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Spectral Moment Estimators: A New Approach to Tone Detection

By J. N. DENENBERG

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Classical tone detectors use narrow bandpass filters to isolate tones. A comparison of a filter's output power against the total input signal power determines whether or not a particular tone is present. This paper considers an alternate method for tone detection. It is based on estimating the first three moments of the signal's band-limited power spectrum. These three moments (zeroth, first, and second) measure the power, power mean frequency, and RMS power bandwidth, respectively, of the signal. If these three moments are available, it is easy to deduce whether or not a tone is present. Since the estimators have a simple digital implementation, this approach should have economic advantages in many applications.

I. INTRODUCTION

The use of tones is proliferating with the growth of the direct-distance-dialing (DDD) network. Tones are used for multifrequency signaling, *Touch-Tone*® signaling, etc.; for milliwatt tone testing, testing in carrier transmission maintenance,[1] etc.; and as audible ring indicators, reorder indicators, etc.

The DDD network is now evolving toward an integrated system of digital transmission and digital switching. Continued use of analog tone detectors in this digital environment is impractical because of the expense of interfacing analog and digital systems. Classical tone detectors do not lend themselves to efficient digital implementation. Therefore, new approaches to tone detection are necessary.

Several methods are currently under investigation including classical digital filters,[2] the discrete Fourier transform,[3,4] and the use of counting techniques.[5] This paper details another approach to tone detection which is amenable to digital implementation: spectral moment estimation.

II. TONE DETECTION

2.1 Classical tone detectors

The classical approach to tone detection uses a narrow bandpass filter, centered at the expected tone frequency, to isolate the tone. A comparison of the power passed by this filter against the power that is input to the filter (and some absolute level requirements) determines the presence or absence of the tone.

This classical decision algorithm is based on the fact that a simple tone has a narrow bandwidth. A tone can, therefore, readily pass through a narrow filter with little loss in power. Several assumptions are implicit. First, a tone's spectrum is narrow only if the tone is present for a long time. If the detection must be made quickly, the filter's bandwidth must be increased to pass all the tone's power. The second assumption concerns the tone's spectral purity. Rapid variations in frequency and amplitude will also effectively widen the bandwidth.

The decision algorithm can be restated in terms of three signal parameters: power, mean frequency, and bandwidth. Thresholds are set by the filter characteristic for the signal's mean frequency and bandwidth (these thresholds are interrelated). Additional limits are placed on signal power. All these criteria must be met if the signal is to be recognized as a tone.

2.2 Spectral moments as tone indicators

A heuristic argument that is similar to that described above can be used to derive an alternate approach to tone detection. Devices can be built to measure the signal's first three spectral moments:* power (P), power mean frequency (f_a), and mean square bandwidth (b^2). These spectral moments are defined in terms of the signal's power spectrum, $S(f)$, as

$$P = \int_f S(f)df,$$

$$f_a = \frac{1}{P}\int_f fS(f)df,$$

* Practical schemes for measuring spectral moments exist. One is described in later sections of this paper.

and

$$b^2 = \frac{1}{P} \int_f (f - f_a)^2 S(f) df,$$

where the region of integration is limited to the range of interest.

Then, for any given tone, thresholds can be set: the power must lie within a required range, the mean frequency must be within tolerances of the expected frequency, and the bandwidth must be less than a specified amount. If these requirements are all satisfied, then the expected tone is declared to be present.

Detection of an unknown tone within a given range of frequencies is also possible. Whenever a tone is present, the bandwidth indication alone can detect it. The other moments can then be used as measurement tools. This demonstrates the power of the spectral moment estimator algorithm.

III. SPECTRAL MOMENT ESTIMATION

3.1 The classical approach

In the past, spectral moments have been estimated by first estimating the spectrum $S(f)$ itself and then substituting this estimate into the definitions of the desired parameters. This classical approach is well documented in the literature.[6]

There are two approaches to the estimation of $S(f)$ from a time-limited segment of the input signal. In one method, it is assumed that the signal is periodic outside the known interval, while in the other, the signal is assumed zero outside that interval. An assumption of periodicity leads to a complex exponential Fourier series that is equal to the input over the known interval. This spectral estimate is discrete in frequency. It consists of weighted impulses at each frequency; the weighting factor for an impulse is the magnitude squared of the corresponding coefficient in the series. Each integral defining the spectral moments becomes a sum.

