(t)$ are really indistinguishable at level ϵ, where

$$U_1(f) = G(f) \tag{2a}$$

and

$$U_2(f) = \begin{cases} G(f), & |f| \leqq W, \\ 0, & |f| > W. \end{cases} \tag{2b}$$

Here, U_1, U_2, and G are the Fourier transforms of u_1, u_2, and g respectively, i.e.,

$$U_1(f) = \int_{-\infty}^{\infty} e^{-i2\pi f t} u_1(t) dt,$$

etc. With band-limited functions so defined, paradoxes such as the one mentioned above are resolved, i.e., that $g(t)$ is band-limited to level $\epsilon > 0$ *does not* imply its predictability.

Let us remark that the quantity ϵ in the above definitions represents a limit on the accuracy of the measuring instruments used to determine the frequency spectrum of a signal. Note that $g(t)$ band-limited to a level ϵ *does not* imply that $c \cdot g(t)$ $(c > 1)$ is also so band-limited, even though $g(t)$ and $c \cdot g(t)$ have the same shape. Thus, Slepian's notion of band-limited signals is distinctly different from the usual notion which defines the bandwidth of a signal as a function of its shape.

In this note, we take another look at a related problem—determining the capacity of the band-limited gaussian channel—in the context of Slepian's bandwidth definition. We show that results obtained by the present author[1] have a particularly elegant statement in this new context.

II. STATEMENT OF THE PROBLEM

The definition of the continuous-time, band-limited, additive gaussian noise channel has the following components:

(*i*) Specification of a set $\mathcal{A}(T, W, P_0)$ of allowable channel input signals that are "approximately band-limited" to $(-W, W)$, approximately time-limited to $(-T/2, T/2)$, and with total energy not exceeding P_0T (so that the average power is $\leqq P_0$).

(*ii*) Specification of the noise.

The channel output is

$$y(t) = s(t) + z(t),$$

where the channel input $s \epsilon \mathcal{A}(T, W, P_0)$, and the noise $z(t)$ is specified by (*ii*).

We take W and P_0 to be fixed parameters. A *code* with parameters (T, M, P_e) is a set of M functions called code words which belong to $\mathcal{A}(T, W, P_0)$, together with a decoder mapping which associates the received signal $y(t)$, $|t| < T/2$, with one of the M code words. With each of the M code words assumed to be *a priori* equally likely to be transmitted, P_e is the probability that the decoder makes an error.

A number $R \geqq 0$, is said to be *permissible* if for every $\lambda > 0$ there is a $T = T(\lambda)$ sufficiently large that there exists a code with parameters (T, M, P_e), where

$$M \geqq e^{RT}, \qquad P_e \leqq \lambda.$$

The channel capacity C is defined as the supremum of permissible R. Reference 1 has a detailed discussion of this problem and its formulation.

In what follows, we shall specify a set $\mathcal{A}(T, W, P_0)$ and also specify the noise. The main result is a formula for C. This model is very similar to Model 4 in Ref. 1.

(*i*) Let $\mathcal{A}(T, W, P_0)$ be the set of functions $s(t)$, $-\infty < t < \infty$, which satisfy

$$\text{(a)}\quad s(t) = 0, \qquad |t| \geqq T/2, \tag{3a}$$

$$\text{(b)}\quad \|s\|^2 \leqq P_0 T, \tag{3b}$$

$$\text{(c)}\quad s(t) \text{ is band-limited to } (-W, W) \text{ at level } \epsilon > 0. \tag{3c}$$

Thus, $\mathcal{A}(T, W_0, P_0)$ is a set of strictly time-limited and approximately band-limited signals.

(*ii*) The noise function $z(t)$, is assumed to be a sample from a gaussian noise process with spectral density

$$N(f) = \begin{cases} N_0/2, & |f| < W, \\ 0, & |f| \geqq W. \end{cases} \tag{4}$$

Let us remark at this point that although we assume in our model that the signal is exactly time-limited to $(-T/2, T/2)$ and the noise is exactly band-limited to $(-W, W)$, our results do not exploit these assumptions. In fact, our results will hold if we introduce appropriate approximations here too.

Finally, we must make the assumption that the decoder function is not capable of distinguishing among signals that are arbitrarily close together. Specifically, we assume that if $y_1(t)$, $y_2(t)$, $-T/2 < t < T/2$ are functions that are mapped by the decoder to distinct code words, then

$$\int_{-T/2}^{T/2} [y_1(t) - y_2(t)]^2 dt \geqq \epsilon'. \tag{5}$$

Inequality (5) is equivalent to requiring that the segments of $y_1(t)$ and $y_2(t)$, $|t| < T/2$ (on which the decoding must be done), are really distinguishable at level ϵ'.* Put another way, the receiver must be insensitive to measurement errors of energy $<\epsilon'/4$.

III. THE RESULT

We state our result as a theorem.

Theorem: For the model defined above,

$$C = W \log\left(1 + \frac{P_0}{N_0 W}\right), \tag{6}$$

provided $\epsilon' > 4\epsilon$.

This result is analogous to the "$2WT$" theorem given by Slepian in Ref. 2. Note that (6) holds for every ϵ and ϵ' provided only that $\epsilon' > 4\epsilon$. Thus, the result is independent of the precision with which we can make measurements.

Proof: The theorem follows immediately from the capacity formula (28) given for Model 4 in Ref. 1. Observe that our $\mathcal{A}(T, W, P_0)$ is identical to the set $a_4(T, W, P_0)$, with $\eta = \epsilon/(P_0 T)$. (Note that no changes in the capacity formula will result when we require the channel input signals to have energy *exactly* $P_0 T$.)

Also note that the right member of ineq. (29) of Ref. 1 should be "$4\nu N_0 WT$." Thus, our assumption in (5) is identical to the assumption of (29) in Ref. 1 with $\nu = \epsilon'/(4N_0 WT)$.

It follows that the capacity formula (28) in Ref. 1 holds; that is, for our model

$$C = W \log\left(1 + \frac{P_0}{N_0 W}\right) + \epsilon(\eta, \nu), \tag{7}$$

where $\epsilon(\eta, \nu) \to 0$, as $\eta, \nu \to 0$, provided

$$\frac{\nu}{\eta} > \frac{P_0}{N_0 W}. \tag{8}$$

Since $\eta = \epsilon/(P_0 T)$ and $\nu = \epsilon'/(4N_0 WT)$, both $\eta, \nu \to 0$ as $T \to \infty$. Further, (8) holds if $\epsilon'/\epsilon > 4$, so that (7) becomes (6) as $T \to \infty$.

REFERENCES

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2. D. Slepian, "On Bandwidth," 1974 Shannon Lecture, presented at the International Symposium on Information Theory, Notre Dame, Indiana, October 21, 1974, Proceedings of the IEEE, *64*, No. 3 (March 1976), pp. 292–300.

* This assumption requires that the space of received signals contain "null zones" which are not in the domain of the decoder mapping. When the received signal belongs to a null zone, the decoder declares an error.