Either spectral estimate may be used in practice. For long measurement intervals (large T), they give identical results. The remainder of this paper is based on the continuous spectral estimate that assumes zero input outside the measurement interval. This choice does not affect the final form of the digital estimators.

3.2 Direct estimation

3.2.1 Motivation

The classical methods for estimating spectral moments require a spectrum analyzer. This expensive apparatus can take various forms including a filter bank, a Fast Fourier Transform (FFT) processor, etc.

However, since many applications require real-time spectrum analysis, the classical methods are often impractical.

Complete information about spectral shape is not required in tone detection. So why take the expensive step of calculating this detailed information? A system that calculates the desired parameters directly from the input data should be more efficient and, therefore, less costly.

Direct spectral parameter estimators exist in the literature. They have been used in applications including communication channel measurement,[7] radar meteorology,[8–10] and frequency-modulation detection.[11]

3.2.2 Signal representation

The received signal $s(t)$ described in Section II is a narrow-band signal. It can be represented by its complex envelope,

$$x(t) = \alpha(t) + j\beta(t), \tag{1}$$

as

$$s(t) = \alpha(t)\cos(2\pi f_0 t) - \beta(t)\sin(2\pi f_0 t), \tag{2}$$

where f_0, the reference frequency, lies near the center of the signal's spectrum. This representation is also well documented in the literature.[12]

Since it is easy* to determine $\alpha(t)$ and $\beta(t)$, the signal's quadrature components, from $s(t)$ and f_0 is determined at the receiver, the signal's spectral moments can be determined from its complex envelope, $x(t)$.

3.2.3 Estimation algorithms

If an estimate, $\hat{S}(f)$, of the signal's power spectrum is based on $x(t)$, the three spectral moment estimates are, classically,

$$\hat{P} = \int_f \hat{S}(f)df, \tag{3}$$

$$\hat{f}_a = \frac{1}{\hat{P}} \int_f f\hat{S}(f)df^\dagger \tag{4}$$

and

$$\hat{b}^2 = \frac{1}{\hat{P}} \int_f [f - \hat{f}_a]^2 \hat{S}(f)df, \tag{5}$$

where the integrals are only computed over frequencies of interest.

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* The method of extracting the signal's quadrature components is discussed in Section IV.

† $\hat{f}_a$ is shifted from f_a by the known quantity f_o since $\hat{S}(f)$ is an estimate of the power spectral density of $x(t)$ instead of $s(t)$.

Earlier work[13] has shown that direct calculation of these three quantities from $x(t)$ is possible. The algorithms are

$$\hat{P} = \frac{1}{T}\int_0^T [\alpha^2(t) + \beta^2(t)]dt, \tag{6}$$

$$\hat{f}_a = \frac{1}{2\pi T\hat{P}}\int_0^T [\alpha(t)\dot{\beta}(t) - \beta(t)\dot{\alpha}(t)]dt, \tag{7}$$

and

$$\hat{b}^2 = \frac{1}{4\pi^2 T\hat{P}}\int_0^T \{[\dot{\alpha}(t)]^2 + [\dot{\beta}(t)]^2\}dt - [\hat{f}_a]^2, \tag{8}$$

where the dot over a quantity indicates the derivative with respect to time.

3.3 Estimate error analysis

Since the direct estimators were derived from the classical estimators, their performance will be equal. Therefore, an analysis of the statistical behavior of the classical estimators can be applied to the direct estimators.

Miller and Rochwarger[14] have done such an analysis. They assume that the error in each measurement is small enough so that a power series expansion converges rapidly. Their results, however, are difficult to apply in practice and do not address the time-limited nature of the measurement. A similar analysis, which takes into account the time-limited nature of the measurement, was done by the author.[13] It also assumes convergence of the power series. In that analysis, the input is a complex sample function from a gaussian random process with total power P, a power spectral density $S(f)$, power mean frequency f_a, and mean square bandwidth b^2. The results are given below.

Estimator bias and variance are of interest for each of the three spectral moments. For the power estimator, the results agree with those found in textbooks.[15] Its expected value is

$$E[\hat{P}] = P. \tag{9}$$

It is therefore unbiased and yields, on the average, the total received power. The power estimator's variance is given by

$$\text{var}\,[\hat{P}] = \frac{1}{T}\int_f S^2(f)df. \tag{10}$$

The expected value of the power mean frequency estimator is

$$E[\hat{f}_a] = (f_a - f_o) - \frac{1}{P^2T}\int_f fS^2(f + f_a)df. \tag{11}$$

It is therefore biased in the general case. An unbiased estimate occurs when the power spectral density is symmetrical about its mean. A similar equation gives the variance of $\hat{f}_a$:

$$\text{var}\,[\hat{f}_a] = \frac{1}{P^2T}\int_f f^2 S^2(f + f_a)df. \tag{12}$$

It is clear from

$$E[\hat{b}^2] = b^2 + \frac{1}{P^2T}\int_f (b^2 - 2f^2)S^2(f + f_a)df \tag{13}$$

that $\hat{b}^2$ is also biased. Its variance is given by

$$\text{var}\,[\hat{b}^2] = \frac{1}{P^2T}\int_f (f^2 - b^2)^2 S^2(f + f_a)df. \tag{14}$$

Note that all the biases and variances are independent of the actual mean frequency. They are functions of the measurement interval, spectral width (b^2), and shape. All of them decrease to zero as the measurement interval increases or as b^2 decreases.

3.4 Computer simulation

The equations in Section 3.3 provide simple theoretical expressions for the bias and variance of the power mean frequency and mean square bandwidth estimators. These expressions are valid for reasonably long time measurements. Since the equations for f_a and b^2 are new, a computer simulation was conducted to substantiate the validity of these expressions. A sequence of independent, complex, gaussian processes was generated. These complex, independent time samples were then transformed by an FFT to yield a flat, or white noise, spectrum on the average. The flat spectrum was then shaped by a gaussian frequency response function. This yielded a set of random spectral coefficients with a gaussian shape in the mean. These frequency coefficients were then used to generate independent estimates of f_{ag} and b_g^2 from each member of the sequence, where f_{ag} and b_g^2 are the mean frequency and variance of the chosen gaussian shape.

Fifty pairs of estimates, $\hat{f}_{ag}$ and $\hat{b}_g^2$, were obtained from 50 independent spectral estimates. Each spectral estimate was derived from 512 complex time samples, thus yielding 512 line resolution. The results of the simulation are shown in Table I where the bar over the quantities of interest indicates the average over the 50 estimates. The mean square error of the quantity of interest, $\epsilon(\ )$, was also obtained by an average over the 50 estimates, i.e., $\epsilon(\hat{f}_a)$ is given by

$$\epsilon(\hat{f}_a) = \overline{(\hat{f}_a - f_a)^2}. \tag{15}$$

Table I — Computer simulation

	$f_{ag} = 0$, $b_g^2 = 400$	$f_{ag} = 128$, $b_g^2 = 400$	Theoretical Values
$\overline{\hat{f}_{ag}}$	−0.07	127.8	0, 128
$\epsilon(\hat{f}_{ag})$	4.05	2.56	2.8
$\overline{\hat{b}_g^2}$	414	395.4	400, 400
$\epsilon(\hat{b}_g)$	1.51	1.48	1.06, 1.06*

* This row is adjusted to be in Hz² using

$$\epsilon(\hat{b}_g) = \frac{\epsilon(\hat{b}_g^2)}{4b_g^2}.$$

Specifically, the gaussian spectral shape is given by

$$S_g(f + f_a) = \frac{1}{\sqrt{2\pi}b_g} \exp\left[-\frac{f^2}{2b_g^2}\right]; \quad -\infty < f < +\infty. \tag{16}$$

Evaluating the equations in Section 3.3 for this example yields

$$E[\hat{f}_a] = f_a, \tag{17}$$

$$\text{var}\,[\hat{f}_a] = \frac{b_g}{4\sqrt{\pi}T}, \tag{18}$$

$$E[\hat{b}^2] = b_g^2, \tag{19}$$

$$\text{var}\,[\hat{b}^2] = \frac{3b_g^3}{8\sqrt{\pi}T}. \tag{20}$$

Note that both estimates are unbiased in this example.

Two cases were simulated. The first had a mean frequency of zero and an RMS bandwidth of 20 Hz. The second case also had a bandwidth of 20 Hz; however, it was centered at 128 Hz. Both estimates were accurate on the average. Their mean square error was predicted with 50 percent accuracy by the equations in Section 3.3.

IV. A DIGITAL TONE DETECTOR

4.1 The application

As noted in Section I, the evolution of the voiceband DDD network toward an integrated system of digital switching and digital transmission makes the use of digital service circuits attractive. In this environment, several assumptions can be made about the form of the input signal. These assumptions are: the sampling rate of the input signal is that of the T-carrier systems (8 kHz), the approximate frequency of the tone is known, and (although the digital encoding is logarithmically compressed in practice) the code is linear. The second assumption allows the tone detector to search for the tone in a narrow band of frequencies and makes the techniques of Sections II and III applicable.

4.2 Quadrature detection

4.2.1 Analysis

In Section III, a narrow-band signal was expressed in terms of its complex envelope as

$$s(t) = \alpha(t) \cos (2\pi f_0 t) - \beta(t) \sin (2\pi f_0 t). \tag{2}$$

Multiplying $s(t)$ by a locally generated cosine wave with a peak amplitude of 2 and a frequency of f_0 yields

$$2 \cos (2\pi f_0 t) \cdot s(t) = \alpha(t)[1 + \cos (4\pi f_0 t)] - \beta(t) \sin 4\pi f_0 t. \tag{21}$$

If this result is passed through a low-pass filter to remove the components near $2f_0$, $\alpha(t)$ is the output. Similarly, if the input is multiplied by another locally generated signal, 90 degrees out of phase from the first reference,

$$-2 \sin (2\pi f_0 t) \cdot s(t) = -\alpha(t) \sin (4\pi f_0 t) + \beta(t)[1 - \cos (2\pi f_0 t)], \tag{22}$$

the result can be filtered to produce $\beta(t)$.

Figure 1 shows a block diagram of the quadrature detection process. The low-pass filters, in addition to eliminating the undesired terms due to mixing, perform an effective bandpass filtering operation on the input. If f_m is the highest frequency passed by the filters, only

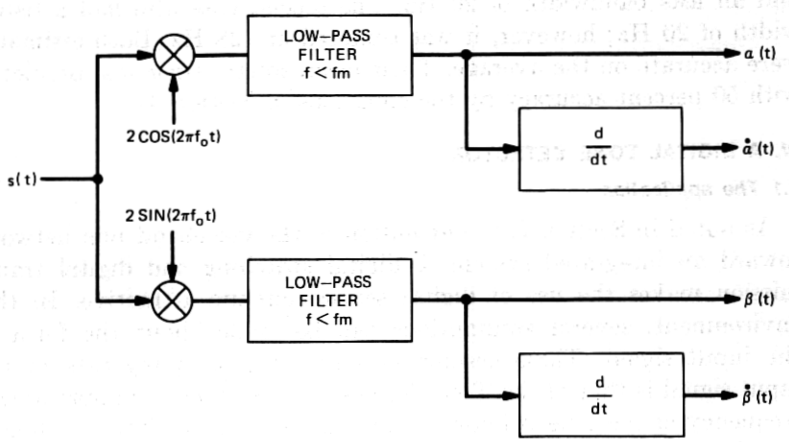

Fig. 1—Quadrature detection.

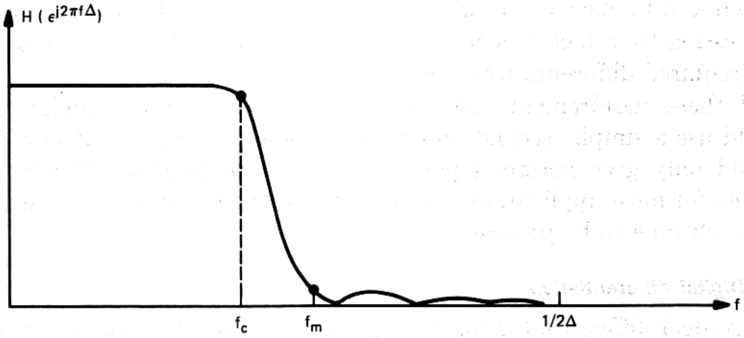

Fig. 2—Typical low-pass filter response.

frequencies in the range

$$f_0 - f_m \leqq f \leqq f_0 + f_m \tag{23}$$

have any effect on the resulting measurements. Differentiation required for the remaining calculations is also shown in Fig. 1. It is not really part of the quadrature detection process and is discussed in Section 4.3.

4.2.2 Low-pass filter requirements

Two parameters of the filters used in quadrature detection are important. They are the cutoff frequency f_c and the maximum frequency f_m. Figure 2 shows a typical low-pass filter frequency characteristic defining its two parameters f_c and f_m.

The parameter f_c sets the frequency limits on the measurement. It should be high enough to allow for any possible allowed frequency offset plus some margin since the tone's bandwidth is spread by the time-limited measurement (on the order of 100 Hz for a 10-ms interval). f_c has an upper limit which is determined by several factors. First, it cannot crowd f_m. This would raise the complexity of the filter and make this approach impractical. It also should not be so high as to allow an unnecessary amount of the background noise to disturb the measurement. An f_c between 200 and 350 Hz will be the usual compromise.

The maximum frequency f_m determines the immunity of the detector to interfering signals. If it is low enough, it also allows the use of resampling which results in considerable savings in the processing after detection. Considerations of immunity usually set an upper limit on f_m of 1 kHz*, while the likelihood of resampling reduces this to 500 Hz. If an f_m of 500 Hz or less is used, the output sample rate can

* One possible application, translating the output of a *Touch-Tone* phone, requires an f_m of 350 Hz for the lower group tone detector. f_c for this detector is at least 175 Hz.

be reduced to once every millisecond. This alone reduces the remaining processing by a factor of 8. It also allows a simple implementation of the required differentiators.

If these requirements are specified, some noncritical applications could use a simple, second-order, digital, low-pass filter. But since this would only give marginal performance, a fourth-order filter is suggested for most applications. Note that two of these filters are required for each tone to be processed.

4.3 Digital differentiation

An ideal differentiator for this application has a frequency response which is proportional to frequency for all frequencies below f_c and rolls off rapidly above f_c. It also has a phase shift of 90 degrees for all frequencies below f_c.

The simplest form for a digital differentiator is the difference between two input samples. This approximates the input's derivative at a time midway between the sample times. More complex differentiator structures are possible; they are usually unnecessary.[16]

In this application, it is necessary to get the derivative at the sample time. Therefore, the differentiator should calculate

$$y_k = x_{k-1} - x_{k+1}, \tag{24}$$

where y_k is the approximate derivative at the time input sample x_k appeared. Note that if only every eighth sample of the input is going to be used in the subsequent processing, the computation of a derivative every millisecond uses two of the other seven samples during that interval.

The accuracy of this approximation can be easily calculated. Taking the z transform of (24), restricting z to the unit circle, and simplifying result in

$$H(\epsilon^{j2\pi f\tau}) = j2 \sin (2\pi f\tau), \tag{25}$$

where τ is the time between samples and f is the frequency input to the differentiator.

In most voiceband tone-detection applications, τ is 125 μs and f is less than 500 Hz.* Therefore, the transfer function can be approximated by the first term of its power series

$$H(\epsilon^{j2\pi f\tau}) \cong j4\pi f\tau. \tag{26}$$

For the numbers stated, this is only a 3-percent error in the worst case. Therefore, the simple differentiator usually suffices.

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* The frequency input to the differentiator is the difference between the signal's frequency and f_o, the reference frequency.

4.4 Calculation of the spectral moment estimates

Quadrature detection, as developed in Section 4.2, yields samples of the input signal's quadrature components $\alpha(t)$ and $\beta(t)$ each millisecond. The differentiators discussed in Section 4.3 give corresponding samples of $\dot{\alpha}(t)$ and $\dot{\beta}(t)$ each millisecond. Calculation of the three estimates from this sequence of four samples each millisecond is straightforward.

The power estimate is given by (6) as:

$$\hat{P} = \frac{1}{T}\int_0^T [\alpha^2(t) + \beta^2(t)]dt. \tag{6}$$

Since $\alpha(t)$ and $\beta(t)$ are only available as samples each millisecond, the integral reduces to an equivalent summation,

$$\hat{P} = \frac{1}{N}\sum_{k=0}^{N-1} [\alpha_k^2 + \beta_k^2], \tag{27}$$

where T, the measurement interval, is given by

$$T = N\tau. \tag{28}$$

Similarly, the estimate of the power times the power mean frequency can be obtained from (7).

$$\widehat{Pf_a} = \frac{1}{2\pi T}\int_0^T [\alpha(t)\dot{\beta}(t) - \beta(t)\dot{\alpha}(t)]dt. \tag{29}$$

The equivalent sum is

$$\widehat{Pf_a} = \frac{1}{4\pi N\tau}\sum_{k=0}^{N-1} [\alpha_k(\beta_{k-1} - \beta_{k+1}) - \beta_k(\alpha_{k-1} - \alpha_{k+1})]. \tag{30}$$

Finally, the estimate giving bandwidth information is derived from (8):

$$\widehat{P(b^2 + f_a^2)} = \frac{1}{4\pi^2 T}\int_0^T \{[\dot{\alpha}(t)]^2 + [\dot{\beta}(t)]^2\}dt. \tag{31}$$

The corresponding equivalent sum is

$$\widehat{P(b^2 + f_a^2)} = \frac{1}{16\pi^2 N\tau^2}\sum_{k=0}^{N-1} [(\alpha_{k-1} - \alpha_{k+1})^2 + (\beta_{k-1} - \beta_{k+1})^2]. \tag{32}$$

Algorithms for the three spectral parameters of interest are now available. $\hat{P}$ is given by (27). Dividing (27) into (30) yields

$$\hat{f}_a = \frac{\widehat{Pf_a}}{\hat{P}}. \tag{33}$$

Similarly, $\hat{b}^2$ is obtained using (27), (30), and (32):

$$\hat{b}^2 = \frac{\widehat{P(b^2 + f_a^2)}}{\hat{P}} - (\hat{f}_a)^2. \tag{34}$$

4.5 Tone detection

The system developed in Sections 4.1 through 4.4 gives estimates of the three spectral parameters P, f_a, and b^2 over a time interval of T seconds. Thresholds must now be set, as discussed in Section II, to determine the presence or absence of a valid tone during that interval. Values for these thresholds are dependent on the particular application and, therefore, are not calculated here. It is suggested, however, that they be determined experimentally rather than through the use of the analysis of Section III. This experimental determination can be made either through simulation or by breadboarding the design. It will give a more reliable result because the approximations needed for the statistical analysis are eliminated.

One final point needs discussion. Many applications have requirements on the measurement interval which are near theoretical limits.[17] Meeting these specifications will require tone detection during overlapping time intervals. This increases the storage requirements of the system.

An approximation can be made which eliminates this increase in storage. Recall that a low-pass filter performs the weighted integral given by

$$y(t) = \int_{-\infty}^{t} x(\lambda)h(t - \lambda)d\lambda, \tag{35}$$

where $h(t)$, the filter's impulse response, is controlled by the filter structure. Therefore, a filter can be used to do the averaging necessary for tone detection. Although this approximation can be made as accurate as desired, the filter's complexity increases rapidly. A simple "lossy integrator" suffices for many applications.

V. CONCLUSIONS

Spectral moment estimators have been used to develop a viable alternative structure for a digital tone detector. The resulting digital system is flexible and can adapt to any tone detection application in which a single tone lies somewhere in a fixed bandwidth. It is easily programmable to cover multiple uses. This system has one additional feature that is not present in most other detector structures: the incoming tone frequency need not be known *a priori*.

VI. ACKNOWLEDGMENTS

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